Superadditive and subadditive integrals

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Outlines

- Motivation and preliminaries
- Superadditive and Subadditive integrals, definition and basic properties
- Relations between the Superadditive and Subadditive integrals with other fuzzy integrals

Motivation of the paper

The concave integral (Lehrer (2009))

Lehrer (2009), introduced a a new integral for capacities, which He called the *concave integral*.

We define an extension of the concave integral, called the *superadditive integral*. We present also the concept of *subadditive integral*.

A motivating example I

Suppose there are three items a, b and c on the markets whose prices are

$$p_a = 1, p_b = 2, p_c = 3$$

Buying or selling

- x_a units of a,
- x_b units of b,
- x_c units of c,

the total price is

$$P(x_a, x_b, x_c) = p_a \times x_a + p_b \times x_b + p_c \times x_c.$$

For example

$$P(10, 15, 5) = 10 \times 1 + 15 \times 2 + 5 \times 3 = 55$$

A motivating example II

Suppose you can buy or sell also combinations of two or three of above items with the following prices

$$p_{ab} = 4, p_{ac} = 5, p_{bc} = 6, p_{abc} = 7.$$

In this case the total price for a seller of (x_a, x_b, x_c) is

$$P^S(x_a, x_b, x_c) =$$

$$max\{p_a \times x'_a + p_b \times x'_b + p_c \times x'_c +$$

$$p_{ab}x'_{ab} + p_{ac}x'_{ac} + p_{bc}x'_{bc} + p_{abc}x'_{abc}$$

such that

$$x'_a + x'_{ab} + x'_{ac} + x'_{abc} \leq x_a,$$

$$x'_b + x'_{ab} + x'_{bc} + x'_{abc} \leq x_b,$$

$$x'_c + x'_{ac} + x'_{bc} + x'_{abc} \leq x_c\}$$

A motivating example III

The total price for a buyer of (x_a, x_b, x_c) is

$$P^B(x_a, x_b, x_c) =$$

$$min\{p_a \times x'_a + p_b \times x'_b + p_c \times x'_c +$$

$$p_{ab}x'_{ab} + p_{ac}x'_{ac} + p_{bc}x'_{bc} + p_{abc}x'_{abc}$$

such that

$$x'_a + x'_{ab} + x'_{ac} + x'_{abc} \geq x_a,$$

$$x'_b + x'_{ab} + x'_{bc} + x'_{abc} \geq x_b,$$

$$x'_c + x'_{ac} + x'_{bc} + x'_{abc} \geq x_c\}$$

A motivating example IV

For example

$$P^S(10, 15, 5) = 3.67 \times p_b + 6.33 \times p_{ac} + 1.33 \times p_{ac} + 3.67 \times p_{abc} = 66.33$$

and

$$P^B(10, 15, 5) = 10 \times p_a + 15 \times p_b + 5 \times p_c = 55$$

A motivating example V

Suppose that one can buy or sell only integer quantities of a, b, c, ab, ac, bc, abc and, moreover that in the market there is the possibility to buy or to sell the following combinations of items a, b abd c with the corresponding prices:

$$p_{aa} = 3, p_{bb} = 3, p_{cc} = 5, p_{aab} = 4, p_{aac}p_{abb} = 5$$

$$p_{aac} = 6, p_{acc} = 7, p_{bbc} = 8, p_{bcc} = 8, p_{aaa} = 4, p_{bbb} = 6, p_{ccc} = 10$$

A motivating example V

In this case the price of (x_a, x_b, x_c) for a seller is

$$P^S(x_a, x_b, x_c) = max\{x'_a p_a + x'_b p_b + x'_c p_c + p_{ab} x'_{ab} + p_{ac} x'_{ac} + p_{bc} x'_{bc} + p_{abc} x'_{abc} +$$

$$p_{aa} x'_{aa} + p_{bb} x'_{bb} + p_{cc} x'_{cc} + p_{aab} x'_{aab} + p_{abb} x'_{abb} + p_{aac} x'_{aac} +$$

$$p_{acc} x'_{acc} + p_{bbc} x'_{bbc} + p_{bcc} x'_{bcc} + p_{aaa} x'_{aaa} + p_{bbb} x'_{bbb} + p_{ccc} x'_{ccc}$$

such that

$$x'_a + x'_{ab} + x'_{ac} + x'_{abc} + 2x'_{aa} + 2x'_{aab} + x'_{abb} + 2x'_{aac} + x'_{acc} + 3x'_{aaa} \leq x_a$$

$$x'_b + x'_{ab} + x'_{bc} + 2x'_{bb} + x'_{abc} + x'_{aab} + 2x'_{abb} + 2x'_{bbc} + x'_{bcc} + 3x'_{bbb} \leq x_b$$

$$x'_c + x'_{ac} + x'_{bc} + 2x'_{cc} + x'_{abc} + x'_{aac} + 2x'_{acc} + x'_{bbc} + 2x'_{bcc} + 3x'_{ccc} \leq x_a$$

$$x'_a, x'_b, x'_c, x'_{ab}, \ldots, x'_{aaa}, x'_{bbb}, x'_{ccc} \in \mathbb{N}$$

A motivating example VI

In this case the price of (x_a, x_b, x_c) for a buyer is

$$P^B(x_a, x_b, x_c) = min\{x'_a p_a + x'_b p_b + x'_c p_c + p_{ab} x'_{ab} + p_{ac} x'_{ac} + p_{bc} x'_{bc} + p_{abc} x'_{abc} +$$

$$p_{aa} x'_{aa} + p_{bb} x'_{bb} + p_{cc} x'_{cc} + p_{aab} x'_{aab} + p_{abb} x'_{abb} + p_{aac} x'_{aac} +$$

$$p_{acc} x'_{acc} + p_{bbc} x'_{bbc} + p_{bcc} x'_{bcc} + p_{aaa} x'_{aaa} + p_{bbb} x'_{bbb} + p_{ccc} x'_{ccc}$$

such that

$$x'_a + x'_{ab} + x'_{ac} + x'_{abc} + 2x'_{aa} + 2x'_{aab} + x'_{abb} + 2x'_{aac} + x'_{acc} + 3x'_{aaa} \geq x_a$$

$$x'_b + x'_{ab} + x'_{bc} + 2x'_{bb} + x'_{abc} + x'_{aab} + 2x'_{abb} + 2x'_{bbc} + x'_{bcc} + 3x'_{bbb} \geq x_b$$

$$x'_c + x'_{ac} + x'_{bc} + 2x'_{cc} + x'_{abc} + x'_{aac} + 2x'_{acc} + x'_{bbc} + 2x'_{bcc} + 3x'_{ccc} \geq x_a$$

$$x'_a, x'_b, x'_c, x'_{ab}, \ldots, x'_{aaa}, x'_{bbb}, x'_{ccc} \in \mathbb{N}$$

A motivating example VII

We have

$$P^S(10, 15, 5) = 10p_{ab} + 5p_{bc} = 10 \times 4 + 5 \times 6 = 70$$

and

$$P^B(10, 15, 5) = 8p_a + p_c + 7p_{bb} + 2p_{cc} + 4p_{aab} = 8\times1+1\times3+7\times7+2\times5+4\times4 = 46$$

Thus we want to study a new integral permitting to model price systems.
In fact, P^S is what we shall call a superadditive integral, while P^B is what we shall call a subadditive integral.

The setting

We present the result in a MCDM framework

- $N = \{1, \dots, n\}$ is the set of *criteria*;
- $\mathbb{R}_+^n$ is identified with the set of possible *alternatives*;
- For all $E \subseteq N$, $\mathbf{1}_E$ is the vector of $\mathbb{R}_+^n$ whose ith component equals 1 if $i \in E$ and equals 0 otherwise.

Definition

An aggregation function $A : \mathbb{R}_+^n \to \mathbb{R}_+$ *is a monotone function such that* $\inf_{\boldsymbol{x} \in \mathbb{R}_+^n} A(\boldsymbol{x}) = 0$ *and* $\sup_{\boldsymbol{x} \in \mathbb{R}_+^n} A(\boldsymbol{x}) = +\infty$.

Properties of an aggregation function A

- *superadditivity* (*subadditivity*), for all $\boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}_+^n$

$$A(\boldsymbol{x}) + A(\boldsymbol{y}) \le A(\boldsymbol{x}+\boldsymbol{y}) \quad [A(\boldsymbol{x}) + A(\boldsymbol{y}) \ge A(\boldsymbol{x}+\boldsymbol{y})]\,;$$

- *homogeneity*, for all $\lambda \in \mathbb{R}_+$ and for all $\boldsymbol{x} \in \mathbb{R}_+^n$

$$A(\lambda\boldsymbol{x}) = \lambda A(\boldsymbol{x});$$

- *concavity* (*convexity*), for all $\lambda \in \mathbb{R}_+$ and for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}_+^n$

$$A(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \ge \lambda A(\boldsymbol{x}) + (1-\lambda)A(\boldsymbol{y})$$

$$[A(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \le \lambda A(\boldsymbol{x}) + (1-\lambda)A(\boldsymbol{y})]\,.$$

Superlinear aggregation functions

An aggregation function A being superadditive (subadditive) and homogeneous is defined *superlinear* (*sublinear*). Observe that any superlinear (sublinear) aggregation function is concave (convex). Of course the inverse is not true.

Definition

A measure on N is a monotone (w.r.t. set inclusion) function $\nu : 2^N \to \mathbb{R}_+$ satisfying the boundary condition $\nu(\emptyset) = 0$.
A capacity μ on N is a measure on N, satisfying $\mu(N) = 1$.

Fuzzy integrals 1

The Choquet integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. the capacity μ is

$$\int^{Ch} \boldsymbol{x} d\mu = \int_0^{max_{i \in N} x_i} \mu(\{i \in N : x_i \geq t\}) dt$$

The Sugeno integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. the measure ν is

$$\int^{Su} \boldsymbol{x} d\nu = max_{i=1}^n \{min(x_i, \nu(\{j \in N : x_j \geq x_i\}))\}$$

The Shilkret integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. the capacity μ is

$$\int^{Sh} \boldsymbol{x} d\mu = max\{x_i \mu(\{j \in N : x_j \geq x_i\}) : i = 1, \ldots, n\}$$

Fuzzy integrals 2

The concave integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. the capacity μ is

$$\int^{cav} \boldsymbol{x} d\mu = sup\{\sum_{T \subseteq N} \alpha_T \mu(T) : \sum_{T \subseteq N} \alpha_T \mathbf{1}_T \leq \boldsymbol{x}, \alpha_T \geq 0 \text{ for all } T \subseteq N\}.$$

The convex integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. the capacity μ is

$$\int^{con} \boldsymbol{x} d\mu = inf\{\sum_{T \subseteq N} \alpha_T \mu(T) : \sum_{T \subseteq N} \alpha_T \mathbf{1}_T \geq \boldsymbol{x}, \alpha_T \geq 0 \text{ for all } T \subseteq N\}.$$

Choquet, concave and convex integrals can be defined, more generally, w.r.t. a bounded measure, rather than a capacity.

Fuzzy integrals 3

A level dependent capacity is a function

$$\mu_{LD} : 2^N \times [0,1] \to [0,1]$$

such that for any $t \in [0,1]$, the following restriction is a capacity

$$\mu_{LD}(\cdot, t) : 2^N \to [0,1]$$

The level dependent Choquet integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ w.r.t. μ_{LD} is

$$\int^{Ch,LD} \boldsymbol{x} d\mu = \int_0^{max_{i \in N} x_i} [\mu(\{i \in N : x_i \geq t\}), t] \, dt.$$

Superadditive and subadditive integrals

The superadditive integral with respect to an aggregation function

Definition

Given an aggregation function A on $\mathbb{R}_+^n$ and $\boldsymbol{x} \in \mathbb{R}_+^n$, if

$$sup\{\sum_{j=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \mathbb{R}_+^n \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\} < +\infty \quad (Int)$$

then the superadditive integral of $\boldsymbol{x}$ with respect to A is

$$\int^{sup} \boldsymbol{x} dA = \sup\{\sum_{j=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \mathbb{R}_+^n \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\}.$$

Note that in definition of superadditive integral, the inequality $\sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}$ can be substituted with equality $\sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{x}$

We say that an aggregation function A is *a feasible base for integration* on $\mathbb{R}^n_+$ if condition (Int) is satisfied for all $\boldsymbol{x} \in \mathbb{R}^n_+$.

Proposition (1)

Let A : be an aggregation function. The following conditions are equivalent

$$\sup\{\sum_{j=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \mathbb{R}^n_+ \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\} < +\infty \ \forall\ \boldsymbol{x} \in \mathbb{R}^n_+ \tag{1}$$

$$sup\{\sum_{j=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \mathbb{R}^n_+ \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \mathbf{1}_N\} < +\infty \tag{2}$$

In words, a necessary and sufficient condition to be the (Int) valid for all $\boldsymbol{x} \in \mathbb{R}^n_+$ is that it holds for the constant vector $\mathbf{1}_N$.

Classes of aggregation functions admitting superadditive extension

Proposition (2)

Let $A : \mathbb{R}_+^n \to \mathbb{R}_+$ *be an aggregation function such that for all* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ *it holds* $A(\boldsymbol{x}) \leq \max_i x_i$ *(i.e. the values aggregation* $A(\boldsymbol{x})$ *cannot be greater than the greatest value). Then, condition (Int) holds for all* $\boldsymbol{x} \in \mathbb{R}_+^n$*.*

Proposition (3)

Let $A : \mathbb{R}_+^n \to \mathbb{R}_+$ *be an aggregation function which is idempotent less then a constant, i.e. there exists* $k \in \mathbb{R}_+$ *such that for all constant vectors* $\boldsymbol{c} = (c, \ldots, c) \in \mathbb{R}_+^n$ *it holds* $A(\boldsymbol{c}) = k \cdot c$*. Then, condition (Int) holds for all* $\boldsymbol{x} \in \mathbb{R}_+^n$*.*

The superadditive integral restricted to $X \subseteq \mathbb{R}^n_+$

Definition

Given an aggregation function A on $\mathbb{R}^n_+$, $X \subseteq \mathbb{R}^n_+$ with $\mathbf{0} \in X$, and $\boldsymbol{x} \in \mathbb{R}^n_+$, then the superadditive integral of $\boldsymbol{x}$ with respect to A and the domain X is defined as follows

$$\int_X^{sup} \boldsymbol{x} dA = \sup\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in X \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\}.$$

- If the domain X is finite, $\int_X^{sup} \boldsymbol{x} dA$ there exists finite for all $\boldsymbol{x} \in \mathbb{R}^n_+$ and for all aggregation functions A.
- In this case, in the definition, the inequality $\sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}$ cannot be substituted with equality $\sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{x}$.

The subadditive integral with respect to an aggregation function

Definition

Given an aggregation function A *on* $\mathbb{R}_+^n$ *and* $\boldsymbol{x} \in \mathbb{R}_+^n$*, the* *subadditive integral* *of* $\boldsymbol{x}$ *with respect to* A *is*

$$\int^{sub} \boldsymbol{x} dA = \inf\{\sum_{j1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \mathbb{R}_+^n \textit{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \geq \boldsymbol{x}\}.$$

We can note that

- any aggregation function admits subadditive extension,
- in definition of subadditive integral, the inequality $\sum_{j=1}^{k} \boldsymbol{y}^j \geq \boldsymbol{x}$ can be substituted with equality $\sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{x}$.

The subadditive integral restricted to $X \subseteq \mathbb{R}^n_+$

Definition

Given an aggregation function A on $\mathbb{R}^n_+$, $X \subseteq \mathbb{R}^n_+$ with $\mathbf{0} \in X$, and $\boldsymbol{x} \in \mathbb{R}^n_+$, then the subadditive integral of $\boldsymbol{x}$ with respect to A and the domain X is defined as follows

$$\int_X^{sub} \boldsymbol{x} dA = \inf\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in X \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \geq \boldsymbol{x}\}.$$

- If the domain X is finite, $\int_X^{sub} \boldsymbol{x} dA$ there exists finite for all $\boldsymbol{x} \in \mathbb{R}^n_+$ and for all aggregation functions A.
- In this case, in the definition, the inequality $\sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}$ cannot be substituted with equality $\sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{x}$.

Finitely decomposable superadditive integral

Definition

An aggregation function A is said with a finitely decomposable superadditive integral if for all $\boldsymbol{x} \in \mathbb{R}_+^n$, there is a finite set

$$S(\boldsymbol{x}) = \{\boldsymbol{y}^j \in \mathbb{R}_+^n, j = 1, \ldots, k\}$$

such that

$$\int^{sup} = \boldsymbol{x} dA = \sum_{\boldsymbol{y}^j \in S(\boldsymbol{x})} A(\boldsymbol{y}^j).$$

For the sake of simplicity in the following we shall denote $\int^{sup} \boldsymbol{x} dA$ by $A^*(\boldsymbol{x})$ and $\int^{sub} \boldsymbol{x} dA$ by $A_*(\boldsymbol{x})$.
Moreover, when we speak of aggregation functions A we implicitly suppose they are feasible basis for integration on $\mathbb{R}_+^n$.

FSTA 2014

Main results on superadditive and subadditive integrals

A characterization of the superadditive integral

Proposition (4)

For any aggregation function A *on* $\mathbb{R}^n_+$,

- $A^*(\mathbf{0}) = 0$, *and*
- $A^*(\boldsymbol{x}) \geq A^*(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$ *such that* $\boldsymbol{x} \geq \boldsymbol{y}$.

Moreover

$$A^*(\boldsymbol{x}) = \min\{C(\boldsymbol{x}) : \; C(\boldsymbol{y}) \geq A(\boldsymbol{y}), \; \forall \; \boldsymbol{y} \in \mathbb{R}^n_+\},$$

where the minimum is taken over all the superadditive aggregation functions on $\mathbb{R}^n_+$.

In words, $A^*(\cdot)$ is the smallest superadditive aggregation function greater or equal than $A(\boldsymbol{x})$.

A characterization of the subadditive integral

Proposition (5)

For any aggregation function A *on* $\mathbb{R}_+^n$*,*

- $A_*(\mathbf{0}) = 0$*, and*
- $A_*(\boldsymbol{x}) \geq A_*(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^n$ *such that* $\boldsymbol{x} \geq \boldsymbol{y}$*.*

Moreover

$$A_*(\boldsymbol{x}) = \max\{C(\boldsymbol{x}) : \ C(\boldsymbol{y}) \leq A(\boldsymbol{y}), \ \forall \ \boldsymbol{y} \in \mathbb{R}_+^n\},$$

where the minimum is taken over all the subadditive aggregation functions on $\mathbb{R}_+^n$*.*

In words, $A_*(\cdot)$ is the greatest subadditive aggregation function smaller or equal than $A(\boldsymbol{x})$.

Additional properties of A^* and A_* (1)

- $A^{**}(\boldsymbol{x}) = A^*(\boldsymbol{x})$ and $A_{**}(\boldsymbol{x}) = A_*(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{R}^n_+$;
- if $A(\boldsymbol{x}) \geq B(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$, then for all $\boldsymbol{x} \in \mathbb{R}^n_+$,

$$A^*(\boldsymbol{x}) \geq B^*(\boldsymbol{x}) \text{ and } A_*(\boldsymbol{x}) \geq B_*(\boldsymbol{x});$$

- if $A(\boldsymbol{x}) \geq B(\boldsymbol{x}) \geq C(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$ and $A^*(\boldsymbol{x}) = C^*(\boldsymbol{x})$, then for all $\boldsymbol{x} \in \mathbb{R}^n_+$

$$B^*(\boldsymbol{x}) = A^*(\boldsymbol{x});$$

- if $A(\boldsymbol{x}) \geq B(\boldsymbol{x}) \geq C(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$ and $A_*(\boldsymbol{x}) = C_*(\boldsymbol{x})$, then for all $\boldsymbol{x} \in \mathbb{R}^n_+$

$$B_*(\boldsymbol{x}) = A_*(\boldsymbol{x}).$$

Additional properties of A^* and A_* (2)

- if $A(\boldsymbol{x}) \geq B(\boldsymbol{x}), C(\boldsymbol{x}) \geq D(x)$ for all $\boldsymbol{x} \in \mathbb{R}_+^n$ and $A^*(\boldsymbol{x}) = D^*(\boldsymbol{x})$, then for all $\boldsymbol{x} \in \mathbb{R}_+^n$ and for all $\lambda \in [0,1]$

$$(\lambda B(\boldsymbol{x}) + (1-\lambda)C(\boldsymbol{x}))^* = A^*(\boldsymbol{x});$$

- if $A(\boldsymbol{x}) \geq B(\boldsymbol{x}), C(\boldsymbol{x}) \geq D(x)$ for all $\boldsymbol{x} \in \mathbb{R}_+^n$ and $A_*(\boldsymbol{x}) = D_*(\boldsymbol{x})$, then for all $\boldsymbol{x} \in \mathbb{R}_+^n$ and for all $\lambda \in [0,1]$

$$(\lambda B(\boldsymbol{x}) + (1-\lambda)C(\boldsymbol{x}))_* = A_*(\boldsymbol{x}).$$

Additional properties of A^* and A_* (3)

- if A is an aggregation functionn with a superadditive integral finitely decomposable, for all $\boldsymbol{x} \in \mathbb{R}_+^n$

$$A^*(\boldsymbol{x}) = max\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in X^{sup}(A) \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\},$$

where

$$X^{sup}(A) = \{z \in \mathbb{R}_+^n : A^*(z) = A(z)\};$$

- if A is with a subdditive integral finitely decomposable, for all $\boldsymbol{x} \in \mathbb{R}_+^n$

$$A^*(\boldsymbol{x}) = min\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in X^{sub}(A) \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \geq \boldsymbol{x}\},$$

where

$$X^{sub}(A) = \{z \in \mathbb{R}_+^n : A_*(z) = A(z)\}.$$

Additional properties of A^* and A_* (4)

- Considering the finite domain $X \subseteq \mathbb{R}_+^n$ with $\mathbf{0} \in X$, then

$$A_X^*(\boldsymbol{x}) = max\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \underline{X^{sup}}(A) \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \leq \boldsymbol{x}\},$$

where

$$\underline{X^{sup}}(A) = \{\boldsymbol{z} \in X^{sup}(A) : \nexists\, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^k \in X^{sup}(A) - \{\mathbf{0}\} : \sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{z}\};$$

considering the finite domain $X \subseteq \mathbb{R}_+^n$ with $\mathbf{0} \in X$, then

$$A_{X*}(\boldsymbol{x}) = min\{\sum_{i=1}^{k} A(\boldsymbol{y}^j) : \boldsymbol{y}^j \in \underline{X^{sub}}(A) \text{ such that } \sum_{j=1}^{k} \boldsymbol{y}^j \geq \boldsymbol{x}\},$$

where

$$\underline{X^{sub}}(A) = \{\boldsymbol{z} \in X^{sub}(A) : \nexists\, \boldsymbol{y}^1, \ldots, \boldsymbol{y}^k \in X^{sub}(A) - \{\mathbf{0}\} : \sum_{j=1}^{k} \boldsymbol{y}^j = \boldsymbol{z}\}.$$

Additional properties of A^* and A_* (5)

- if A and B are aggregation functions on $\mathbb{R}_+^n$ with a finite domain $X \subseteq \mathbb{R}_+^n$ with $\mathbf{0} \in X$, and

$$A_X^*(\boldsymbol{x}) = B_X^*(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}_+^n,$$

  then

$$B_X(\boldsymbol{x}) \geq \underline{A}_X(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}_+^n,$$

  with

$$\underline{A}_X(\boldsymbol{x}) = \max\{A(\boldsymbol{y}) : \boldsymbol{x} \geq \boldsymbol{y} \text{ with } \boldsymbol{y} \in \underline{X^{sup}}(A)\};$$

- if A and B are aggregation functions on $\mathbb{R}_+^n$ with a finite domain X, then $A^*(\boldsymbol{x}) = B^*(\boldsymbol{x})$ if and only if for all $\boldsymbol{x} \in \mathbb{R}_+^n$

$$\underline{A}(\boldsymbol{x}) \leq B(\boldsymbol{x}) \leq A^*(\boldsymbol{x});$$

Additional properties of A^* and A_* (6)

- for all $\boldsymbol{x} \in \mathbb{R}^n_+$
$$A^*(\boldsymbol{x}) \geq \widetilde{A}(\boldsymbol{x}) \leq A_*(\boldsymbol{x})$$
where
$$\widetilde{A}(\boldsymbol{x}) = \lim_{n \to +\infty} nA\left(\frac{1}{n}\boldsymbol{x}\right);$$
moreover, $\widetilde{A}$ is an homogeneous aggregation function;
- for all aggregation functions A and for all $\boldsymbol{x} \in \mathbb{R}^n_+$,
$$\widetilde{A}(\boldsymbol{x}) = \widetilde{\widetilde{A}}(\boldsymbol{x});$$
- for all aggregation functions A and B, if $A(\boldsymbol{x}) \geq B(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$,
$$\widetilde{A}(\boldsymbol{x}) \geq \widetilde{B}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n_+.$$

Additional properties of A^* and A_* (7)

For all aggregation functions A and for all $\boldsymbol{x} \in \mathbb{R}_+^n$

$$A^*(\boldsymbol{x})$$

$$\geq$$

$$\widetilde{(A^*)}(\boldsymbol{x}) = \left(\widetilde{A}\right)^*(\boldsymbol{x})$$

$$\geq$$

$$\widetilde{A}(\boldsymbol{x})$$

$$\geq$$

$$\left(\widetilde{A}\right)_*(\boldsymbol{x}) = \widetilde{(A_*)}(\boldsymbol{x})$$

$$\geq$$

$$A_*(\boldsymbol{x})$$

Additional properties of A^* and A_* (8)

For all aggregation functions A, if B is a superlinear aggregation function such that for all $\boldsymbol{x} \in \mathbb{R}_+^n$ $A^*(\boldsymbol{x}) \geq B(\boldsymbol{x})$, then

$$\widetilde{(A^*)}(\boldsymbol{x}) = \left(\widetilde{A}\right)^* (\boldsymbol{x}) \geq B(\boldsymbol{x}) \forall \boldsymbol{x} \in\in \mathbb{R}_+^n$$

i.e. $\widetilde{(A^*)}(\boldsymbol{x}) = \left(\widetilde{A}\right)^* (\boldsymbol{x})$ is the maximal superlinear function majorizing $A^*(\boldsymbol{x})$;

Additional properties of A^* and A_* (9)

For all aggregation functions A, if B is a sublinear aggregation function such that for all $\boldsymbol{x} \in \mathbb{R}_+^n$ $A_*(\boldsymbol{x}) \leq B(\boldsymbol{x})$, then

$$\widetilde{(A_*)}(\boldsymbol{x}) = \left(\widetilde{A}\right)_* (\boldsymbol{x}) \leq B(\boldsymbol{x}) \forall \boldsymbol{x} \in\in \mathbb{R}_+^n$$

i.e. $\widetilde{(A_*)}(\boldsymbol{x}) = \left(\widetilde{A}\right)_* (\boldsymbol{x})$ is the minimal superlinear function minorizing $A_*(\boldsymbol{x})$.

Additional properties of A^* and A_* (10)

- if A is homogeneous, then A^* is superlinear;
- if A^* is homogeneous, then

$$A^*(\boldsymbol{x}) = \min\{C(\boldsymbol{x}) : C(\boldsymbol{y}) \geq A(\boldsymbol{y}) \ \ \forall \boldsymbol{y} \in \mathbb{R}_+^n\}$$

  where the minimum is taken over all the superlinear aggregation functions on $\mathbb{R}_+^n$
- A^* is homogeneous iff there exists $W \subseteq \mathbb{R}_+^n$ such that

$$A^*(\boldsymbol{x}) = \min\{\sum_{i=1}^{n} x_i w_i : \boldsymbol{w} \in W\};$$

Additional properties of A^* and A_* (11)

- if A is homogeneous, then A_* is sublinear;
- if A_* is homogeneous, then

$$A_*(\boldsymbol{x}) = \max\{C(\boldsymbol{x}) : C(\boldsymbol{y}) \geq A(\boldsymbol{y}) \ \ \forall \boldsymbol{y} \in \mathbb{R}^n_+\}$$

  where the maximum is taken over all the sublinear aggregation functions on $\mathbb{R}^n_+$
- A_* is homogeneous iff there exists $W \subseteq \mathbb{R}^n_+$ such that

$$A^*(\boldsymbol{x}) = \max\{\sum_{i=1}^{n} x_i w_i : \boldsymbol{w} \in W\};$$

Additional properties of A^* and A_* (12)

For all aggregation functions A,

$$A^*(\boldsymbol{x}) \geq \left(\widetilde{A}\right)^*(\mathbf{1}_N) \int^{cav} \boldsymbol{x} d\mu \geq \left(\widetilde{A}\right)^*(\mathbf{1}_N) \int^{con} \boldsymbol{x} d\mu \geq A_*(\boldsymbol{x})$$

with

$$\mu(T) = \frac{\left(\widetilde{A}\right)^*(\mathbf{1}_T)}{\left(\widetilde{A}\right)^*(\mathbf{1}_N)}$$

for all $T \subseteq N$;
we can say also

$$A^*(\boldsymbol{x}) \geq \int^{cav} \boldsymbol{x} d\widetilde{\mu} \geq \int^{con} \boldsymbol{x} d\widetilde{\mu} \geq A_*(\boldsymbol{x})$$

with

$$\widetilde{\mu}(T) = \left(\widetilde{A}\right)^*(\mathbf{1}_T)$$

for all $T \subseteq N$;

Additional properties of A^* and A_* (13)

- if A is subadditive, then, for all $\boldsymbol{x} \in \mathbb{R}_+^n$,

$$A^*(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i$$

with

$$w_i = \widetilde{A}(\mathbf{1}_{\{i\}});$$

if A is superadditive, then, for all $\boldsymbol{x} \in \mathbb{R}_+^n$,

$$A_*(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i$$

with w_i defined as above;

Additional properties of A^* and A_* (14)

for any aggregation function A on $\mathbb{R}_+^n$

$$\left(A^*\right)_*(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i$$

with

$$w_i = \widetilde{\left(A^*\right)}\left(\mathbf{1}_{\{i\}}\right),$$

and

$$\left(A_*\right)^*(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i$$

with

$$w_i = \widetilde{\left(A_*\right)}\left(\mathbf{1}_{\{i\}}\right),$$

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Links of superadditive integral with other fuzzy integrals

Next theorem links the Choquet, Sugeno and Shilkret integrals to the concave integral via the superadditive integral.

Theorem

1. *If* $A(\boldsymbol{x}) = \int^{Ch} \boldsymbol{x} d\mu$, *then* $A^*(\boldsymbol{x}) = \int^{cav} \boldsymbol{x} d\mu$;
2. *if* $A(\boldsymbol{x}) = \int^{Su} \boldsymbol{x} d\nu$, *then* $A^*(\boldsymbol{x}) = \int^{cav} \boldsymbol{x} d\mu(\nu)$, *with*
$$\mu(\nu(T)) = \begin{cases} 1 & \text{if } \nu(T) > 0 \\ 0 & \text{if } \nu(T) = 0; \end{cases}$$
3. *if* $A(\boldsymbol{x}) = \int^{Sh} \boldsymbol{x} d\mu$, *then* $A^*(\boldsymbol{x}) = \int^{cav} \boldsymbol{x} d\mu$;

The concave integral of *Lehrer (2009)* is a superadditive integral. Next theorem collects new properties.

Theorem

The concave integral satisfies the following properties:

1. $\int^{cav} \boldsymbol{x} d\mu = \int^{cav} \boldsymbol{x} d\underline{\mu}$, with $\underline{\mu}(T) = \max\{\mu(S) : S \subseteq T$ and $S \in \mathcal{N}(\mu)\}$ where
$$\mathcal{N}(\mu) = \{T \subseteq N : \mu(T) = \int^{cav} \mathbf{1}_T d\mu\};$$
2. $\int^{cav} \boldsymbol{x} d\mu = \int^{cav} \boldsymbol{x} d\underline{\underline{\mu}}$, with $\underline{\underline{\mu}}(T) = \max\{\mu(S) : S \subseteq T$ and $S \in \underline{\mathcal{N}(\mu)}\}$ where
$$\underline{\mathcal{N}(\mu)} = \{T \in \mathcal{N}(\mu) : \nexists \text{ a partition of } T, \mathcal{T} = (T_1, \ldots, T_k),$$
$$\text{such that } : \textstyle\sum_{T_i \in \mathcal{T}} \mu(T_i) = \int^{cav} \mathbf{1}_T d\mu\};$$

3. for all $\boldsymbol{x} \in \mathbb{R}^n_+$, $\int^{cav} \boldsymbol{x} d\mu^\circ = \int^{cav} \boldsymbol{x} d\mu$ for all measure μ° such that
$$\underline{\underline{\mu}}(T) \leq \mu^\circ(T) \leq \overline{\mu}(T) \text{ for all } T \subseteq N,$$
with $\overline{\mu}(T) = \int^{cav} \mathbf{1}_T d\mu$;
4. if for the two capacities, μ_1 and μ_2 on 2^N we have
$$\int^{cav} \boldsymbol{x} d\mu_1 = \int^{cav} \boldsymbol{x} d\mu_2$$
for all $\boldsymbol{x} \in \mathbb{R}^n_+$, then
$$\int^{cav} \boldsymbol{x} d(\lambda\mu_1 + (1-\lambda)\mu_2) = \int^{cav} \boldsymbol{x} d\mu_1$$
for all $\boldsymbol{x} \in \mathbb{R}^n_+$(i.e. if μ is a capacity on 2^N, then $\mathcal{C}(\mu)$, being the set of all the capacities on 2^N giving the same concave integral of μ, is convex).

The following Theorem 2 links the superadditive integral to the level dependent Choquet integral.

Theorem

Suppose that $A(\cdot) = \int^{Ch,LD} \cdot d\mu_{LD}$*, with the level dependent capacity* $\mu_{LD}(S,t)$ *being non-increasing with respect to* t *for all* $S \subseteq N$*, then for all* $\boldsymbol{x} \in \mathbb{R}^n_+$

$$A^*(\boldsymbol{x}) = \int^{cav} \boldsymbol{x} d\mu, \qquad \textit{where } \mu(S) = \mu_{LD}(S,0), \ \forall S \subseteq N.$$

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Thanks for your attention