Local Finiteness in t-Norm Based Algebras

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Background

Starting point:

The local finiteness of bimonoids is an interesting property for weighted automata.

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Problem: Which t-norm based bimonoids have this property?

Some results:

S.G.: *Local and relativized local finiteness in t-norm based structures*, Information Sciences **228** (2013), 26–36.

Preliminaries

A finite set $G = \{b_1, \ldots, b_m\}$ generates in an abelian semigroup $\mathfrak{A} = (A, *)$ the set

$$\langle G \rangle_{\mathfrak{A}} = \{b_1^{k_1} * \ldots * b_m^{k_m} \mid k_1, \ldots, k_m \in \mathbb{N}\} .$$

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Definition

An algebraic structure $\mathfrak{A}$ is **locally finite** iff each of its finite subsets G generates a finite subalgebra $\langle G \rangle_{\mathfrak{A}}$ only.
A **t-norm** is a binary operation in $[0, 1]$ which makes it into an ordered abelian monoid with 1 as unit.
A **t-conorm** is a binary operation in $[0, 1]$ which makes it into an ordered abelian monoid with 0 as unit.

More Preliminaries

Remark:
A continuous t-norm T has an **ordinal sum representation** $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$ with non-overlapping intervals $[l_i, r_i]$, order automorphisms φ_i of the unit interval, and $T_i = T_{\mathrm{L}}$ or $T_i = T_{\mathrm{P}}$.

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Such an order automorphisms h shall be called **rational based** iff its restriction $h \restriction \mathbb{Q}$ is an order automorphism of the rational unit interval $[0, 1] \cap \mathbb{Q}$.

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Proposition

A t-conorm monoid $([0, 1], S_T, 0)$ *is locally finite iff its corresponding t-norm monoid* $([0, 1], T, 1)$ *is.*

Some Definitions

Definition

For any $a \in [0,1]$ *and any t-norm* $*$ *let the* $*$-**order** *of* a *be the smallest integer* $0 \neq n \in \mathbb{N}$ *such that* $a^n = a^{n+1}$, *if such an integer exists. Otherwise* a *shall be* **of infinite** $*$-**order**.

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And a t-norm $*$ *shall be* **weakly nilpotent** *iff each* $a \in [0,1[$ *is a nilpotent element, i.e. has a power* n *with* $a^n = 0$*.*

Local Finiteness for t-Norm Monoids 1

Theorem

Let G be a finite subset of a t-norm monoid $\mathfrak{A} = ([0,1], *, 1)$. $\langle G \rangle_{\mathfrak{A}}$ is finite iff G consists of elements of finite order only.

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$(\Leftarrow)$: In this case one has for a suitable n

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$(\Rightarrow)$: But if there is a $b \in G$ of infinite order, then $\langle G \rangle_{\mathfrak{A}}$ is infinite. $\square$

Local Finiteness for t-Norm Monoids 2

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The product monoid $([0,1], T_{\mathsf{P}}, 1)$ *is not locally finite.*
If T *is weakly nilpotent then the monoid* $([0,1], T, 1)$ *is locally finite.*

Continuous t-Norms

Theorem

A t-norm monoid $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite iff* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands,*

Continuous t-Norms

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A t-norm monoid $([0,1], T, 1)$ *with a continuous t-norm* T *is locally finite iff* T *does only have locally finite summands in its representation as ordinal sum of archimedean summands, i.e. iff* T *does not have a product-norm isomorphic summand in this representation.*

t-Norm Bimonoids

Definition

A bimonoid is an algebraic structure $\mathfrak{A} = (A, *_1, *_2, e_1, e_2)$ *such that both* $(A, *_1, e_1)$ *and* $(A, *_2, e_2)$ *are monoids.*

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The Gödel-bimonoid $([0,1], T_{\mathrm{G}}, S_{\mathrm{G}}, 1, 0)$ *is locally finite.*

Proposition

The product-bimonoid $([0,1], T_{\mathrm{P}}, S_{\mathrm{P}}, 1, 0)$ *is not locally finite.*

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Iteration of this construction yields an infinite descending sequence of irrationals from $[0,1]$. Hence $\langle \alpha_0 \rangle$ is infinite.

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Remark:

The reference to an irrational number is essential;
as is the simultaneous availability of the operations T_{L} and S_{L}.

t-Norm Bimonoids

Proposition

The rational Łukasiewicz-bimonoid $([0,1] \cap \mathbb{Q}, T_{\mathsf{L}}, S_{\mathsf{L}}, 1, 0)$ *is locally finite.*

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Proposition

The t-norm bimonoid $([0,1], T_{\mathsf{nM}}, S_{\mathsf{nM}}, 1)$ *based upon the nilpotent minimum*

$$T_{\mathsf{nM}}(x,y) = \begin{cases} \min\{x,y\}\,, & \textit{if } x+y>1 \\ 0 & \textit{otherwise} \end{cases}$$

is locally finite.

t-Norm Bimonoids

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Example

The t-norm bimonoid $([0,1], T^*, S_{T^*}, 1, 0)$ *with the continuous t-norm*

$$T^* = \sum_{i \in \{1\}} ([\frac{1}{2}, 1], T_{\mathsf{L}}, \varphi^*)$$

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$$T^* = \sum_{i \in \{1\}} ([\frac{1}{2}, 1], T_{\mathrm{L}}, \varphi^*)$$

and the order isomorphism $\varphi^* : [\frac{1}{2}, 1] \longrightarrow [0,1]$ *given by* $\varphi^*(x) = 2x - 1$ *is locally finite.*

t-Norm Bimonoids

Here T^* acts on the square $u_\mathrm{r} = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ as (an isomorphic copy of) T_L, and acts as the min-operation otherwise.

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NB: The particular choice of the order isomorphism φ^* is unimportant here.

A general result

Theorem

Suppose that T is a continuous t-norm such that

- *T has an ordinal sum representation $T = \sum_{i \in I}([l_i, r_i], T_i, \varphi_i)$ without product-isomorphic summands,*

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- *for each Łukasiewicz summand $([l_k, r_k], T_{\mathrm{L}}, \varphi_k)$ the interval $[1 - r_k, 1 - l_k]$ does* **not overlap** *with any domain interval $[l_i, r_i]$ for a Łukasiewicz summand $([l_i, r_i], T_{\mathrm{L}}, \varphi_i)$, $i \in I$.*

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Then the t-norm bimonoid $([0, 1], T, S_T, 1, 0)$ is locally finite.

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Problem:

What in the case of overlap ?

Overlap situations

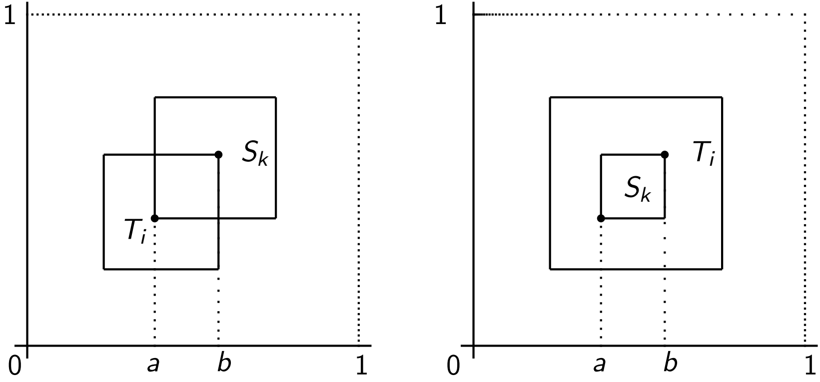

Figure: $\langle T_i, S_k \rangle$-overlap: partial (left) and total (right)

Overlap situations

Proposition

If in a t-norm bimonoid $\mathfrak{A}$ its t-norm T has a summand $([l_i, r_i], T_{\mathrm{L}}, h_i)$ with a rational-based order automorphism h_i and with full self-overlap, then $\mathfrak{A}$ is not locally finite.

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Here **full self-overlap** of the i-th summand T_i means that the domains of T_i and $S_i = S_{T_i}$ coincide.

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Example

For each $0 < a < \frac{1}{2}$ *the t-norm* $T = ([a, 1-a], T_{\mathrm{L}}, id)$ *has full self-overlap and determines, thus, a bimonoid which is not locally finite.*

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Proposition

Suppose to have partial $\langle T_i, S_k \rangle$*-overlap with rational borders of the overlap interval* $[a, b]$*, and that* T_i, T_k *are zoomed versions of* T_{L}*. Then each irrational* $c \in [a, b]$ *is of infinite* $\mathfrak{A}$*-order.*

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Proposition

Suppose to have partial $\langle S_k, T_i \rangle$-overlap in the t-norm bimonoid $\mathfrak{A}$ together with $T_i(b, b) \leq a$, then each $c \in [a, b]$ is of finite $\mathfrak{A}$-order.

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Proposition

Suppose to have total $\langle T_i, S_k\rangle$-overlap in the t-norm bimonoid $\mathfrak{A}$. If $T_i(b,b) \leq a$ and the T_i-domain does not overlap with another S_j-domain, $j \neq k$, then each $c \in [a,b]$ is of finite $\mathfrak{A}$-order.

Overlap situations

Example

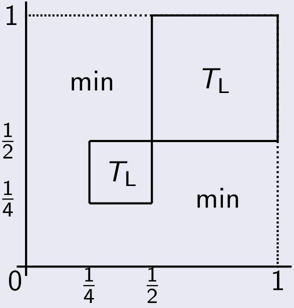
The t-norm

$$([0, \frac{2}{3}], T_{\mathrm{L}}) \oplus ([\frac{2}{3}, 1], T_{\mathrm{L}})$$

determines a locally finite bimonoid, as does the t-norm

$$T = ([\frac{1}{4}, \frac{1}{2}], T_{\mathrm{L}}, id) \oplus ([\frac{1}{2}, 1], T_{\mathrm{L}}, id)$$

explained in the figure on the right.



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Proposition

Suppose to have total $\langle T_i, S_k \rangle$*-overlap in the t-norm bimonoid* $\mathfrak{A}$*. Let the* T_i*-range totally overlap with just the* S_j*-ranges for* $j \in J$*, and let be* b *the supremum of all* $1 - l_j$ *for* $j \in J$*. If each one of these* S_j*-ranges is covered by one of the intervals* $[T_i(b, b), b]$*,* $[T_i(b, b, b), T_i(b, b)], \ldots,$ *then each* $c \in [l_i, r_i]$ *is of finite* $\mathfrak{A}$*-order.*

Example

$$T = ([\frac{1}{6}, \frac{2}{6}], T_{\text{L}}, id) \oplus ([\frac{2}{6}, \frac{1}{2}], T_{\text{L}}, id) \oplus ([\frac{1}{2}, 1], T_{\text{L}}, id)$$

Overlap situations

Thank You