On the category of (L, M)-rough sets with variable range

Aleksandrs Eļkins[1], Aleksandrs Šostak[1],[2]

[1]Department of Mathematics, University of Latvia
[2]Institute of Computer Science and Mathematics, University of Latvia

January 26 - January 31, 2014
12th International Conference on Fuzzy Set Theory and Applications
Liptovský Ján, Slovakia

Introduction

The starting point

The starting point of our interest in subject considered in this talk are rough sets introduced in 1982 by Z.Pawlak and later generalized in different directions as fuzzy rough sets and rough fuzzy sets by D. Dubois and H. Prade and other authors. As noticed by J. Kortelainen and J. Jarvinen, J. Hao and Q. Liu, K. Qui and Z. Pei and some other authors there are natural interrelations between categories of fuzzy rough sets and categories of fuzzy topology. Aiming to develop an approach which would be a common background for different categories of fuzzy rough sets, categories related to fuzzy topology as well as some other categories in the field of "Fuzzy Mathematics" A.Šostak introduced the concept of an approximate system.

Brief historical remark

In the middle of XX century - interest to investigate uncertainties whose nature is not probabilistic:

1. L.A. Zadeh Fuzzy sets. Information and Control. 8: 338-353, 1965
2. J.A. Goguen L-fuzzy sets. Journal of Mathematical Analysis and Applications 18(1):145-174, 1967
3. Z. Pawlak Rough sets. International Journal of Parallel Programming 11 (5): 341-356, 1981
4. D. Dubois, H. Prade Rough fuzzy sets and fuzzy rough sets, Internat. J. General Systems, 17 (2-3), 191-209, 1990
5. Y.Y. Yao A comparative study of fuzzy sets and rough sets, Inf. Sci., 109, 227-242, 1998

Development of theory of approximate systems

The concept of an approximate systems was introduced in

- A.Šostak, On approximative fuzzy operators, 1^{st} Czech-Latvian Seminar on Fuzzy Sets and Soft Computing, Trojanice, Czech Republic, Abstracts 7-8, 2008
- A.Šostak, Towards the theory of M-approximate systems: Fundamentals and examples, Fuzzy Sets and Syst. 161, 2440-2461, 2010
- A.Šostak, Towards the theory of approximate systems: Variable range categories, Proceedings ICTA2011 Islamabad, Pakistan July 4-10, 2011, Cambridge Scientific Publ., pp. 265-284, 2012

Approximate systems was introduced as ajoint framework for the study of categories related to (fuzzy) rough sets and (fuzzy) topology.

Context of research I: Involved structures

- Lattice $\mathbb{L}$, lattice L, lattice $\mathbb{L} = L^X$
  $\mathbb{L}$ is an infinitely distributive complete lattice.
- Lattice M is a complete lattice.

Upper M-approximation operator

Definition

Given an infinitely distributive complete lattice $\mathbb{L}$, a complete lattice M. Then a mapping $u : \mathbb{L} \times M \to \mathbb{L}$ is called an upper M-approximation operator if it satisfying the following conditions:

1. $u(0_{\mathbb{L}}, \alpha) = 0_{\mathbb{L}}, \forall \alpha \in M$;
2. $u(a \vee b, \alpha) = u(a, \alpha) \vee u(b, \alpha)\ \forall a, b \in \mathbb{L}, \forall \alpha \in M$;
3. $\alpha \leq \beta \Longrightarrow u(a, \alpha) \leq u(a, \beta)\ \ \forall a \in \mathbb{L}, \forall \alpha, \beta \in M$;
4. $a \leq u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$;
5. $u(u(a, \alpha), \alpha) = u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$.

Upper M-approximation operator

Definition

Given an infinitely distributive complete lattice $\mathbb{L}$, a complete lattice M. Then a mapping $u : \mathbb{L} \times M \to \mathbb{L}$ is called an upper M-approximation operator if it satisfying the following conditions:

1. $u(0_{\mathbb{L}}, \alpha) = 0_{\mathbb{L}}, \forall \alpha \in M$;
2. $u(a \vee b, \alpha) = u(a, \alpha) \vee u(b, \alpha)\ \forall a, b \in \mathbb{L}, \forall \alpha \in M$;
3. $\alpha \leq \beta \Longrightarrow u(a, \alpha) \leq u(a, \beta)\ \ \forall a \in \mathbb{L}, \forall \alpha, \beta \in M$;
4. $a \leq u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$;
5. $u(u(a, \alpha), \alpha) = u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$.

Lower M-approximation operator

Definition

Given an infinitely distributive complete lattice $\mathbb{L}$, a complete lattice M. Then a mapping $l : \mathbb{L} \times M \to \mathbb{L}$ is called a lower M-approximation operator if it satisfying the following conditions:

1. $l(1_{\mathbb{L}}, \alpha) = 1_{\mathbb{L}}, \forall \alpha \in M$;
2. $l(a \wedge b, \alpha) = l(a, \alpha) \wedge l(b, \alpha)\ \forall a, b \in \mathbb{L}, \forall \alpha \in M$;
3. $\alpha \leq \beta \implies l(a, \alpha) \geq l(a, \beta)\ \ \forall a \in \mathbb{L}, \forall \alpha, \beta \in M$;
4. $a \geq l(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$;
5. $l(l(a, \alpha), \alpha) = l(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$.

Lower M-approximation operator

Definition

Given an infinitely distributive complete lattice $\mathbb{L}$, a complete lattice M. Then a mapping $l : \mathbb{L} \times M \to \mathbb{L}$ is called a lower M-approximation operator if it satisfying the following conditions:

1. $l(1_{\mathbb{L}}, \alpha) = 1_{\mathbb{L}}, \forall \alpha \in M$;
2. $l(a \wedge b, \alpha) = l(a, \alpha) \wedge l(b, \alpha)\ \forall a, b \in \mathbb{L}, \forall \alpha \in M$;
3. $\alpha \leq \beta \implies l(a, \alpha) \geq l(a, \beta)\ \ \forall a \in \mathbb{L}, \forall \alpha, \beta \in M$;
4. $a \geq l(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$;
5. $l(l(a, \alpha), \alpha) = l(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$.

Definition of the approximate system

Definition

The quadruple $(\mathbb{L}, M, l, u)$ is approximate system if $u : \mathbb{L} \times M \to \mathbb{L}$ and $l : \mathbb{L} \times M \to \mathbb{L}$ are respectively an upper M-approximation operator and a lower M-approximation operator.

Category AS

Let **AS** be the family of all approximate systems $(\mathbb{L}, M, u, l)$. The objects of this category are quadruple $(\mathbb{L}, M, l, u)$, as morphisms we take pairs $(f, \varphi) : (\mathbb{L}_1, M_1, u_1, l_1) \to (\mathbb{L}_2, M_2, u_2, l_2)$, where $f : \mathbb{L}_2 \to \mathbb{L}_1$ is a morphism of infinitely distributive complete latices $\varphi : M_2 \to M_1$ is a morphism in the category of complete lattices.

1. $u_1(f(b), \varphi(\beta)) \leq f(u_2(b, \beta))\ \forall b \in \mathbb{L}_2,\ \forall \beta \in M_2$
2. $f(l_2(b, \beta)) \leq l_1(f(b), \varphi(\beta))\ \forall b \in \mathbb{L}_2, \forall \beta \in M_2$

Monoidal type structure

Definition

A complete infinitely distributive lattice

$$(L, \leq, \wedge, \vee)$$

with the smallest and the largest elements 0_L and 1_L respectively.
$* : L \times L \to L$ are commutative associative monotone operations on L, such that $1_L * \alpha = \alpha$, $0_L * \alpha = 0_L$ for every $\alpha \in L$ and distributing over arbitrary joins.
There is a further binary operation - residium $\mapsto$ on a lattice

$$\alpha \mapsto \beta = \bigvee \{\gamma \mid \gamma * \alpha \leq \beta, \gamma \in L\}$$

for every $\alpha, \beta \in L$.

Many-valued (L, M)-relations

Definition

A many-valued (L, M)-relation ρ on X is a non-decreasing on the third coordinate mapping $\rho : X \times X \times M \to L$.
An (L, M)-relation ρ on X is called

1. reflexive if $\rho(x, x, \alpha) = 1_L, \forall x \in X, \forall \alpha \in M$,
2. serial if for all $x \in X \ \ \exists y \in X$ s.t. $\rho(x, y, \alpha) = 1, \forall \alpha \in M$,
3. symmetric if $\rho(x, y, \alpha) = \rho(y, x, \alpha) \forall x, y \in X, \forall \alpha \in M$,
4. transitive if $\rho(x, y, \alpha) * \rho(y, z, \beta) \leq \rho(x, z, \alpha \wedge \beta)$ for all $x, y, z \in X, \forall \alpha, \beta \in M$.

Many-valued (L, M)-relations

Definition

A many-valued (L, M)-relation ρ on X is a non-decreasing on the third coordinate mapping $\rho : X \times X \times M \to L$.
An (L, M)-relation ρ on X is called

1. reflexive if $\rho(x, x, \alpha) = 1_L, \forall x \in X, \forall \alpha \in M$,
2. serial if for all $x \in X \ \exists y \in X$ s.t. $\rho(x, y, \alpha) = 1, \forall \alpha \in M$,
3. symmetric if $\rho(x, y, \alpha) = \rho(y, x, \alpha) \forall x, y \in X, \forall \alpha \in M$,
4. transitive if $\rho(x, y, \alpha) * \rho(y, z, \beta) \leq \rho(x, z, \alpha \wedge \beta)$ for all $x, y, z \in X, \forall \alpha, \beta \in M$.

(L, M)-rough sets

Let a cl-monoid L be given, let X be a set, $A \in L^X$.
Further let an (L, M)-relation $\rho : X \times X \times M \to L$ on a set X be given.

1. We construct a lower approximation operator: $l_\rho \colon L^X \times M \to L^X$

$$l_\rho(A, \alpha)(x) = \inf_{x' \in X} (\rho(x, x', \alpha) \mapsto A(x')).$$

$$a \mapsto b = \bigvee \{c \mid c * a \le b, c \in L\}$$

2. We construct an upper approximation operator: $u_\rho \colon L^X \times M \to L^X$

$$u_\rho(A, \alpha)(x) = \sup_{x' \in X} (\rho(x, x', \alpha) * A(x')).$$

3. We call the family of triples $\{A, l_\rho(A, \alpha), u_\rho(A, \alpha) | \forall \alpha \in M\}$ an (L, M)-rough set and study its properties.

(L, M)-rough sets

Let a cl-monoid L be given, let X be a set, $A \in L^X$.
Further let an (L, M)-relation $\rho : X \times X \times M \to L$ on a set X be given.

1. We construct a lower approximation operator: $l_\rho \colon L^X \times M \to L^X$

$$l_\rho(A, \alpha)(x) = \inf_{x' \in X} (\rho(x, x', \alpha) \mapsto A(x')).$$

$$a \mapsto b = \bigvee \{c \mid c * a \leq b, c \in L\}$$

2. We construct an upper approximation operator: $u_\rho \colon L^X \times M \to L^X$

$$u_\rho(A, \alpha)(x) = \sup_{x' \in X} (\rho(x, x', \alpha) * A(x')).$$

3. We call the family of triples $\{A, l_\rho(A, \alpha), u_\rho(A, \alpha) | \forall \alpha \in M\}$ an (L, M)-rough set and study its properties.

(L, M)-rough sets

Let a cl-monoid L be given, let X be a set, $A \in L^X$.
Further let an (L, M)-relation $\rho : X \times X \times M \to L$ on a set X be given.

1. We construct a lower approximation operator: $l_\rho : L^X \times M \to L^X$

$$l_\rho(A, \alpha)(x) = \inf_{x' \in X} (\rho(x, x', \alpha) \mapsto A(x')).$$

$$a \mapsto b = \bigvee \{c \mid c * a \leq b, c \in L\}$$

2. We construct an upper approximation operator:
$u_\rho : L^X \times M \to L^X$

$$u_\rho(A, \alpha)(x) = \sup_{x' \in X} (\rho(x, x', \alpha) * A(x')).$$

3. We call the family of triples $\{A, l_\rho(A, \alpha), u_\rho(A, \alpha) | \forall \alpha \in M\}$ an (L, M)-rough set and study its properties.

(L, M)-rough sets

Let a cl-monoid L be given, let X be a set, $A \in L^X$.
Further let an (L, M)-relation $\rho : X \times X \times M \to L$ on a set X be given.

1. We construct a lower approximation operator: $l_\rho \colon L^X \times M \to L^X$

$$l_\rho(A, \alpha)(x) = \inf_{x' \in X} (\rho(x, x', \alpha) \mapsto A(x')).$$

$$a \mapsto b = \bigvee \{c \mid c * a \leq b, c \in L\}$$

2. We construct an upper approximation operator:
$u_\rho \colon L^X \times M \to L^X$

$$u_\rho(A, \alpha)(x) = \sup_{x' \in X} (\rho(x, x', \alpha) * A(x')).$$

3. We call the family of triples $\{A, l_\rho(A, \alpha), u_\rho(A, \alpha) | \forall \alpha \in M\}$ an (L, M)-rough set and study its properties.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0,1\}$ with $\rho(x,y,0_M) = \rho(x,y,1_M) \forall x,y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x,\alpha) = \{x' \in X \mid \rho(x,x',\alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A,0_M) = \{x \in X \mid P_\rho(x,0_M) \subseteq A\} = \{x \in X \mid P_\rho(x,1_M) \subseteq A\} = l_\rho(A,1_M)$
5. Define an upper approximation of A: $u_\rho(A,0_M) = \{x \in X \mid P_\rho(x,0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x,1_M) \cap A \neq \emptyset\} = u_\rho(A,1_M)$
6. The triples $(A, l_\rho(A,0_M), u_\rho(A,0_M))$ and $(A, l_\rho(A,1_M), u_\rho(A,1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: Classical rough sets

In case $L = \{0_L, 1_L\}$ and $M = \{0_M, 1_M\}$ we obtain Pawlak model of rough sets:

Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci., 11 (1982), 341-356

1. Let $A \subseteq X$ be a set under research.
2. Let an equivalence relation $\rho : X \times X \times M \to \{0, 1\}$ with $\rho(x, y, 0_M) = \rho(x, y, 1_M) \forall x, y \in X$ be given.
3. Relation ρ partitions the set X into equivalence classes on M-level: $P_\rho(x, \alpha) = \{x' \in X \mid \rho(x, x', \alpha) = 1\}, \alpha \in \{0_M, 1_M\}$.
4. Define a lower approximation of A: $l_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \subseteq A\} = \{x \in X \mid P_\rho(x, 1_M) \subseteq A\} = l_\rho(A, 1_M)$
5. Define an upper approximation of A: $u_\rho(A, 0_M) = \{x \in X \mid P_\rho(x, 0_M) \cap A \neq \emptyset\} = \{x \in X \mid P_\rho(x, 1_M) \cap A \neq \emptyset\} = u_\rho(A, 1_M)$
6. The triples $(A, l_\rho(A, 0_M), u_\rho(A, 0_M))$ and $(A, l_\rho(A, 1_M), u_\rho(A, 1_M))$ are rough sets in Pawlak's sense.

Example: (L, M)-rough sets with Łukasiewicz t-norm

We obtain model of (L, M)-rough sets:

1. Let $A \in L^X$ be a set under research .
2. Let an (L, M)-relation $\rho : X \times X \times M \to L$ be given.
3. Let Łukasiewicz t-norm $T_L(a, b) = max(a + b - 1, 0)$ be used.
4. Define a lower approximation of A:
   $l_\rho(A, \alpha)(x) = \inf_{x' \in X}(\rho(x, x', \alpha) \mapsto A(x')) =$
   $\inf_{x' \in X}(min(1 - \rho(x, x', \alpha) + A(x'), 1))$
5. Define an upper approximation of A:
   $u_\rho(A, \alpha)(x) = \sup_{x' \in X}(\rho(x, x', \alpha) * A(x')) =$
   $\sup_{x' \in X}(max(\rho(x, x', \alpha) + A(x') - 1, 0))$
6. The triples $\{(A, l_\rho(A, \alpha), u_\rho(A, \alpha)) \mid \forall \alpha \in M\}$ are all (L, M)-rough sets defined with (L, M)-relation ρ, using Łukasiewicz t-norm

Order

REL (L) be the family of all (L, M)-relations on sets, where lattice L is fixed. Consequently **REL** (X, L, M) be the family of all (L, M)-relations with fixed L and M on fixed set X.
We introduce an oder on the family **REL** (X, L, M) by extending it from the order of lattice L:

$$\rho \leq \sigma \Longleftrightarrow \rho(x, x', \alpha) \leq \sigma(x, x', \alpha) \text{ for every } x, x' \in X, \forall \alpha \in M$$

Theorem

Given two (L, M)-relations $\rho, \sigma : X \times X \times M \to L$ on a set X

$$\rho \leq \sigma \Longleftrightarrow (l_\rho, u_\rho) \succeq (l_\sigma, u_\sigma).$$

Category of sets with (L, M)-relations

Let L be fixed. Given sets equipped with (L, M)-relations (X, M_1, ρ), (Y, M_2, σ), we consider mappings $f : X \to Y$ respecting this relations:

$$\sigma(f(x), f(x'), \alpha) \geq \rho(x, x', \varphi(\alpha)) \quad \forall x, x' \in X$$

and $\forall \alpha \in M_2$ In the result we obtain a category $\mathbf{REL}(L)$.

Conditions on M-approximation operator

Definition

Mapping $u : \mathbb{L} \times M \to \mathbb{L}$ is called an upper M-approximation operator if it satisfying the following conditions:

1. $u(0_{\mathbb{L}}, \alpha) = 0_{\mathbb{L}}, \forall \alpha \in M$;
2. $u(a \vee b, \alpha) = u(a, \alpha) \vee u(b, \alpha)\ \forall a, b \in \mathbb{L}, \forall \alpha \in M$;
3. $\alpha \leq \beta \Longrightarrow u(a, \alpha) \leq u(a, \beta)\ \ \forall a \in \mathbb{L}, \forall \alpha, \beta \in M$;
4. $a \leq u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$;
5. $u(u(a, \alpha), \alpha) = u(a, \alpha)\ \forall a \in \mathbb{L}, \forall \alpha \in M$.

(L, M)-rough sets as approximate system

Let $\mathfrak{S}(L) = \{(L^X, M, u_\rho, l_\rho) \mid X \in \mathbf{SET}, M \in \mathbf{CLAT}\}$ that is $\mathfrak{S}(L)$ is the family of approximate systems induced by (L, M)-relations.

Theorem

Every $\Sigma_{(X,M,\rho)} = (L^X, M, u_\rho, l_\rho) \in \mathfrak{S}(X, L)$ is indeed an M-valued approximate system if (L, M)-relation ρ is transitive and reflexive.

Order of approximate structures on fixed lattice L

Definition

Given two approximate systems, X is fixed. $(L^X, M, u_{\rho_1}, l_{\rho_1})$ and $(L^X, M, u_{\rho_2}, l_{\rho_2})$, then $(L^X, M, u_{\rho_1}, l_{\rho_1}) \preceq (L^X, M, u_{\rho_2}, l_{\rho_2})$ iff $l_{\rho_1} \leq l_{\rho_2}$ and $u_{\rho_1} \geq u_{\rho_2}$.

Link, L is fixed

Theorem

By assigning to $(L, X, \rho, M) \in Ob(\mathbf{REL}(L))$ *an approximate system* $\Sigma_\rho = (L^X, M, u_\rho, l_\rho)$ *and assigning to each morphism* $f : (L, X_1, \rho_1, M_1) \to (L, X_2, \rho_2, M_2)$ *of* $\mathbf{REL}(L)$ *the morphism* $F : \Sigma_{\rho_1} \to \Sigma_{\rho_2}$ *defined as pair* $(f_L^{\leftarrow}, \varphi)$, *where* $f_L^{\leftarrow} : L^{X_2} \to L^{X_1}$ $\varphi : M_2 \to M_1$ *, we obtain an isomorphism functor* $\Phi : \mathbf{REL}(L) \to \mathbf{ASR}(L, \mathcal{M})$ *from the category* $\mathbf{REL}(L)$ *into the subcategory* $\mathbf{ASR}(L, \mathcal{M})$ *of category* $\mathbf{AS}$ *.*

$\mathbf{ASR}(L, \mathcal{M})$ is the category of variable-range approximate systems induced by (L, M)-relations on non-empty sets. $f_L^{\leftarrow}$ is backward operator, $f_L^{\leftarrow}(B) = B \circ f, B \subset L^{X_2}$

Category $\mathbf{ASR}(\mathcal{L}, M)$

Let $\mathbf{ASR}(\mathcal{L}, M)$ be the family of all (L, M)-rough sets with different base L. The objects of this category are triples (X, L, M, ρ), as morphisms we take pairs $(f, \psi) : (X_1, L_1, M, \rho_1) \to (X_2, L_2, M, \rho_2)$, where $f : X_1 \to X_2$ is a mapping and $\psi : L_2 \to L_1$ is a morphism in the category of cl-monoids.

1. $(\rho_1(x, x', \alpha) \leq (\rho_2(f(x), f(x'), \alpha) \forall x, x' \in X_1, \forall \alpha \in M$
2. $\psi(\rho_1(x, x', \alpha) \leq (\rho_2(f(x), f(x'), \alpha) \forall x, x' \in X_1, \forall \alpha \in M$

Conclusion

1. **ASR**$(L, \mathcal{M})$
2. **ASR**$(\mathcal{L}, M)$

Thank you for your attention!