Balanced Implications

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Introduction

- We contribute to the theory of implications in non-classical logics related by adjointness.
- Our aim is to investigate implications in (L, P)-valued logics.
- $(L, \leq)$ and $(P, \leq)$ stand for two (complete lattices) posets, interpreting two, possibly different, types of *truth values*.
- We expound some motivations behind the use of two posets L and P in the definition of implications, as a generalization of the usual one-poset approach.
- We will introduce balanced implications and investigate them in relation to tied implications.
- We provide new characterizations of balanced implications and tied implications, and we explore the close relationship between these two notions.

Implications and their adjoints

An implication, on two posets P and L, is a function $\Rightarrow: P \times L \to L$ with the following basic intuitive demands:

- $\Rightarrow$ is antitone in the left and isotone in the right argument,
- $\Rightarrow$ satisfies the boundary condition $1_P \Rightarrow z = z$,
- $\Rightarrow$ has an adjoint $\supset$ in the left argument, i.e., $\supset: L \times L \to P$ satisfies $\forall a \in P, \ \forall y, z \in L:$

$$\textbf{Adjointness:} \quad y \leq_L a \Rightarrow z \quad \text{iff} \quad a \leq_P y \supset z. \tag{1}$$

- The function $\supset: L \times L \to P$ is called a *comparator*.
- $\supset$ is antitone in the left argument, isotone in the right argument and

$$\textbf{Comparator axiom}: \quad y \supset z = 1_P \quad \text{iff} \quad y \leq_L z. \tag{2}$$

The ordered pair $(\Rightarrow, \supset)$ is called an *adjoint pair* on (L, P).
In the case $P = L$, we say on P to mean on (P, P).

Implications and their adjoints

- The adjoint $\supset$ of $\Rightarrow$ exists iff the implication $\Rightarrow$ satisfies for all indexed families $\{a_j\}$ in P

$$\sup_j a_j \Rightarrow z = \inf_j \left(a_j \Rightarrow z\right), \tag{3}$$

- It is then uniquely given by

$$y \supset z = \sup \{a \in P \mid \; y \leq_L a \Rightarrow z\}. \tag{4}$$

- The comparator satisfies a condition analogous to (3).

Motivations

- On the one hand, the condition Adjointness is a main tool in building a useful calculus for implications; generating universally valid inequalities.
- It is natural to request an implication between truth values with same semantics to satisfy the comparator axiom (2).
- It is equally natural to restrict the need for the comparator axiom (2) of implications between truth values of differing semantics.
- Those belong to independently chosen lattices, which may or may not coincide.
- The two-posets approach comes at no price at all. An algebraic proof in this framework is an exact replica of the corresponding proof in the one-poset situation.

Motivations

- The three notions, vagueness, uncertainty and truthlikeness, constitute the basic axes of approximate reasoning models.
- We may say that vagueness, uncertainty and truthlikeness, until few years ago, were not clearly differentiated from each other, possibly because they are usually coded by real numbers from the unit interval $[0, 1]$.
- In the last years, much effort has been devoted to clarify the conceptual differences among these notions.
- For more information: (L.Godo, R. Rodriguez (2008), Logical approaches to fuzzy similarity-based reasoning: an overview) where the authors clarify the distinctive features of each notion.
- These notions frequently coexist in the same applications.

Motivations

- For example, rule-based medical decisions support systems, like CADIAG-2 [Adlassnig], handle various types of graded information.
  "IF highly increased amylase activities,
  THEN pancreatic cancer with the degree of confirmation 0.95".
- Certainty-qualified rules:
  "If x is A, then y is B is a-certain"
- Certainty rules:
  "The more true x is A, the more certain y is B"
  e.g., "The younger a man, the more certainly he is single"
- We might consider also rules like this:
  "if A is true then B is close to be true"
- Challenge: Find an appropriate formal (logical) framework to represent knowledge and reason with it.
- I believe that these notions may be formalized and combined under a homogeneous framework which should be an appropriate extension of fuzzy logic in the narrow sense.

Our approach admits more concrete examples of adjointness pairs, that would otherwise have been excluded, see Example 1, 2 and 3.

Example

The smallest implication on (L, P) is

$$a \Rightarrow z = \begin{cases} 1_L, & a = 0_P \\ z, & a > 0_P \end{cases}$$

The smallest comparator on (L, P) is

$$y \supset z = \begin{cases} 1_P, & y \leq z \\ 0_P, & y \not\leq z \end{cases}$$

The ordered pair $(\Rightarrow, \supset)$ is an adjoint pair on (L, P).

Example

Let $(\Rightarrow, \supset)$ be a an adjoint pair on a complete lattice P, let W be a set of two or more elements, and let P^W be the product lattice. Then an adjointness pair $(\Rightarrow^*, \supset^*)$ on (P^W, P) is defined as follows: for all $a \in P$, $y, z \in P^W$ and $w \in W$

$$\begin{aligned}(a \Rightarrow^* z)(w) &= a \Rightarrow z(w)\\ y \supset^* z &= \inf_{w \in W} (y(w) \supset z(w)).\end{aligned}$$

Example

Let W be any non-empty set; let $\otimes$ be a t-norm; let $P \subset [0,1]$ and containing 0 and 1 and closed under $\otimes$; and let $S : W \times W \longrightarrow P$ be a $\otimes$-similarity relation (i.e., reflexive, symmetry, separable, and $\otimes$-transitive). Then an ordered pair $(\Rightarrow, \supset)$ on $(2^W, P)$ is defined as follows: for all $a \in P$, $y, z \in 2^W$ and $w \in W$

$$a \Rightarrow z = \begin{cases} \top, & a = 0 \\ \{w : S(w,u) \geq a \text{ for some } u \in z\}, & a > 0 \end{cases}$$

$$y \supset z = \begin{cases} 1, & y \leq z \\ \inf_{u \in y} \sup_{v \in z} S(u,v), & y \nleq z \end{cases}$$

Under certain additional assumptions like for instance the finiteness of W, the ordered pair $(\Rightarrow, \supset)$ is an adjoint pair on $(2^W, P)$.

Faithful implications

Definition

An implication $\Rightarrow$ is said to be *faithful* if it satisfies:

$$(\forall a, b \in P) \;\; (\text{if } a \neq b, \text{ then } (\exists z \in L): \; a \Rightarrow z \neq b \Rightarrow z) \,. \qquad (5)$$

A comparator $\supset$ is said to be *full* if for every $a \in P$ there is $(y, z) \in L^2$ such that $a = y \supset z$.

Faithfulness is closely related to the following *closure operator* on P:

$$\hat{a} = \inf_{z \in L} ((a \Rightarrow z) \supset z) \,, \quad a \in P. \qquad (6)$$

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). The operator $\hat{}$ becomes the identity function id_P if and only if $\Rightarrow$ is faithful.

Faithful implications

The following is the characterization of comparators that are full.

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P) (completeness is not assumed). Then, the comparator $\supset$ is full if and only if for all b in P there is z in L such that $(b \Rightarrow z) \supset z = b$.

- Consequently, if the comparator $\supset$ is full, then the implication $\Rightarrow$ is faithful.

In general, the comparators associated with faithful implications are not full.

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P), in which the range of $\supset$ is finite and P is a chain. Then the comparator $\supset$ is full if and only if the implication $\Rightarrow$ is faithful.

Faithful implications

- The Faithfulness of $\Rightarrow$ would be a favorable situation.
- By allowing P to differ from L, we gain some flexibility in its handling.
- Suppose that an implication $\Rightarrow$ is not faithful. Then
- we can reduce P to a smaller complete lattice $\overline{P}$ through the order-preserving retraction $\widehat{\ }: P \longrightarrow \overline{P} \subset P \quad (6)$.
- The restriction $\Rightarrow: \overline{P} \times L \longrightarrow L$ will be faithful.
- The connectives $\Rightarrow, \supset$ will be unchanged basically, because we have $\widehat{a} \Rightarrow z = a \Rightarrow z$ for all a, z,
- The comparator $\supset$ will not be affected by this restriction, because $\overline{P}$ contains the range of $\supset$.

Faithful implications

Example

Consider the following two binary operations on the unit interval $[0, 1]$:

$$a \Rightarrow c = \max\{1 - 2a, c\}$$

$$b \supset c = \begin{cases} 1, & b \leq c \\ (1-b)/2, & b > c \end{cases}$$

It is easy to say that the ordered pair $(\Rightarrow, \supset)$ is an adjoint pair on $[0, 1]$. And we have for all $a \in P = [0, 1]$:

$$\inf_{z \in [0,1]} ((a \Rightarrow z) \supset z) = \begin{cases} 1, & a \in [1/2, 1] \\ a, & a \in [0, 1/2) \end{cases}$$

Thus, the inequality $a \leq_P \inf\limits_{z \in P} ((a \Rightarrow z) \supset z)$ is not identity for this adjoint pair, unless $P = [0, 1/2) \cup 1$.

Balanced implications

Definition

We say that an adjoint pair $(\Rightarrow, \supset)$ on (L, P) is *balanced* if the following inequality holds for all $x, y, z, w \in L$:

$$x \supset y \leq_P ((x \supset z) \Rightarrow w) \supset ((y \supset z) \Rightarrow w) \tag{7}$$

We also say that the implication $\Rightarrow$ is balanced.

Definition

An adjoint pair $(\Rightarrow, \supset)$ on (L, P) is said to satisfy the *mixed exchange principle* (MEP for short) if there exists an implication-like operation $\rightarrow$ on P (i.e., $\rightarrow$ is antitone in the left argument and isotone in the right argument) such that the following identity holds for all $y, z \in L$ and $a \in P$:

$$y \supset (a \Rightarrow z) = a \rightarrow (y \supset z) \tag{8}$$

We also say that the adjoint pair $(\Rightarrow, \supset)$ with $\rightarrow$ satisfy MEP.

Balanced implications

Definition

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). $\twoheadrightarrow$ is the binary operation on P given by

$$a \twoheadrightarrow b \overset{def}{=} \inf_{z \in L} ((b \Rightarrow z) \supset (a \Rightarrow z)), \ a, b \in P. \tag{9}$$

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). Then the following are equivalent:

(i) $(\Rightarrow, \supset)$ *is balanced.*

(ii) $(\Rightarrow, \supset)$ *with* $\twoheadrightarrow$ *satisfy the MEP.*

(iii) $(\Rightarrow, \supset)$ *with* $\twoheadrightarrow$ *satisfy the following two inequalities:*

$$x \supset y \leq_P (y \supset z) \twoheadrightarrow (x \supset z). \tag{10}$$

$$a \twoheadrightarrow b \leq_P (b \Rightarrow z) \supset (a \Rightarrow z). \tag{11}$$

Balanced implications

- Balance, which is phrased solely in terms of the adjoint pair $(\Rightarrow, \supset)$, is shown to be a necessary and sufficient condition for MEP of $(\Rightarrow, \supset)$.
- MEP can be seen as a weakened form of the exchange principle.
- It holds for several types of implications used in algebraic structures known in literatures, among which
  - *BCK-algebras* ,
  - *Pseudo-BCK-algebras* and
  - *Residuated algebras*.
- We studied properties of that $\twoheadrightarrow$ when $\Rightarrow$ is balanced.

Balanced implications

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). Then

(i) *For all a, b in P: $b \leq_P a \twoheadrightarrow b$.*

(ii) $1_P \twoheadrightarrow b = b$ *if and only if* $\Rightarrow$ *is faithful.*

(iii) *If* $\Rightarrow$ *is faithful, then* $a \twoheadrightarrow b = 1_P$ *if and only if* $a \leq_P b$.

(iv) $\Rightarrow$ *is balanced iff the following MEP holds:* $\forall a \in P$ *and* $y, z \in L$

$$y \supset (a \Rightarrow z) = a \twoheadrightarrow (y \supset z) \tag{12}$$

(v) *If* $\Rightarrow$ *is balanced and the comparator* $\supset$ *is full, then* $\twoheadrightarrow$ *becomes the unique binary operation on* P *that satisfies, with* $(\Rightarrow, \supset)$*, the MEP.*

(vi) *If* $\Rightarrow$ *is balanced then* $\Rightarrow$ *satisfies the exchange principle iff* $\twoheadrightarrow$ *satisfies the following adjointness condition,* $\forall a, b, c \in P$:

$$a \leq_P b \twoheadrightarrow c \text{ iff } b \leq_P a \twoheadrightarrow c \tag{13}$$

Balanced implications

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). If $\Rightarrow$ is balanced and satisfies the exchange principle, and the comparator $\supset$ is full, then the tuple $(P, \twoheadrightarrow, 1_P)$ is a BCK-algebra.

Consequently, $\twoheadrightarrow$ satisfies also the following properties:

(i) $\twoheadrightarrow$ satisfies for all a, b, c *in* P*:*

$$a \twoheadrightarrow (b \twoheadrightarrow c) = b \twoheadrightarrow (a \twoheadrightarrow c)\,. \tag{14}$$

$$a \twoheadrightarrow b \leq_P (b \twoheadrightarrow c) \twoheadrightarrow (a \twoheadrightarrow c). \tag{15}$$

(ii) For all indexed families $\{a_i\}_{i \in I}$ in P we have $\bigvee_{i \in I} a_i \twoheadrightarrow c = \bigvee_{i \in I} (a_i \twoheadrightarrow c)$.

Balanced implications and the law of importation

Definition

A binary operation $\star$ on P is said to *tie* an implication $\Rightarrow: P \times L \to L$ if the following identity holds:

$$(\forall a, b \in P)\,(\forall z \in L) \quad (((a \star b) \Rightarrow z) = (a \Rightarrow (b \Rightarrow z)))\,, \tag{16}$$

We say that $\Rightarrow$ is *tied*.

- Tiedness extends to multiple-valued logic the equivalence, in classical logic (known as the *law of importation*), of the following two statements:

  "If (X and Y) then C", and "If X then (if Y then C)".
- It holds for several types of implications used in fuzzy logic, among which the *residuated implications* and *S-implications* are two types.
- We will investigate balanced implications in relation to tied implications.

Example

Let n be a strong negation on a complete lattice P and let $(P, \leq, \otimes, \to, 1_P)$ be (commutative) residuated lattice on P. Then an adjointness pair $(\Rightarrow, \supset)$ on P is defined as follows:

$$\begin{aligned} a \Rightarrow c &= n(a \otimes n(c)) \\ b \supset c &= n(c) \to n(b) \end{aligned}$$

This $\Rightarrow$ is called *S-type implication* on P. It is easy to see that $\otimes$ ties $\Rightarrow$. For instance, when $\otimes$ is Min and n is the usual negation $b \mapsto 1 - b$ on the unit interval $[0, 1]$, $\Rightarrow$ is the *Kleene-Dienes implication* and $\supset$ is the *contrapose-Gödel implication*, given by

$$\begin{aligned} a \Rightarrow c &= \max\{1 - a, c\} \\ b \supset c &= \begin{cases} 1, & b \leq c \\ 1 - b, & b > c \end{cases} \end{aligned}$$

Balanced implications and the law of importation

- There have been many papers, both theoretical and showing usefulness of tied implications in approximate reasoning in the recent past.
- We point out that the study tied implications was started algebrically by Abdel-Hamid and Morsi (2003).
- It was then adopted and formulated syntactically, within the first order logic of tied implications, by El-Zekey (joint work with Morsi, Lotfallah) in FSS (2006).
- However, in this work, we don't request the implication $\Rightarrow$ to have an adjoint $\& : P \times L \to L$ in the right argument; that is,

$$\forall a \in P,\ \forall y, z \in L : y \leq_L a \Rightarrow z \quad \textit{iff} \quad a \& y \leq_L z. \tag{17}$$

- This work is a continuation of the mentioned work.

Balanced implications and the law of importation

Theorem

Let $(\Rightarrow, \supset)$ *be an adjoint pair on* (L, P) *(completeness is not assumed). If* $\Rightarrow$ *is tied, then it is balanced.*

- Balance is the strongest necessary condition we could formulate for the tiedness of an implication $\Rightarrow$.
- The following question is well justified: **Is balance equivalent to tiedness?**

Balanced implications and the law of importation

Definition

An implication-like operation with condition (P) (i.e. *with product*), is an implication-like operation $\to$ on P satisfying the condition (P):

(P) For all $a, b \in P$ there exists

$$a \odot b \overset{notation}{=} \min\{c \in P | a \leq_P b \to c\}.$$

Proposition

Let $\to$ be a binary operation on a poset $(P, \leq_P)$. Then the following statments are equivalent:

(i) *The operation $\to$ is an implication-like operation with condition (P).*

(ii) *There exists an isotone binary operation $\odot$ on P such that the condition (RP) holds, where:*

(RP) *for all a, b, c in P : $a \odot b \leq_P c$ iff $a \leq_P b \to c$.*

Balanced implications and the law of importation

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P) (completeness is not assumed), and let $\rightarrow$ be an implication-like operation with condition (P) on P (i.e. with product $\odot$). Then the following statements are equivalent:

- **(i)** *The adjoint pair $(\Rightarrow, \supset)$ with $\rightarrow$ satisfy the MEP.*
- **(ii)** *The binary operation $\odot$ ties $\Rightarrow$.*

Theorem

Suppose an implication $\Rightarrow$, of an adjoint pair $(\Rightarrow, \supset)$ on complete lattice (L, P), is tied, faithful and satisfies the exchange principle, then there exists a commutative residuated lattice $(P, \leq, \otimes, \rightarrow, 1_P)$ on P such that its conjunction $\otimes$ is the unique binary operation on P that ties $\Rightarrow$ and its residuum $\rightarrow$ satisfies with $(\Rightarrow, \supset)$ the MEP.

Balanced implications and the law of importation

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P) in which $\Rightarrow$ is faithful. Then the following statements are equivalent:

(i) $\Rightarrow$ *is tied.*

(ii) $\twoheadrightarrow$ *satisfies the condition (P) and its product* $\odot$ *ties* $\Rightarrow$*.*

(iii) $(\Rightarrow, \supset)$ *with* $\twoheadrightarrow$ *satisfy the MEP and* $\twoheadrightarrow$ *satisfies the condition (P).*

(v) $\Rightarrow$ *is balanced and* $\twoheadrightarrow$ *satisfies the condition (P).*

Consequently, in adjointness pair with faithful and balanced implications, $\Rightarrow$ is tied iff $\twoheadrightarrow$ satisfies the condition (P).

Balanced implications and the law of importation

Example

Consider the following example (by Radosław Łukasik, FSS 2010). Let $\to$ be a binary operation on $[0,1]$ given by

$$a \to b = \begin{cases} 1 & a \leq b \\ 1 - a + b & 0 < b < a \leq 1 \\ 0 & b = 0, 0 < a \leq 1 \end{cases}$$

Note that in this example $L = P = [0,1]$ and $\Rightarrow = \supset = \to$.
The implication $\to$ is full (and hence faithful) and satisfies the exchange principle (EP), and hence $(\to, \to)$ is a balanced adjoint pair on $[0,1]$. But $\to$ is not right continuous in the second variable which is equivalent to say that $\to$ does not satisfy condition (p). Then the implication $\Rightarrow = \to$ is not tied.

Some special cases

If L has a bottom element $\perp$, we define two functions $\neg : P \to L$ and $\# : L \to P$ by:

$$\neg a = a \Rightarrow \perp \quad (18)$$

$$\# x = x \supset \perp \quad (19)$$

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P) in which $\perp$ is the bottom element of L and, for all $a \in P$, the identity $\#\neg a = a$ holds. Then $\Rightarrow$ becomes tied if and only if it is balanced.

Some special cases

Definition

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P). We say that $\Rightarrow$ is *continuous in the left argument* if it satisfies

$$\inf_j b_j \Rightarrow z = \sup_j (b_j \Rightarrow z) \tag{20}$$

for all nonempty indexed families $\{b_j\}$ in P and for all $z \in L$.

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on complete lattices (L, P) in which $\Rightarrow$ is continuous in the left argument. Then $\Rightarrow$ becomes tied if and only if it is balanced.

Theorem

Let $(\Rightarrow, \supset)$ be an adjoint pair on (L, P) in which the range of $\supset$ is finite and P is a chain. Then $\Rightarrow$ becomes tied if and only if it is balanced.

Conclusions and future work

- We investigated implications on (L, P) where L and P stand for two posets, interpreting two, possibly different, types of *truth values*.
- We expounded some motivations behind the use of two posets as a generalization of the usual one-poset approach.
- We provide new characterizations of balanced implications and tied implications, and we explore the close relationship between these two notions.
- Perhaps, the most important consequence of such a study of balanced implications is that a new algebraic semantic for a non-classical logic arise.
- Such a formal system can serve as a combined calculus for a pair of two, possibly different, types of uncertainty.

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THANKS