The General Nilpotent System

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Outline

Motivation and background

Nilpotent Connective Systems

Fuzzy set theory

Proper choice of fuzzy connectives

Nilpotent operators

Nilpotent t-norms and t-conorms: preferable properties

Operators – Systems

Instead of pure operators, we examine connective SYSTEMS

Our results

Consistent nilpotent connective systems which are not isomorphic to Łukasiewicz system

Basic preliminaries

Negations

Definition

A decreasing function $n(x) : [0,1] \to [0,1]$ is a fuzzy negation, if $n(0) = 1$ and $n(1) = 0$.

Definition

A fuzzy negation is

- strict, if it is strictly decreasing and continuous
- strong, if it is involutive, i.e. $n(n(x)) = x$ for $\forall x \in [0,1]$.

Trillas' theorem

$n(x)$ is a strong negation if and only if

$$n(x) = f_n(x)^{-1}(1 - f_n(x)),$$

where $f_n(x) : [0,1] \to [0,1]$ is continuous and strictly increasing.

Triangular norms and conorms

(Schweizer and Sklar, 1960)

t-norm T (Conjunction)

A t-norm T is a function on $[0,1]^2$ that satisfies, for all $x, y, z \in [0,1]$:

- $T(x,1) = x$ (neutral element 1);
- $x \leq y \Rightarrow T(x,z) \leq T(y,z)$ (monotonicity);
- $T(x,y) = T(y,x)$ (commutativity);
- $T(T(x,y),z) = T(x,T(y,z))$ (associativity).

t-conorm S (Disjunction)

- $S(x,0) = x$ (neutral element 0) plus the other three properties above.

Representable t-norms and t-conorms

A continuous t-norm T is said to be

- Archimedean if $T(x,x) < x$ holds for all $x \in (0,1)$,
- strict if T is strictly monotone i.e. $T(x,y) < T(x,z)$ whenever $x \in [0,1]$ and $y < z$, and
- nilpotent if there exist $x, y \in (0,1)$ such that $T(x,y) = 0$.

A continuous t-conorm S is said to be

- Archimedean if $S(x,x) > x$ holds for every $x, y \in (0,1)$,
- strict if S is strictly monotone i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1]$ and $y < z$, and
- nilpotent if there exist $x, y \in (0,1)$ such that $S(x,y) = 1$.

Additive generators

A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm iff it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0)), \quad x,y \in [0,1]. \tag{1}$$

A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm iff it has a continuous additive generator, i.e. there exists a continuous strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$S(x,y) = s^{-1}(\min(s(x) + s(y), s(1)), \quad x,y \in [0,1]. \tag{2}$$

Additive generators

A t-norm T is strict if and only if $t(0) = \infty$.

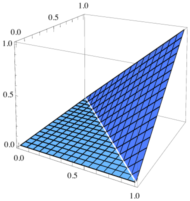
A t-norm T is nilpotent if and only if $t(0) < \infty$.

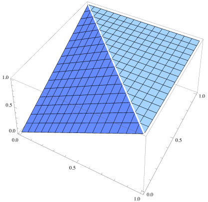
A t-conorm S is strict if and only if $s(1) = \infty$.

A t-conorm S is nilpotent if and only if $s(1) < \infty$.

Nilpotent operators

Preferable Properties





$$\cong T_L(x,y) = \max(x+y-1, 0)$$

$$\cong S_L(x,y) = \min(x+y, 1)$$

Law of contradiction $c(x, n(x)) = 0$

Excluded middle $d(x, n(x)) = 1$

Coincidence of residual and S-implications

...

Notations

Normalized generator functions – uniquely determined

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}.$$

$$f_c(x), f_d(x), f_n(x) : [0,1] \to [0,1].$$

Cutting operation

$$[x] = \begin{cases} 0 & \textit{if} \quad x < 0 \\ x & \textit{if} \quad 0 \leq x \leq 1 \\ 1 & \textit{if} \quad 1 < x \end{cases}$$

Cutting function

$$c(x,y) = f_c^{-1}[f_c(x) + f_c(y)]$$
$$d(x,y) = f_d^{-1}[f_d(x) + f_d(y)]$$

$$\mathrm{Min}(x,y) = f_c^{-1}\left[f_c(x) + [f_c(y) - f_c(x)]\right]$$

$$\mathrm{Max}(x,y) = n\left(f_c^{-1}\left[f_c(n(x)) + [f_c(n(y)) - f_c(n(x))]\right]\right)$$

$$T_L(x,y) = [x+y-1]$$
$$S_L(x,y) = [x+y]$$

Connective Systems

Connective Systems

- strong negation $n(x)$
- conjunction $c(c, y)$
- disjunction $d(x, y)$

Definition

The triple $(c(x, y), d(x, y), n(x))$, where $c(x, y)$ is a t-norm, $d(x, y)$ is a t-conorm and $n(x)$ is a strong negation, is called a connective system.

Definition

A connective system is nilpotent, if the conjunction is a nilpotent t-norm, and the disjunction is a nilpotent t-conorm.

Definition

Two connective systems $(c_1(x,y), d_1(x,y), n_1(x))$ and $(c_2(x,y), d_2(x,y), n_2(x))$ are isomorphic, if there exists a bijection $\phi : [0,1] \to [0,1]$ such that

$$\phi^{-1}\left(c_1\left(\phi(x), \phi(y)\right)\right) = c_2(x,y)$$

$$\phi^{-1}\left(d_1\left(\phi(x), \phi(y)\right)\right) = d_2(x,y)$$

$$\phi^{-1}\left(n_1\left(\phi(x)\right)\right) = n_2(x)$$

Definition

A connective system is called Łukasiewicz system, if it is isomorphic to $([x+y-1], [x+y], 1-x)$, i.e. it has the form $(\phi^{-1}[\phi(x)+\phi(y)-1], \phi^{-1}[\phi(x)+\phi(y)], \phi^{-1}[1-\phi(x)])$.

Consistency of Connective Systems

Classification Property:

- law of contradiction:

$$c(x, n(x)) = 0,$$

- excluded middle:

$$d(x, n(x)) = 1.$$

De Morgan Laws:

$$c(n(x), n(y)) = n(d(x, y))$$

or

$$d(n(x), n(y)) = n(c(x, y)).$$

IS IT SENSIBLE TO USE MORE THAN ONE GENERATOR FUNCTIONS?

Results

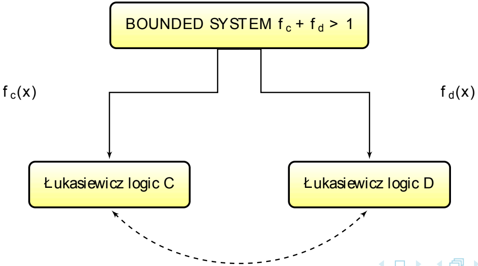
Nilpotent Connective Systems

Among nilpotent systems,

Łukasiewicz connective system (T_L, S_L, standard negation) is characterized by $f_c(x) + f_d(x) = 1, \quad \forall x \in [0,1]$.

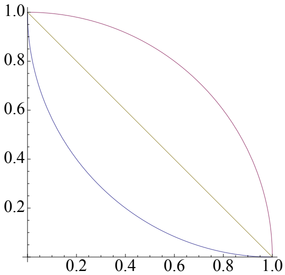
For $f_c(x) + f_d(x) < 1, \quad \forall x \in [0,1]$: the system is not consistent.

For $f_c(x) + f_d(x) > 1, \forall x \in [0,1]$: BOUNDED SYSTEMS



Bounded Systems

Three different negations



$$n_d(x) = f_d^{-1}(1 - f_d(x)) < n(x) < n_c(x) = f_c^{-1}(1 - f_c(x)).$$

$$\boxed{f_c(x) + f_d(x) = 1 \Leftrightarrow n_c(x) = n(x) = n_d(x)}$$

Consistency

Theorem

In a connective structure the classification property holds if and only if

$$n_d(x) \leq n(x) \leq n_c(x), \quad \forall x \in [0, 1].$$

Consistency

Theorem

In a connective structure the classification property holds if and only if

$$n_d(x) \leq n(x) \leq n_c(x), \quad \forall x \in [0,1].$$

Theorem

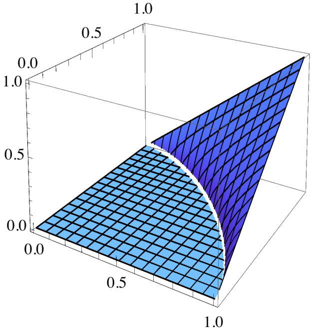
In a connective system the De Morgan law holds if and only if

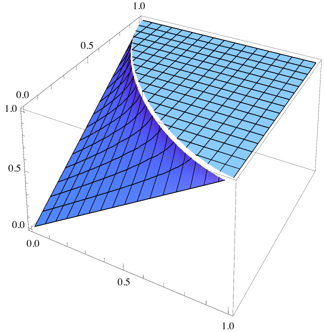
$$n(x) = f_c^{-1}\left(f_d(x)\right) = f_d^{-1}\left(f_c(x)\right), \quad \forall x \in [0,1].$$

Consistency

1. If $c(x,y), d(x,y)$ and $n(x)$ fulfil the De Morgan identity and the classification property (i.e. they form a consitent system), then $f_c(x) + f_d(x) \geq 1, \quad \forall x \in [0,1]$.
2. If $f_c(x) + f_d(x) \geq 1, \quad \forall x \in [0,1]$ and the De Morgan law holds, then the classification property also holds (which now means that the system is consistent).

Example for $f_c(x) + f_d(x) > 1$





For $f_c(x) := 1 - x^\alpha, f_d(x) := 1 - (1-x)^\alpha, n(x) := 1 - x, \quad \alpha \in (1, \infty)$.

- the connective system is consistent,
- $f_c(x) + f_d(x) > 1, \quad \forall x \in [0,1]$
  (or equivalently $n_d(x) < n(x) < n_c(x), \quad \forall x \in [0,1]$).

Rational generators

For the Dombi functions (from 'pliant systems')

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}$$

$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$$

the following statements are equivalent:

1. The connective structure defined by the Dombi functions satisfies the De Morgan law.
2. $$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}.$$

Rational generators

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$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}$$

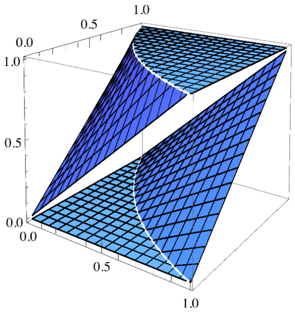
$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$$

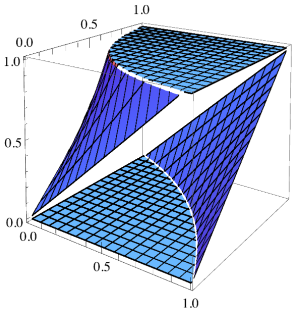
the following statements are equivalent:

1. $n_d(x) < n(x) < n_c(x)$
2. $\nu_d < \nu < \nu_c$
3. Given that the De Morgan property holds, $f_c(x) + f_d(x) > 1$, or equivalently $\nu_c + \nu_d < 1$.

Examples
Rational generators



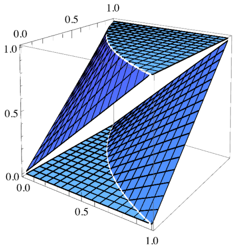
$\nu_c = 0.6$ and $\nu_d = 0.4$

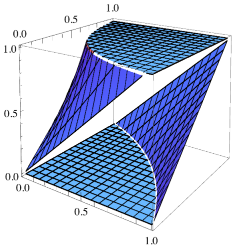


$\nu_c = 0.4$ and $\nu_d = 0.2$

Conclusion and further work

Conclusion and further work





- Consistent nilpotent connective systems using more than one generator functions (not new operators, new system)
- $$f_c(x) + f_d(x) > 1$$ $$n_d(x) < n(x) < n_c(x)$$
- 2 naturally derived thresholds (ν_c, ν_d)
- Further work: implication, equivalence

THANK YOU FOR YOUR ATTENTION!

"Life is like riding a bicycle.
To keep your balance you must keep moving."
/Einstein/