Copulas, Stochastic Precedence, and Stochastic Ordering for Capacities

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Outline

Notation

- X_1, X_2 real random variables on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $\mathbb{P}_{X_1,X_2}$ probability measure over $\mathbb{R} \times \mathbb{R}$ induced by (X_1, X_2)
- F_{X_1,X_2} df and G_1, G_2 marginal dfs
- F_{X_1,X_2} non-necessarily absolutely continuous
- For notational simplicity's sake, $G_1, G_2 \in \mathcal{G}$
- $\mathcal{G}$ class of probability distribution functions on $\mathbb{R}$, continuous and strictly increasing (in the domain where positive and smaller than one)

Stochastic Precedence...

X_1 *stochastically precedes* X_2 ($X_1 \preceq_{sp} X_2$) if

$$\mathbb{P}(X_1 \leq X_2) \geq 1/2$$

Interest of $X_1 \preceq_{sp} X_2$ for applications pointed out several times (e.g. [Arcones et al.(2002)Arcones, Kvam, and Samaniego], [Boland et al.(2004)Boland, Singh, and Cukic], [Navarro and Rubio(2010)]).

$X_1 \preceq_{sp} X_2$ perceived as less restrictive and sometimes more realistic than usual *stochastic ordering*

$$X_1 \preceq_{st} X_2 : G_1(t) \geq G_2(t), \forall t \in \mathbb{R}$$

...and Stochastic Order

Actually, in several cases

$$X_1 \preceq_{st} X_2 \Rightarrow X_1 \preceq_{sp} X_2$$

In particular when X_1, X_2 stochastically independent

In some cases of stochastic dependence

$$X_1 \preceq_{st} X_2 \nRightarrow X_1 \preceq_{sp} X_2$$

Remark: $X_1 \preceq_{sp} X_2$ is not a stochastic order (it is not transitive)

Purposes:

- Analysis of relations between $\{X_1 \leq_{sp} X_2\}$ and $\{X_1 \leq_{st} X_2\}$
- Analysis of a special class of copulas

Motivations:

- Target-Based Utilities ([Bordley and LiCalzi(2000), Castagnoli and Calzi(1996)]) where *Fixed-Copula Target Based Models have been* introduced in [De Santis et al.(2013)De Santis, Fantozzi, and Spizzichino]
- Apparent paradoxes arising about stochastic comparisons among waiting times to occurrences of *words* in random sequences of *letters* from an alphabet (see [Chen and Zame(1979), De Santis and Spizzichino(2012a), De Santis and Spizzichino(2012b)])

Outline

Stochastic Precedence of level γ

For a more complete analysis: $X_1 \leq_{sp}^{(\gamma)} X_2$

Definition 1

For $\gamma \in [0,1]$, X_1 stochastically precedes X_2 at level γ if $\mathbb{P}(X_1 \leq X_2) \geq \gamma$

For $G_1, G_2 \in \mathcal{G}$ and C copula

$$\eta(C, G_1, G_2) := \mathbb{P}(X_1 \leq X_2)$$

where X_1 and X_2 such that

$$F_{X_1,X_2}(x_1, x_2) = C\left[G_1(x_1), G_2(x_2)\right]$$

The condition $\{X_1 \preceq_{sp}^{(\gamma)} X_2\}$ becomes

$$\eta(C, G_1, G_2) \geq \gamma$$

Definition 2

For $\gamma \in [0, 1]$, $\mathcal{L}_\gamma$ class of all copulas $C \in \mathcal{C}$ such that

$$\eta(C, G_1, G_2) \geq \gamma \tag{1}$$

for all $G_1, G_2 \in \mathcal{G}$ with $G_1 \preceq_{st} G_2$

Proposition 3

Let $G_1, G_1', G_2, G_2' \in \mathcal{G}$. Then

$$G_2 \preceq_{st} G_2' \Rightarrow \eta(C, G_1, G_2) \leq \eta(C, G_1, G_2')$$

$$G_1 \preceq_{st} G_1' \Rightarrow \eta(C, G_1, G_2) \geq \eta(C, G_1', G_2)$$

This Proposition allows us

- to characterize the class $\mathcal{L}_\gamma$
- to find lower and upper bounds for the quantity $\eta(C, G_1, G_2)$

Outline

A characterization of the class $\mathcal{L}_\gamma$ 1

A basic fact: $\eta(C, G, G)$ only depend on the copula C

Proposition 4

For any pair of distribution functions $G', G'' \in \mathcal{G}$, one has

$$\eta(C, G', G') = \eta(C, G'', G'') \tag{2}$$

As a consequence of Proposition 4, for $G \in \mathcal{G}$ we can introduce the symbol

$$\eta(C) := \eta(C, G, G) \tag{3}$$

A characterization of the class $\mathcal{L}_\gamma$ 2

The quantity $\eta(C)$ characterizes the class $\mathcal{L}_\gamma$, $0 \le \gamma \le 1$, in fact

Theorem 5

$C \in \mathcal{L}_\gamma$ if and only if $\eta(C) \ge \gamma$.

Then

$$\mathcal{L}_\gamma = \{C \in \mathcal{C} : \eta(C) \ge \gamma\} \tag{4}$$

with

$$\eta(C) = \inf_{G_1, G_2 \in \mathcal{G}} \{\eta(C, G_1, G_2) : G_1 \preceq_{st} G_2\}. \tag{5}$$

Here the infimum is a minimum and it is attained when $G_1 = G_2$.

Extension to $D(\mathbb{R})$

The definition of $\eta(C, G_1, G_2)$ can be extended to the case

$$G_1, G_2 \in D(\mathbb{R}) \quad \text{(space of all probability distribution functions on } \mathbb{R})$$

The class $\mathcal{G}$ has however a special role

Theorem 6

Let $C \in \mathcal{C}$, $G, H \in D(\mathbb{R})$ with $G \preceq_{st} H$, then $\eta(C, G, H) \geq \eta(C)$

Now

$$\eta(C) = \inf_{G_1, G_2 \in D(\mathbb{R})} \{\eta(C, G_1, G_2) : G_1 \preceq_{st} G_2\}. \tag{6}$$

The minimum of $\eta(C, G, H)$, for $G, H \in D(\mathbb{R})$, is attained at (C, G, G), for any $G \in \mathcal{G} \subset D(\mathbb{R})$. In fact, for $G', G'' \in D(\mathbb{R})$, one can have

$$\eta(C, G', G') \neq \eta(C, G'', G'')$$

Outline

TBA utility functions

- The **Target-Based Approach** has been introduced by [Bordley and LiCalzi(2000)] and [Castagnoli and Calzi(1996)] for the single-attribute case
- Some related ideas were already around in the previous economic literature
- Several interesting developments appeared in the subsequent years, concerning the multi-attribute case (see in particular [Bordley and Kirkwood(2004), Castagnoli and LiCalzi(2006), Tsetlin and Winkler(2006), Tsetlin and Winkler(2007)])
- The TBA gives rise to easily-understandable and practically useful interpretations of several notions of Utility Theory

$U : \mathbb{R} \rightarrow \mathbb{R}$ bounded and right-continuous utility function
By suitable normalization, not restrictive to assume

$$U : \mathbb{R} \rightarrow [0, 1]$$

Definition 7

The Target-Based Approach amounts to regarding $U(x)$ as the probability distribution function of a real-valued random variable T (the "Target")

$\Xi := \{X_a\}_{a \in A}$ a family of random variables

$X_a \in \Xi$ *prospect* or *lottery*, random variable with distribution F_a

The Decision Maker is expected to "optimally" select $X_a \in \Xi$
$U : \mathbb{R} \to \mathbb{R}$ utility function describing the DM's attitude toward Risk

According to the Expected Utility Principle, DM should maximize:

$$\mathbb{E}\left[U\left(X_a\right)\right] = \int_{\mathbb{R}} U(x) dF_a(x)$$

Outline

The case of independence

T random variable such that

$$T \perp\!\!\!\perp X$$

$$U(x) = F_T(x)$$

i.e.: T "Target"

Then, for $X_a \in \Xi$,

$$\mathbb{E}\left(U(X_a)\right) = \mathbb{E}\left(F_T(X_a)\right) = \int \mathbb{P}\left(T \le x\right) F_a(dx) = \mathbb{P}\left(T \le X_a\right)$$

Conclusion about TBA under independence

TBA requires:

- To fix a utility function in terms of the probability distribution function of a Target T
- To select X_a in such a way that

$$\mathbb{E}\left[U\left(X_a\right)\right] = \mathbb{P}\left(T \leq X_a\right)$$

is as large as possible!

Outline

TBA can be rendered more general than the expected utility approach by allowing for stochastic dependence between targets and prospects

TBA considers more general decision rules by admitting possibility of stochastic correlation between T and the prospect X

In the jointly absolutely continuous case we can consider

$$\upsilon(x) := \mathbb{P}(T \leq x | X = x) \tag{7}$$

so that

$$\mathbb{P}(T \leq X) = \int_{\mathbb{R}} \upsilon(x) f_X(x)\, dx = \mathbb{E}\left[\upsilon(X)\right] \tag{8}$$

- If X, T are not independent, $\upsilon(x)$ does not coincide with the distribution function $F_T(x)$
- $\upsilon(x)$ possibly non-everywhere increasing in some cases of strong correlation between T and X (then not a bona-fine utility function)
- Then TBA more general than EUP. Too general, however, to allow stochastic dependence to change with the choice of X
- On the other hand: stochastic independence too restrictive in real economic problems

Formalization of independence between any single X and a fixed target T: Choice restricted only to prospects X's such that connecting copula of (T, X) equal to the *product copula*

$$C_T(u, v) = \Pi(u, v) = u \cdot v$$

An appropriate generalization

Definition 8

Fixed-Copula Target-Based Model when, for any T and for any prospect X, $C_{T,X}(u, v)$ remains equal to a fixed $C_T(u, v)$

Different reasons of interest for such a definition can be found in the frame of the TBA:

- Fixed-Copula Target-Based Models can be natural e.g. in micro-economic scenarios, where stochastic dependence between X's and T is induced by underlying variables, with macro-economic meaning (such as inflation, sovereign wealth funds, volatility indexes of a market,...) which are in common to any pair (T, X)
- Results about the class L_γ admit a direct interpretation in the TBA context

In particular:
T, X', X'' random variables with marginal distributions F_T, $F_{X'}$, and $F_{X''}$ respectively and let the pairs $(T, X'), (T, X'')$ **share the same connecting copula** C.

Proposition 9

$X' \preceq_{st} X''$ implies

$$\mathbb{P}(T \leq X') \leq \mathbb{P}(T \leq X'') \tag{9}$$

In other words stochastic comparison between two prospects X' and X'' implies that X'' is preferred to X' for any target T, provided (T, X') and (T, X'') share the same connecting copula C

Above implication not necessarily true in the case of pairs $(T, X'), (T, X'')$ with two different connecting copulas

Warning is in order even under fixed-Copula TBA Model however

Due to conceptual possibility of simultaneous conditions

$$T \leq_{st} X; \qquad X \leq_{sp}^{(\gamma)} T$$

thoughtful choice of the copula C_T is needed

In particular:
C_T should belong to $\mathcal{L}_\gamma$ with γ large enough
(an adequate level of non-exchangeability of C_T is required)

By Theorem 5, only an appropriate lower bound for $\mathbb{P}(T \leq X)$ with T and X identically distributed is need to check the condition

$$C \in \mathcal{L}_\gamma$$

Outline

The Case of Capacities over $\mathbb{R}^2_+$

We aim to replace the probability measure $\mathbb{P}$ with a capacity μ.
Given a set T and an algebra Σ of subsets of T, a capacity on (T, Σ) is a set function $\mu : \Sigma \to \mathbb{R}$ s.t.

(i) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$;
(ii) $\mu(\emptyset) = 0$; $\mu(T) = 1$.

We define, for $T = \mathbb{R}^2_+$ and $\Sigma = \mathcal{B}^2_+$, The distribution function $G_\mu : \mathbb{R}^2_+ \to [0, 1]$ under the capacity μ:

$$G_\mu(x, y) = \mu([0, x] \times [0, y])$$

non-decreasing in each dimension due to monotonicity of μ; marginals

$$G_{\mu_1}(x) = \mu_1([0, x]) = \mu([0, x] \times \mathbb{R}_+)$$

$$G_{\mu_2}(y) = \mu_2([0, y]) = \mu(\mathbb{R}_+ \times [0, y])$$

First questions:

- What can be extended from probabilities to capacities?
- Can we define an analogous to the class $\mathcal{L}_\gamma$?
- Extensions of interest in the economic context - TBA?
- Examples?

Extension of Stochastic Ordering and Stochastic Precedence

Easy to extend the properties of stochastic ordering

$$G_{\mu_1} \preceq_{st} G_{\mu_2} \quad \text{iff} \quad G_{\mu_1}(x) \geq G_{\mu_2}(x) \quad \forall x \geq 0$$

For stochastic precedence of level γ $(0 \leq \gamma \leq 1)$ we want to check if the quantity

$$\mu(A) \geq \gamma \quad \text{with} \quad A = \{(x, y) \in \mathbb{R}^2_+ : x \leq y\}$$

Analogously to $\mathcal{L}_\gamma$ we can define

$$\mathcal{M}_\gamma = \{G_\mu \in \mathcal{C}(\mathbb{R}^2_+, \mathcal{B}^2_+) : \mu(A) \geq \gamma, \quad G_{\mu_1} \preceq_{st} G_{\mu_2}\}$$

In this case we will write $G_{\mu_1} \preceq^{(\gamma)}_{sp} G_{\mu_2}$

Outline

Remarks

Remark 4.1

The definition of stochastic ordering makes sense in any case

Remark 4.2

The value of $\mu(A)$ makes sense when the measure μ is symmetric, i.e. $\mu(A^c) = 1 - \mu(A)$

Otherwise the class $\mathcal{M}_\gamma$ could be ill-defined

Properties

A capacity $\mu : \Sigma \to [0,1]$ is convex (2-monotone, supermodular) if, $\forall A, B \in \Sigma$

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$$

For 2-dimensional capacities this condition has different consequences:

- the distribution function G_μ is 2-increasing
- G_μ satisfies the Fréchet bounds

$$\max\left(G_{\mu_1}(x) + G_{\mu_2}(y) - 1, 0\right) \leq G_\mu(x,y) \leq \min(G_{\mu_1}(x), G_{\mu_2}(y))$$

- there exists $C_\mu : [0,1]^2 \to [0,1]$, a *generalized copula* of μ, s.t.

$$C_\mu(G_{\mu_1}(x), G_{\mu_2}(y)) = G_\mu(x,y)$$

with C_μ grounded, 2–increasing, and with uniform margins.

Examples - 1

$\mathbb{P}'$ and $\mathbb{P}''$ probability measures;

G' and G'' d.f. with marginals $G'_1 \preceq_{st} G'_2$ and $G''_1 \preceq_{st} G''_2$

$C'(G'_1, G'_2), C''(G''_1, G''_2) \in \mathcal{L}_\gamma$

$$\mu = \mathbb{P}' \vee \mathbb{P}''$$

So G_μ belongs to $\mathcal{M}_\gamma$:

$$\mu(A) = \mathbb{P}'(A) \vee \mathbb{P}''(A) \geq \gamma \quad \text{and} \quad G_{\mu_1} \preceq_{st} G_{\mu_2}$$

If $\mathbb{P}'$ and $\mathbb{P}''$ convex then μ is convex and the copula C_μ exists.

Examples - 2

$\mathbb{P}$ probability measure; $G_1 \preceq_{st} G_2$ and $C(G_1, G_2) \in \mathcal{L}_\gamma$;

$\phi : [0, 1] \to [0, 1]$ continuous and strictly increasing

$$\mu = \phi \circ \mathbb{P}$$

Then $\mu_1 = \phi \circ G_1 \preceq_{st} \phi \circ G_2 = \mu_2$

$$\mu(A) = \phi \circ \mathbb{P}(A) \geq \gamma \text{ if } \phi \text{ concave}$$

and $G_\mu \in \mathcal{M}_\gamma$

If $C \in \mathcal{L}^c_\gamma$ and ϕ convex, then $G_\mu \in \mathcal{M}^c_\gamma$ and C_μ exists

Comparison to fixed-Copula TBA

In fixed-Copula TBA Model we saw there is the conceptual possibility of simultaneous conditions

$$T \preceq_{st} X; \qquad X \preceq_{sp}^{(\gamma)} T$$

thoughtful choice of the copula C_T is needed

For the case of capacities we may have

$$G_{\mu_T} \preceq_{st} G_{\mu_X}; \qquad G_{\mu_X} \preceq_{sp}^{(\gamma)} G_{\mu_T}$$

with $G_\mu = C_\mu(G_{\mu_T}, G_{\mu_X})$, if exists.

In particular: C_μ should belong to $\mathcal{M}_\gamma$ with γ large enough
(an adequate level of asymmetry of C_μ is required)

For r.v. T, X' and X'', $X' \preceq_{st} X''$ implies

$$\mathbb{P}(T \leq X') \leq \mathbb{P}(T \leq X'')$$

for (T, X') and (T, X'') sharing the same connecting copula C.

For capacities one can have two measures G'_μ and G''_μ with marginals

$$G_{\mu_1} \quad \text{and} \quad G'_{\mu_2} \preceq_{st} G''_{\mu_2}$$

with same copula C_μ that satisfy

$$G'_\mu(A) \leq G''_\mu(A)$$

conclusions

Open questions:

- conditions under which define $\eta(C_\mu, G_{\mu_1}, G_{\mu_2})$ as integral;
- what about $\eta(C_\mu) = \eta(C_\mu, G_\mu, G_\mu)$?;
- extended definition of $\mathcal{M}_\gamma$;
- wider study of utilities like in TBA under measures of capacity;

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Thanks for your attention!