On M-valued L-fuzzy syntopogenous structures

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Topology, Uniformity and Proximity

Three important general topology structures:

Topological spaces

Topology: $T \subseteq 2^X$ such that

1. $\emptyset, X \in T$;
2. $U, V \in T \Rightarrow U \cap V \in T$;
3. $U_i \in T \forall i \in I \Rightarrow \bigcup_i U_i \in T$

(X, T) is called topological space.

Topology, Uniformity and Proximity

Proximity spaces

Proximity: $\delta \subseteq 2^X \times 2^X$ such that

1. $(\emptyset, X) \notin \delta$
2. $(A, B) \in \delta \iff (B, A) \in \delta$;
3. $(A, B \cup C) \in \delta \iff (A, B) \in \delta$ or $(A, C) \in \delta$
4. $(A, B) \notin \delta$ then $\exists C, D$: $(A, C) \notin \delta$, $(B, D) \notin \delta$ and $C \cup D = X$.

(X, δ) is called proximity space.

Topology, Uniformity and Proximity

$V \subset X \times X$ an entourage of diagonal, if $\Delta \subset V$ and $V = -V$, where $\Delta = \{(x,x) : x \in X\}$ and $-V = \{(x,y) : (y,x) \in V\}$.
$\mathcal{D}_X$ family of all entourages of the diagonal.

Uniform space

Uniformity: $\mathcal{U} \subseteq \mathcal{D}_X$ such that

1. If $V \in \mathcal{U}$ and $V \subset W \in \mathcal{D}_X$, then $W \in \mathcal{U}$
2. If $V_1, V_2 \in \mathcal{U}$, then $V_1 \cap V_2 \in \mathcal{U}$
3. For every $V \in \mathcal{U}$ exists $W \in \mathcal{U}$ such that $2W \subset V$
4. $\bigcap \mathcal{U} = \Delta$.

$(X, \mathcal{U})$ is called uniform space.

Syntopogeneous structures in Topology

Motivation

Syntopogeneous structure is a concept which allows to develop a unified approach to all three topological structures:

- Topological spaces (open sets, closed sets, closure, limit points, neighbourhood of point and so on)
- Proximity spaces (as nearness structure)
- Uniform spaces (as uniformity type properties in metric spaces)

Syntopogeneous structures were introduced by A. Csaszar in 1963.

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Syntopogeneous structures in Topology

Situation in fuzzy Topology

In fuzzy topology we have developed theories of

1. Fuzzy topologies;
2. Fuzzy proximities;
3. Fuzzy uniformities

Problem

Find the appropriate concepts for fuzzy syntopogeneous structures.

Syntopogeneous structures in Topology

Syntopogeneous structures in fuzzy Topology

- First time introduced by A.K.Katsaras and C.G.Petalas in 1983. Concept: $[0,1]^X \times [0,1]^X \to \{0,1\}$
- Expanded by A.Šostaks 1997. Concept: $[0,1]^X \times [0,1]^X \to [0,1]$
- Our object: $L^X \times L^X \to L$, where L complete lattice

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Topogeneous orders (in crisp case)

Topogeneous order

Topogeneous order on a set X is a relation σ on it powerset 2^X such that

- $(\emptyset, \emptyset), (X, X) \in \sigma$
- If $M' \subseteq M$ and $N \subseteq N'$ and $(M, N) \in \sigma$ then $(M', N') \in \sigma$.
- If $(M, N) \in \sigma$ then $M \subseteq N$
- $(M_1 \cup M_2, N) \in \sigma \iff (M_1, N), (M_2, N) \in \sigma$.
- $(M, N_1 \cap N_2) \in \sigma \iff (M, N_1), (M, N_2) \in \sigma$.

Syntopogeneous structures

Syntopogeneous structures

A family $\mathcal{S}$ of topogeneous orders on a set X is called a syntopogeneous structure if

1. $\mathcal{S}$ is directed, that is
$\sigma_1, \sigma_2 \in \mathcal{S} \Longrightarrow \exists \sigma \in \mathcal{S}$ such that $\sigma_1 \cup \sigma_2 \subseteq \sigma$;
2. $\forall \sigma \in \mathcal{S}\ \exists \sigma' \in \mathcal{S}$ such that $\sigma' \circ \sigma' \supseteq \sigma$,
where $(M, N) \in \sigma_1 \circ \sigma_2$, if $\exists P \in 2^X$ such that $(M, P) \in \sigma_1$ and $(P, N) \in \sigma_2$.

Approach to the fuzzy version

How to define fuzzy semi-topogeneous order

L-fuzzy semi-topogeneous order on a set X is a L-fuzzy relation σ on its L powerset L^X, that is $\sigma : L^X \times L^X \to L$ such that

- $\sigma(0_L, 0_L) = \sigma(1_L, 1_L) = 1_L$
- If $M' \leq M$ and $N \leq N'$ then $\sigma(M, N) \leq \sigma(M', N')$.
- If $\sigma(M, N)$ then $M \tilde{\subseteq} N$

- As the substitute for the last property we take

$$M \tilde{\subseteq} N = \inf_{x \in X} (M(x) \mapsto N(x)) =: M \Im N$$

where $\mapsto$ is an implicator on L

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Implicator

Different authors axiomatize different properties for implicator. For our merits we take the following axioms for $\mapsto L \times L \to L$:

- $a_1 \vee a_2 \mapsto b = (a_1 \mapsto b) \wedge (a_2 \mapsto b)$;
- $a \mapsto b_1 \wedge b_2 = (a \mapsto b_1) \wedge (a \mapsto b_2)$;
- $0 \mapsto a = 1_L$ for every $a \in L$ (left boundary condition);
- $1 \mapsto a = a$ for every $a \in L$ (left neutrality)
- $(a \mapsto 0) \mapsto (b \mapsto 0) = b \mapsto a$.

Remark: Note that properties (1)and (2) are stronger version of $a \mapsto b$ is non-increasing on the first argument and $a \mapsto b$ is non-decreasing on the second argument.

Remark: From (5) and (4) we have the following important double negation property: $(a \mapsto 0) \mapsto 0 = a$ for every $a \in L$. Thus $a \mapsto 0$ is an order reversing involution and we write $a^c = a \mapsto 0$.

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L-fuzzy semi-topogeneous orders

Thus let X be a set, L a complete lattice and $\mapsto: L \times L \to L$

Semi-topogeneous order on a set X is a L-fuzzy relation σ on its L-powerset L^X, that is $\sigma : L^X \times L^X \to L$ such that

(1to) $\sigma(0_L, 0_L)) = \sigma(1_L, 1_L) = 1_L$

(2to) If $M' \leq M$ and $N \leq N'$ then $\sigma(M, N) \leq \sigma(M', N')$.

(3to) $\sigma(M, N) \leq M \mathfrak{I} N$.

Special properties of fuzzy semi-topogeneous orders

- Fuzzy semi-topogeneous order is called topogeneous if

  (4to) $\sigma(M_1 \vee M_2, N) = \sigma(M_1 N) \wedge \sigma(M_2, N)$.
  (5to) $\sigma(M, N_1 \wedge N_2) = \sigma(M, N_1) \wedge \sigma(M, N_2)$
- Fuzzy topogeneous order is called perfect if

  (6to) $\sigma(\bigvee_i M_i, N) = \bigwedge_i \sigma(M_i, N)$.
- Fuzzy topogeneous order is called biperfect if it is perfect and

  (7to) $\sigma(M, \bigwedge_i N_i) = \bigwedge_i \sigma(M, N_i)$
- Fuzzy semitopogeneous order is called symmetric if

  (8to) $\sigma(M, N) = \sigma(N^c, M^c)$

L-fuzzy syntopogeneous structures

Definition

An L-fuzzy syntopogeneous structure on a set X is a family $\mathcal{S}$ of L-fuzzy topogeneous orders on X such that

- $\mathcal{S}$ is directed, that is given two L-fuzzy topogeneous orders $\sigma_1, \sigma_2 \in \mathcal{S}$ there exists $\sigma \in \mathcal{S}$ such that $\sigma_1 \vee \sigma_2 \leq \sigma$;
- For every $\sigma \in \mathcal{S}$ there exists $\sigma' \in \mathcal{S}$ such that $\sigma \leq \sigma' \circ \sigma'$, where $\sigma_1 \circ \sigma_2 = \{\bigvee(\sigma_1(M, P) \wedge \sigma_2(P, N) : P \in L^X\}$.

$(X, \mathcal{S})$ is called syntopogeneous space.

L-fuzzy topogeneous order construction from L-fuzzy semi-topogeneous order

Proposition

Let $M, N \in L^X$ be L-fuzzy semi-topogeneous order σ on X, let a mapping $\sigma^T : L^X \times L^X \to L$ be defined by the equality

$$\sigma^T(M, N) = sup\{\bigwedge_{i,j} \sigma(M_i, N_j) : \bigvee_{i=1}^{m} M_i = M, \bigwedge_{j=1}^{n} N_j = N\}.$$

Then σ^T is an L-fuzzy topogeneous order.

Similary perfect and biperfect topogeneous orders can be obtained from topogeneous orders.

L-fuzzy topogeneous order construction from *L*-fuzzy semi-topogeneous order

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L-fuzzy topologies and perfect *L*-fuzzy topogeneous orders

Theorem

Let $\sigma : L^X \times L^X \to L$ be a perfect topogeneous fuzzy order. Then the mapping : $\mathcal{T} : L^X \to L$ defined by

$$\mathcal{T}_\sigma(M) = \sigma(M, M), M \in L^X$$

is an *L*-fuzzy topology.
Conversely, given an *L*-fuzzy topology $\mathcal{T} : L^X \to L$ on X, the mapping $\sigma_\mathcal{T} : L^X \times L^X \to L$ defined by the equality

$$\sigma_\mathcal{T}(M, N) = \bigvee\{\mathcal{T}(P) : M \leq P \leq N, P \in L^X\}$$

is a perfect topogeneous fuzzy order. Besides $\mathcal{T}_{\sigma_\mathcal{T}} = \mathcal{T}$ and $\sigma_{\mathcal{T}_\sigma} = \sigma$ for every *L*-fuzzy topology $\mathcal{T}$ and every perfect *L*-fuzzy topogeneous

L-fuzzy proximities and L-fuzzy symmetric topogeneous orders

Theorem

Let $\sigma : L^X \times L^X \to L$ be a symmetric L-fuzzy topogeneous order on X. Then the mapping $\delta_\sigma : L^X \times L^X \to L$ defined by

$$\delta(A, B) = \sigma(A, B^c) \mapsto 0$$

is an L-fuzzy proximity on X. Conversely, let $\delta : L^X \times L^X \to L$ be an L-fuzzy proximity. Then with following equality we gain

$$\sigma(A, B) = \delta(A, B^c) \mapsto 0$$

a symmetric L-fuzzy topogeneous order on X. Besides $\delta_{\sigma_\delta} = \delta$ and $\sigma_{\delta_\sigma} = \sigma$ for every symmetric L-fuzzy topogeneous order σ and for any L-fuzzy proximity δ.

L-fuzzy uniformities and L-fuzzy biperfect topogeneous orders

Theorem

To every biperfect L-fuzzy topogeneous order σ on X there corresponds an element $U_\sigma \in \mathcal{D}_X$ defined by

$$U_\sigma(A, \alpha) = \bigwedge\{B \mid \sigma(A, B) \geq \alpha\}.$$

Conversely, to every U there corresponds a biperfect L-fuzzy topogeneous order defined by

$$\sigma_U(A, B) = \bigvee\{\alpha \mid U(A, \alpha) \leq B\}.$$

Moreover, $U_{\sigma_U} = U$ and $\sigma_{U_\sigma} = \sigma$ for each biperfect L-fuzzy topogeneous order σ and each $U \in \mathcal{D}_X$.

Problem

From contrapositive simmetry

- $(a \mapsto 0) \mapsto (b \mapsto 0) = b \mapsto a$

and left neutrality

- $1 \mapsto a = a$ for every $a \in L$

follows double negation law

- $(a \mapsto 0) \mapsto 0 = a$ for every $a \in L$

that could be too restrictive

We have to ask...

What is the influence of this property in syntopogeneous structures theory?

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Work in progress: L-fuzzy proximities and L-fuzzy symmetric topogeneous orders

L-fuzzy proximity

Let L be a completely distributive lattice with implicator $\mapsto: L \times L \to L$. A mapping $d : L^X \times L^X \to L$ is called an L-fuzzy proximity on X if it satisfies the following conditions:

(FP1) $d(0_X, 1_X) = 0$;

(FP2) $d(A, B) = d(B, A)$;

(FP3) $d(A, B \vee C) = d(A, B) \vee d(A, C)$;

(FP4) $d(A, B) \geq (A \hookrightarrow B^c)^c$;

(FP5) $d(A, B) \geq \inf\{d(A, E) \vee d(B, E^c) \mid E \in L^X\}$.

The pair (X, d) is called an L-fuzzy proximity space.

Some directions for further research

- To study relation between L-fuzzy uniformities and L-fuzzy biperfect topogeneous orders in case without contrapositive symmetry.
- To develop the asymmetric version of the concept of a syntopogenous structure and to work out the corresponding theory.
- To study categorical properties of the category of L-fuzzy syntopogenous spaces and its relations with different subcategories.
- To define and to study (L, M)-syntopogenous structures, that is when (semi-)topogeneous orders are defined on the L-powersets of a set X and take their values in, probably, a different lattice M.

Thank you for your attention!

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