FUSION FUNCTIONS AND THEIR DIRECTIONAL MONOTONICITY

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The main aim of our contribution is to open the theory of directional monotonicity of fusion functions.

An aggregation function is a function $A\colon [0,1]^n \to [0,1]$ which is monotone (increasing in each component) and satisfies the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$.

In real data processing, there are several other fusion techniques which do not satisfy the condition of monotonicity, e.g., implications, the mode function or some types of means.

Wilkin and Beliakov (2013) studied weakly monotone means. A weak monotonicity - monotonicity in direction given by vector $\vec{r} = (1, 1, \ldots, 1)$.

In our contribution we introduce and study fusion functions and a general notion of directional monotonicity of fusion functions.

2 Fusion functions and $\vec{r}$-monotonicity

3 Properties of $\vec{r}$-monotone fusion functions

4 Fusion functions and their $\mathcal{D}^{\vec{}}$-sets

5 $\mathcal{D}^{\vec{}}$-sets of linear/piece-wise linear functions

6 Concluding remarks

Definition

Let $n \in \mathbb{N}$, $n \geq 2$ and $\mathbf{I} = [0,1]$. A fusion function is an arbitrary function $F\colon \mathbf{I}^n \to \mathbf{I}$.

Definition

Let $\vec{r}$ be a real n-dimensional vector, $\vec{r} \neq \vec{0}$. A fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ is $\vec{r}$-increasing ($\vec{r}$-decreasing) if for all points $\mathbf{x} \in \mathbf{I}^n$ and all $c > 0$ such that $\mathbf{x} + c\vec{r} \in \mathbf{I}^n$, it holds

$$F(\mathbf{x} + c\vec{r}) \geq F(\mathbf{x}) \quad (F(\mathbf{x} + c\vec{r}) \leq F(\mathbf{x})).$$

Remark

(i) *For any $\alpha > 0$ the increasing/decreasing monotonicity of a fusion function F in direction $\alpha\vec{r}$ is equivalent to the same property in direction $\vec{r}$.*

(ii) *If a fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ has the first-order partial derivatives with respect to all variables then it has the directional derivative in any direction $\vec{r}$ at each point $\mathbf{x} \in \mathbf{I}^n$ admitting a positive constant $c > 0$ such that $\mathbf{x} + c\vec{r} \in \mathbf{I}^n$. Consequently:*

F is $\vec{r}$-increasing ($\vec{r}$-decreasing) if and only if its directional derivative in direction $\vec{r}$ is non-negative (non-positive) at each admissible point $\mathbf{x} \in \mathbf{I}^n$.

Example

The Lukasiewicz implication $I_L : \mathbf{I}^2 \to \mathbf{I}$, *given by*

$$I_L(x, y) = \min\{1, 1 - x + y\}$$

is $(1, 1)$*-increasing.*
For all points $(x, y) \in \mathbf{I}^2$ *and all* $c > 0$*, such that*
$(x, y) + c(1, 1) = (x + c, y + c) \in \mathbf{I}^2$,
we have

$$\begin{aligned} I_L(x + c, y + c) &= \min\{1, 1 - x - c + y + c\} \\ &= \min\{1, 1 - x + y\} = I_L(x, y). \end{aligned}$$

Note that I_L *is also a* $(1, 1)$*-decreasing fusion function.*

Example

The Reichenbach implication $I_R \colon \mathbf{I}^2 \to \mathbf{I}$, *given by*

$$I_R(x, y) = 1 - x + xy,$$

is neither $(1, 1)$*-increasing nor* $(1, 1)$*-decreasing.*

- *Let* $\vec{r} = (1, 1)$, $\mathbf{x} = (0.1, 0.2)$, $c = 0.2$, *i.e.,* $\mathbf{x} + c\vec{r} = (0.3, 0.4)$. *Then* $(0.3, 0.4) > (0.1, 0.2)$ *but* $I_R(0.3, 0.4) = 0.82 < I_R(0.1, 0.2) = 0.92 \ \Rightarrow\ I_R$ *is not* $(1, 1)$*-increasing.*
- *If* $\mathbf{x} = (0.4, 0.5)$, $c = 0.2$, *i.e.,* $\mathbf{x} + c\vec{r} = (0.6, 0.7)$, *then* $(0.6, 0.7) > (0.4, 0.5)$ *and* $I_R(0.6, 0.7) = 0.82 > I_R(0.4, 0.5) = 0.8 \ \Rightarrow\ I_R$ *is not* $(1, 1)$*-decreasing.*

Example

Let $F\colon \mathbf{I}^2 \to \mathbf{I}$,

$$F(x,y) = x - (\max\{0, x-y\})^2 .$$

Then F *is a continuous fusion function, but not an aggregation function. This function can be used in decision making.* F *is, e.g.,* $(1,1)$*-increasing,* $(0,1)$*-increasing, but it is neither* $(1,0)$*-increasing nor* $(1,0)$*-decreasing.*

Example

The weighted Lehmer mean $L_\lambda\colon \mathbf{I}^2 \to \mathbf{I}$, $L_\lambda(x,y) = \frac{\lambda x^2+(1-\lambda)y^2}{\lambda x+(1-\lambda)y}$ *(with convention* $\frac{0}{0}=0$*), where* $\lambda \in]0,1[$, *is* $(1-\lambda,\lambda)$*-increasing.*

Example

The function $F\colon \mathbf{I}^2 \to \mathbf{I}$,

$$F(x,y) = \begin{cases} 0 & \text{if } \ \mathrm{x} = \mathrm{y} = 1/2 \\ 1/2 & \text{if } \{\mathrm{x},\mathrm{y}\} \cap \{0,1\} \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

is not $\vec{r}$*-monotone in any direction* $\vec{r}$*.*

Remark

Let $\vec{e}_i = (\epsilon_1, \ldots, \epsilon_n)$, *where* $\epsilon_i = 1$ *and* $\epsilon_j = 0$ *for each* $j \neq i$, $i, j = 1, \ldots, n$.

The increasing monotonicity of a fusion function F is equivalent to the $\vec{e}_i$*-increasing monotonicity of F for all i.*

Aggregation functions $A\colon \mathbf{I}^n \to \mathbf{I}$ *are fusion functions, which are* $\vec{e}_i$*-increasing for each* $i = 1, \ldots, n$, *and satisfying the boundary conditions* $A(\mathbf{0}) = 0$ *and* $A(\mathbf{1}) = 1$.

Implications $I\colon \mathbf{I}^2 \to \mathbf{I}$ *are* $\vec{e}_1$*-decreasing and* $\vec{e}_2$*-increasing fusion functions, satisfying the boundary conditions* $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

We summarize several important properties of directionally monotone fusion functions:

Proposition

A fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ is $\vec{r}$-decreasing if and only if F is $(-\vec{r})$-increasing.

$$V_n = \{\vec{r} \in \mathbb{R}^n \mid \vec{r} \neq \vec{0}\}.$$

Theorem

Let $\vec{r}$ and $\vec{s}$ be vectors in V_n, such that for a given $a, b \geq 0$, $a + b > 0$, it holds that for each point $\mathbf{x}$ in $\mathbf{I}^n$ and $c > 0$ such that $\mathbf{x} + c\vec{u}$ is in $\mathbf{I}^n$, where $\vec{u} = a\vec{r} + b\vec{s}$, the points $\mathbf{x} + ca\vec{r}$ or $\mathbf{x} + cb\vec{s}$ are also in $\mathbf{I}^n$. Then if a fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ is both $\vec{r}$- and $\vec{s}$-increasing then it is also $\vec{u}$-increasing.

Corollary

Let $\vec{r} = (r_1, \ldots, r_n)$ and $\vec{s} = (s_1, \ldots, s_n)$ be vectors in V_n satisfying the condition

$$d_{\vec{r},\vec{s}} := \mathrm{card}\{i \in \{1, \ldots, n\} \mid r_i s_i \geq 0\} \geq n - 1$$

and let a fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ be $\vec{r}$-increasing and $\vec{s}$-increasing. Then F is $\vec{u}$-increasing for each $\vec{u} = a\vec{r} + b\vec{s}$ with $a, b \geq 0$, $a + b > 0$.

Theorem

Let a fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ have the first-order partial derivatives with respect to each variable. If F is $\vec{r}$-increasing and $\vec{s}$-increasing for some directions $\vec{r}$ and $\vec{s}$, then F is also $\vec{u}$ increasing for each $\vec{u} = a\vec{r} + b\vec{s}$, where $a \geq 0$, $b \geq 0$, $a + b > 0$.

Proposition

Let $F\colon \mathbf{I}^n \to \mathbf{I}$ be an $\vec{r}$-increasing fusion function. If $\varphi\colon \mathbf{I} \to \mathbf{I}$ is an increasing (decreasing) function then $G = \varphi \circ F$ is an $\vec{r}$-increasing (decreasing) fusion function.

Proposition

Let $F_1, \ldots, F_k \colon \mathbf{I}^n \to \mathbf{I}$ be $\vec{r}$-increasing fusion functions and let $F \colon \mathbf{I}^k \to \mathbf{I}$ be an increasing function. Then the function $G = F(F_1, \ldots, F_k)$ is an $\vec{r}$-increasing fusion function.

Remark

The $\vec{r}$-increasing monotonicity of F is not sufficient for the $\vec{r}$-increasing monotonicity of G.
Consider $(1,1)$-increasing functions F, F_1, $F_2 \colon [0,1]^2 \to [0,1]$,
$F_1(x,y) = x - (\max\{0, x-y\})^2, \quad F_2(x,y) = I_L(x,y),$
$F(x,y) = \frac{x^2+y^2}{x+y}$ (the Lehmer mean for $\lambda = 1/2$).

Put $G = F(F_1, F_2)$. As $G(x,x) = \frac{x^2+1}{x+1}$, it holds $G(0,0) = 1$, $G(1/2, 1/2) = 5/6$, and $G(1,1) = 1$, which shows that G is not $(1,1)$-increasing.

Proposition

Let $F\colon \mathbf{I}^n \to \mathbf{I}$ be an increasing fusion function and $f_i\colon [0,1] \to [0,1]$, $i = 1, \ldots, n$, be monotone functions. If a function $G\colon \mathbf{I}^n \to \mathbf{I}$ is defined by

$$G(x_1, \ldots, x_n) = F\left(f_1(x_1), \ldots, f_n(x_n)\right),$$

then G is $\vec{r}$-increasing for each vector $\vec{r} = (r_1, \ldots, r_n)$ with the property:
$\forall\ i \in \{1, \ldots, n\}$, $\ r_i \geq 0$ if f_i is increasing and $r_i \leq 0$ if f_i is decreasing.

Proposition gives a sufficient condition for the r-increasing monotonicity of G. The function G can also be $\vec{r}$-increasing for some other vectors $\vec{r}$, as is shown in the next example.

Example

- *Consider implications* $I_S\colon [0,1]^2 \to [0,1]$, $I_S(x,y) = S(n(x),y)$, *where* S *is a t-conorm and* n *a negation.*
All I_S *are certainly* $\vec{r}$*-increasing for all* $\vec{r} = (r_1, r_2) \in V_2$ *with* $r_1 \leq 0$ *and* $r_2 \geq 0$.
But, e.g., the Lukasiewicz implication $I_L = I_{S_L}$*, obtained for* $S_L(x,y) = \min\{1, x+y\}$ *and* $n(x) = 1 - x$*, is also* $(1,1)$*-increasing and* $(-1,-1)$*-increasing.*
- *Each aggregation function* $A\colon [0,1]^2 \to [0,1]$ *is* $\vec{r}$*-increasing at least for all* $\vec{r} = (r_1, r_2) \in V_2$ *with* $r_1 \geq 0$ *and* $r_2 \geq 0$.
But, e.g., the weakest aggregation function A_* *is* $\vec{r}$*-increasing for all* $\vec{r} = (r_1, r_2)$ *with the exception of the vectors with* $r_1 \leq 0$ *and* $r_2 \leq 0$.

Proposition

Let $F\colon \mathbf{I}^n \to \mathbf{I}$ be an $\vec{r}$-increasing fusion function and let functions $g, f_i \in \{id_{\mathbf{I}}, 1 - id_{\mathbf{I}}\}$, $i = 1, \ldots, n$. If a function $H\colon \mathbf{I}^n \to \mathbf{I}$ is defined by

$$H(x_1, \ldots, x_n) = g\left(F\left(f_1(x_1), \ldots, f_n(x_n)\right)\right),$$

then H is $\vec{s}$-increasing where

$$\vec{s} = (-1)^{g(0)}\left((-1)^{f_1(0)} r_1, \ldots, (-1)^{f_n(0)} r_n\right).$$

Corollary

If a fusion function F is $\vec{r}$-increasing then its dual F^d, given by $F^d(\mathbf{x}) = 1 - F(\mathbf{1} - \mathbf{x})$, is also $\vec{r}$-increasing.

Let $\vec{\mathcal{D}}(F)$ be the set of all directions $\vec{r} \in V_n$ in which a fusion function F is increasing. Clearly, for each n-ary aggregation function A, $\vec{\mathcal{D}}(A) \supseteq V_n^+ = \{\vec{r} \in V_n \mid r_1 \geq 0, \ldots, r_n \geq 0\}$.

Example

Let $L_{1/2} \colon \mathbf{I}^2 \to \mathbf{I}$ be the Lehmer mean given by

$$L_{1/2}(x, y) = \frac{x^2 + y^2}{x + y}.$$

We have proved that the only vectors with respect to which the function $L_{1/2}$ is $\vec{r}$-increasing are vectors $\vec{r} = \alpha(1, 1)$ with $\alpha > 0$, i.e., $\vec{\mathcal{D}}(L_{1/2}) = \{\alpha(1, 1) \mid \alpha > 0\}$. The function $L_{1/2}$ is weakly monotone only.

It can be shown that in general, for the weighted Lehmer mean L_λ, $\lambda \in]0, 1[$, it holds $\vec{\mathcal{D}}(L_\lambda) = \{\alpha(1 - \lambda, \lambda) \mid \alpha > 0\}$.

Example

- *Consider the Lukasiewicz t-norm* T_L. *As*

$$T_L(\mathbf{x} + c\vec{r}) = \max(0, x + y - 1 + c(r_1 + r_2)),$$

for all admissible $\mathbf{x} \in \mathbf{I}^2$ *and* $c > 0$,

$$T_L(\mathbf{x} + c\vec{r}) \geq T_L(\mathbf{x}) \Leftrightarrow r_1 + r_2 \geq 0, \text{ i.e.,}$$

$$\vec{\mathcal{D}}(T_L) = \{\vec{r} = (r_1, r_2) \mid r_2 \geq -r_1\}.$$

The set $\vec{\mathcal{D}}(T_L)$ *is a superset of the minimal set of vectors* $\{\vec{r} = (r_1, r_2) \mid r_1 \geq 0, r_2 \geq 0\}$ *with respect to which each aggregation function is* $\vec{r}$*-increasing.*

- *Similarly, for the Lukasiewicz implication* I_L *we have* $\vec{\mathcal{D}}(I_L) = \{\vec{r} = (r_1, r_2) \mid r_2 \geq r_1\} \supset \{\vec{r} = (r_1, r_2) \mid r_1 \leq 0, r_2 \geq 0\}$.

In many applications, linear/piece-wise linear fusion functions are considered.

Lemma

A linear function $F\colon \mathbf{I}^n \to \mathbf{I}$ is $\vec{r}$-increasing if and only if for a fixed connected set $E \subseteq \mathbf{I}^n$ with positive volume, $\vec{F}|_E$ is $\vec{r}$-increasing.

A continuous fusion function F is piece-wise linear if and only if it can be obtained by a patchwork technique with linear functions B_j, $j = 1, \ldots, k$.

Lemma

If $F\colon \mathbf{I}^n \to \mathbf{I}$ is a continuous piece-wise linear fusion function determined by linear functions $B_1, \ldots, B_k$, then $\vec{\mathcal{D}}(F) = \bigcap_{j=1}^{k} \vec{\mathcal{D}}(B_j)$.

Proposition

Let a fusion function $F\colon \mathbf{I}^n \to \mathbf{I}$ *be linear, i.e.,* $F(\mathbf{x}) = b + \sum_{i=1}^{n} a_i x_i$.

Then

$$\vec{\mathcal{D}}(F) = \{(r_1, \ldots, r_n) \in V_n \mid \sum_{i=1}^{n} a_i r_i \geq 0\}.$$

Corollary

Let $\vec{w} = (w_1, \ldots, w_n)$ *be a weighting vector, i.e., for each* $i \in \{1, \ldots, n\}$, $w_i \geq 0$ *and* $\sum_{i=1}^{n} w_i = 1$. *Let* $W_{\vec{w}}\colon \mathbf{I}^n \to \mathbf{I}$ *be a weighted arithmetic mean,* $W_{\vec{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$. *Then*

$$\vec{\mathcal{D}}(W_{\vec{w}}) = \left\{(r_1, \ldots, r_n) \in V_n \mid \sum_{i=1}^{n} w_i r_i \geq 0\right\}.$$

Example

A continuous piece-wise linear fusion function $F\colon \mathbf{I}^2 \to \mathbf{I}$, *given by*

$$F(x, y) = \min\{x, y, 1 - x, 1 - y\},$$

was applied in several works.

F can be seen as a fusion function obtained by a patchwork technique related to the projections P_1, P_2 *and functions* $1 - P_1$ *and* $1 - P_2$.
Thus

$$\vec{\mathcal{D}}(F) = \vec{\mathcal{D}}(P_1) \cap \vec{\mathcal{D}}(P_2) \cap \vec{\mathcal{D}}(1 - P_1) \cap \vec{\mathcal{D}}(1 - P_2) = \emptyset.$$

This means that there is no direction $\vec{r}$ *in which* F *is* $\vec{r}$*-monotone.*

Special piece-wise linear fusion functions are, e.g., OWA operators and the Choquet integral

Corollary

Let $A_{\vec{w}} \colon \mathbf{I}^n \to \mathbf{I}$ be an OWA operator corresponding to a weighting vector $\vec{w}$. Then

$$\begin{aligned}\vec{\mathcal{D}}(A_{\vec{w}}) &= \bigcap_{\sigma\in\Omega} \vec{\mathcal{D}}\left(W_{\sigma(\vec{w})}\right)\\ &= \bigcap_{\sigma\in\Omega} \left\{\vec{r} = (r_1, \ldots, r_n) \in V_n \mid \sum_{i=1}^{n} w_{\sigma(i)} r_i \geq 0\right\},\end{aligned}$$

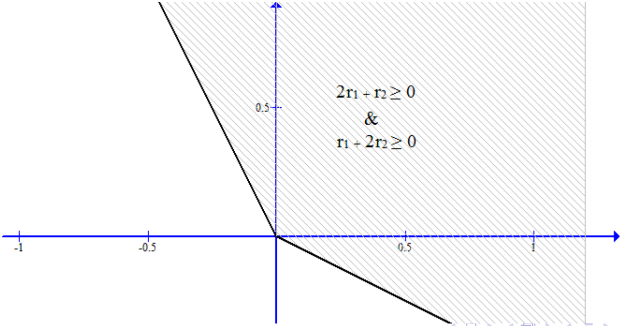
where Ω is the set of all permutations $\sigma \colon [n] \to [n]$ and $\sigma(\vec{w}) = (w_{\sigma(1)}, \ldots, w_{\sigma(n)})$.

$$\vec{\mathcal{D}}(A_{\vec{w}}) = \{\vec{r} = (r_1, \ldots, r_n) \in V_n \mid \forall \sigma \in \Omega : \sum_{i=1}^{n} w_{\sigma(i)} r_i \geq 0\}.$$

Example

Consider the OWA operator A,
$A(x,y) = \frac{1}{3}\min\{x,y\} + \frac{2}{3}\max\{x,y\}$. *Then*

$$\begin{aligned}\vec{\mathcal{D}}(A) &= \vec{\mathcal{D}}\left(W_{(1/3,2/3)}\right) \cap \vec{\mathcal{D}}\left(W_{(2/3,1/3)}\right)\\ &= \{(r_1,r_2)\in V_2 \mid r_1+2r_2 \geq 0,\ 2r_1+r_2 \geq 0\}.\end{aligned}$$



Corollary

Let Ch_μ *be the Choquet integral corresponding to a capacity* μ*. Then*

$$\begin{aligned}\vec{\mathcal{D}}(\mathrm{Ch}_\mu) &= \bigcap_{\sigma\in\Omega} \vec{\mathcal{D}}\left(W_{\sigma(\mu)}\right) \\ &= \bigcap_{\sigma\in\Omega} \left\{\vec{r} = (r_1, \ldots, r_n) \in V_n \mid \sum_{i=1}^{n} w^{\mu}_{\sigma,i} r_i \geq 0\right\}.\end{aligned}$$

$$\vec{\mathcal{D}}(\mathrm{Ch}_\mu) = \left\{\vec{r} = (r_1, \ldots, r_n) \in V_n \mid \forall\sigma \in \Omega : \sum_{i=1}^{n} w^{\mu}_{\sigma,i} r_i \geq 0\right\}.$$

Example

Consider the capacity $\mu\colon 2^{[2]} \to \mathbf{I}$, $\mu(\{1\}) = 0.3$, $\mu(\{2\}) = 0.4$ *and the Choquet integral with respect to* μ, *i.e.,*

$$\mathrm{Ch}_\mu(x,y) = \begin{cases} 0.3x + 0.7y & \text{if } x \geq y, \\ 0.6x + 0.4y & \text{if } x \leq y. \end{cases}$$

Then

$$\vec{\mathcal{D}}(\mathrm{Ch}_\mu) = \{(r_1, r_2) \in V_2 \mid 0.3r_1 + 0.7r_2 \geq 0,\ 0.6r_1 + 0.4r_2 \geq 0\}.$$

Remark

Note that the Sugeno and Shilkret integrals, hierarchical Choquet integral, twofold integral are also continuous piece-wise linear functions, which allows to determine the set of all directions in which they are increasing.

- We have introduced and discussed fusion functions on the unit interval $\mathbf{I}$ and their directional monotonicity. Note that this property is related to the directional derivative (if it exists).
- Our results generalize the results of Wilkin and Beliakov (2013) concerning the so-called weak monotonicity, which is in our terminology $(1, 1)$-monotonicity.
- All introduced notions can easily be rewritten for any subinterval of the extended real line. However, not all of our results have the same form when open or unbounded intervals are considered.
- The aim of this contribution has been the opening the theory of fusion functions and directional monotonicity, and thus a deeper study of directional monotonicity in all mentioned cases will be the topic of our next investigations.

THANK YOU FOR YOUR ATTENTION !