Group decision making: From penalty functions to interval-valued fuzzy sets

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- Penalty functions on Cartesian product of lattices
- Cartesian product of lattices
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Group Decision Making (GDM) Problem

We have a set of options or alternatives

$$X = \{x_1, \cdots, x_p\} \quad (p \geq 2)$$

and a set of criteria or experts

$$E = \{e_1, \cdots, e_n\} \quad (n \geq 2)$$

each of whom provides his/her preferences over the set of alternatives

The problem is to find a solution, which will be an alternative or a set of alternatives, which is (are) the most accepted one(s) by the whole set of experts.

GDM

- Most of resolution procedures are based in comparing alternatives pairwise, with each expert providing his/her preferences in a matrix as follows:

$$r_{el} = \begin{pmatrix} - & r_{(el)12} & \cdots & r_{(el)1p} \\ r_{(el)21} & - & \cdots & r_{(el)2p} \\ \cdots & \cdots & - & \cdots \\ r_{(el)p1} & \cdots & \cdots & - \end{pmatrix}$$

- Usually there exists uncertainty $\longrightarrow$ fuzzy sets.
- Vague concepts as, for instance, that of majority $\longrightarrow$ fuzzy sets.

GDM

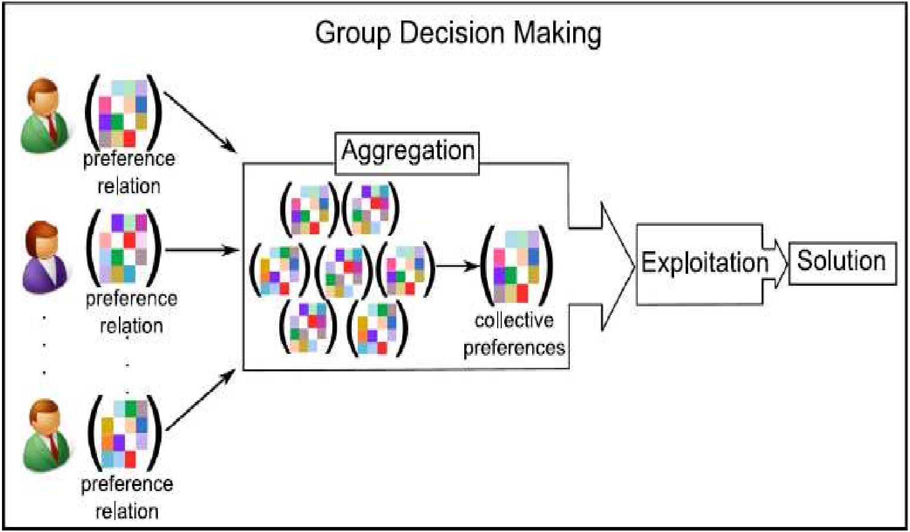
Typically, a selection process for GDM consists:

(1) Uniform representation of information.

(2) Application of a selection procedure. This procedure consists of two phases:

   (2.1) Aggregation phase. A collective preference structure is built from the set of individual homogeneous preference structures.

   (2.2) Exploitation phase. A given method is applied to the collective preference structure to obtain a selection of alternatives.


Group Decision Making
preference relation
preference relation
preference relation
Aggregation
collective preferences
Exploitation
Solution

Aggregation to be used: penalty functions

Definition

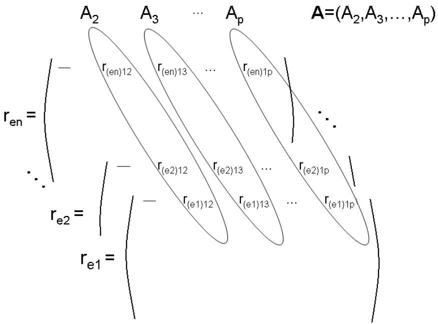
A function $M : [a,b]^n \to [a,b]$ *is an aggregation function if it is monotone increasing in each component and* $M(\mathbf{a}) = M(a, \cdots, a) = a$ *and* $M(\mathbf{b}) = M(b, \cdots, b) = b$

Definition

An aggregation function M *is averaging (or compensative or a mean) if*

$$\min(\mathbf{x}) = \min(x_1, \cdots, x_n) \leq M(x_1, \cdots, x_n) \leq \max(x_1, \cdots, x_n) = \max(\mathbf{x})$$

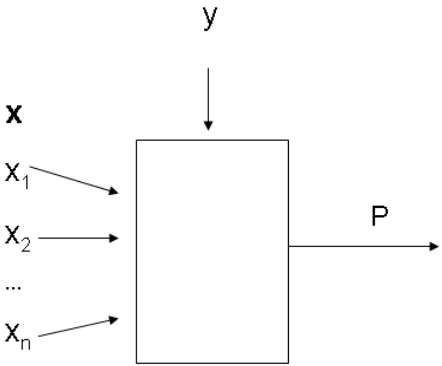
Aggregation phase: Up to now, the aggregation we have used for every A_i has been the same.



Data must impose:

1. The aggregation to be used
2. The way in which we represent information.

Aggregation to be used: penalty functions



Aggregation to be used: penalty functions

Definition

A penalty function is a function

$$P : [a, b]^{n+1} \to \mathbb{R}^+ = [0, \infty]$$

such that:

1. $P(\mathbf{x}, y) = 0$ *if* $x_i = y$ *for every* $i = 1, \cdots, n$;
2. $P(\mathbf{x}, y)$ *is quasiconvex in* y *for every* $\mathbf{x}$; *that is,*

$$P(\mathbf{x}, \lambda \cdot y_1 + (1 - \lambda) \cdot y_2) \leq \text{máx}(P(\mathbf{x}, y_1), P(\mathbf{x}, y_2))$$

- T. Calvo, R. Mesiar, R. Yager, A quantitative weights and aggregation, IEEE Transactions on Fuzzy Systems 12 (2004) 6269
- T. Calvo, G. Beliakov, Aggregation functions based on penalties, Fuzzy Sets and Systems 161 (10) (2010) 14201436.

Aggregation to be used: penalty functions

We call function based on the penalty function P (or penalty-based function) to the function

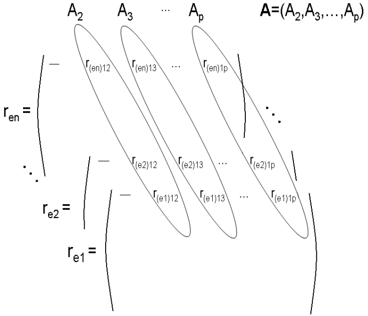
$$f(\mathbf{x}) = \arg\min_{y} P(\mathbf{x}, y),$$

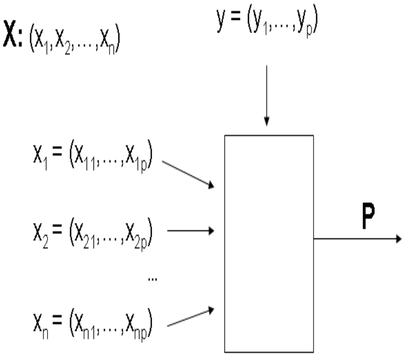
ifi y is the unique minimum, and $y = \frac{c+d}{2}$ if the set of minima is the interval $[c, d]$

Theorem

Every averaging aggregation function can be represented as a penalty-based function in the sense of the previous definition.

Penalty functions on a Cartesian product of lattices





Cartesian product of lattices

Definition

A poset $(P, \leq)$ is a set P with a relation $\leq$ which is reflexive, antisymmetric and transitive. A chain in a poset is a totally ordered set. The length of a finite chain is equal to the number of elements in that chain minus one.

Definition

Let L be a set. A lattice $\mathcal{L} = (L, \leq, \wedge, \vee)$ is a poset with respect to the partial order $\leq$ in L and the operations $\wedge$ and $\vee$ which satisfy the properties of absortion, idempotency, commutativity and associativity. That is, a bounded poset such that every pair of elements has a unique minimal upper bound and a unique maximal lower bound en L.

Cartesian product of lattices

Proposition

Let $\mathcal{L}_1 = (L_1, \leq_1, \wedge_1, \vee_1)$ *and* $\mathcal{L}_2 = (L_2, \leq_2, \wedge_2, \vee_2)$ *be two lattices. The Cartesian product*

$$\mathcal{L}_1 \times \mathcal{L}_2 = (L_1 \times L_2, \leq, \wedge, \vee)$$

with $\leq$ *defined by*

$$(x_1, x_2) \leq (y_1, y_2) \text{ iff } x_1 \leq y_1 \text{ and } x_2 \leq y_2$$

and

$$\wedge\left((x_1, x_2), (y_1, y_2)\right) = \left(\wedge_1(x_1, y_1), \wedge_2(x_2, y_2)\right)$$
$$\vee\left((x_1, x_2), (y_1, y_2)\right) = \left(\vee_1(x_1, y_1), \vee_2(x_2, y_2)\right)$$

is a lattice.

Remarks:

- In this work we use the Cartesian product of **finite** chains $\mathcal{C}$ with supremum and infimum.
- All the finite chains of the same length n are isomorphic to each other and isomorphic to the chain $\mathcal{C} = 0 \leq 1 \leq 2 \leq \cdots, \leq n-1$
- The Cartesian product of chains needs not be a chain.

Theorem

Let $\mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$. Let a and b be two elements of $\mathcal{L}_m$ such that $a \leq b$. Then all the minimal chains that join a and b have the same length.

Corollary

Let $a, b \in \mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$. *Then all the minimal chains which join* $\wedge(a, b)$ *and* $\vee(a, b)$ *have the same length.*

If $\mathcal{L}_m$ is a Cartesian product of m chains, then the distance between $x, y \in \mathcal{L}_m$ may be defined as the length of the chain $\mathcal{C}$ with minimal element $a = \wedge(x, y)$ and maximal element $b = \vee(x, y)$. That is:

$$d(x, y) = length(\mathcal{C}) - 1$$

This definition is equivalent to:

$$d(x, y) = \sum_{i=1}^{m} d_i(x_i, y_i) = \sum_{i=1}^{m} |x_i - y_i|$$

Natural distance

Restricted lattice dissimilarity functions

Let's consider the lattice $\mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$. Take

$$1_{\mathcal{L}_m} = (\vee(\mathcal{C}_1), \cdots, \vee(\mathcal{C}_m))$$
$$0_{\mathcal{L}_m} = (\wedge(\mathcal{C}_1), \cdots, \wedge(\mathcal{C}_m))$$

Definition

Let $\mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$. A mapping

$$\delta_R : \mathcal{L}_m \times \mathcal{L}_m \to \mathcal{L}_m$$

is a restricted lattice dissimilarity function if

1. $\delta_R(x, y) = \delta_R(y, x)$ *for each* $x, y \in \mathcal{L}_m$*;*
2. $\delta_R(x, y) = 1_{\mathcal{L}_m}$ *iff for every* $i = 1, \cdots, m$
$$x_i = \mathit{Sup}(\mathcal{C}_i) \text{ and } y_i = \mathit{Inf}(\mathcal{C}_i)$$
or
$$x_i = \mathit{Inf}(\mathcal{C}_i) \text{ and } y_i = \mathit{Sup}(\mathcal{C}_i)$$
3. $\delta_R(x, y) = 0_{\mathcal{L}_m}$ *iff* $x = y$*;*
4. *If* $x \leq y \leq z$ *then* $\delta_R(x, y) \leq \delta_R(x, z)$ *and* $\delta_R(y, z) \leq \delta_R(x, z)$*.*

Restricted lattice dissimilarity functions

Proposition

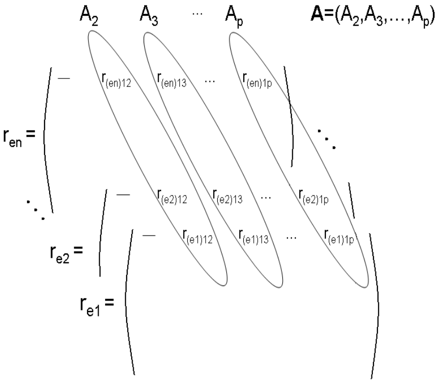
Let $x, y \in \mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$ *and let* $\delta_{R_i} : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$ *be a restricted lattice dissimilarity function for each* i*. Then the mapping*

$$\delta_R(x, y) = (\delta_{R_1}(x_1, y_1), \cdots, \delta_{R_m}(x_m, y_m))$$

is a restricted lattice dissimilarity function.

L-Multisets: $\mathcal{FS}(U)^m$ is the class of sets $\mathbf{A} = (A_1, \cdots, A_m)$ with $A_i : U \to \mathcal{C}_i$

$$\mathbf{A}(u_i) = (A_1(u_i), \cdots, A_m(u_i)) \text{ for every } u_i \in U$$



Lattice distances

L-Multisets: $\mathcal{FS}(U)^m$ is the class of sets $\mathbf{A} = (A_1, \cdots, A_m)$ with $A_i : U \to \mathcal{C}_i$

$$\mathbf{A}(u_i) = (A_1(u_i), \cdots, A_m(u_i)) \text{ for every } u_i \in U$$

Definition

Let $\mathcal{L}_m = (\mathcal{C}_1 \times \cdots \times \mathcal{C}_m, \leq, \wedge, \vee)$*. A mapping*

$$\Omega : \mathcal{FS}(U)^m \times \mathcal{FS}(U)^m \to \mathcal{L}_m$$

is a lattice distance in $\mathcal{FS}(U)^m$ *if*

1. $\Omega(\mathbf{A}, \mathbf{B}) = \Omega(\mathbf{B}, \mathbf{A})$ *for every* $\mathbf{A}, \mathbf{B} \in \mathcal{FS}(U)^m$*;*
2. $\Omega(\mathbf{A}, \mathbf{B}) = 0_{\mathcal{L}_m}$ *iff* $A_i = B_i$ *for every* $i = 1, \cdots, m$*;*
3. $\Omega(\mathbf{A}, \mathbf{B}) = 1_{\mathcal{L}_m}$ *iff* A_i *y* B_i *are sets such that* u_j
   $A_i(u_j) = \vee(\mathcal{C}_i)$ *and* $B_i(u_j) = \wedge(\mathcal{C}_i)$ *or*
   $A_i(u_j) = \wedge(\mathcal{C}_i)$ *and* $B_i(u_j) = \vee(\mathcal{C}_i)$*;*
4. *If* $\mathbf{A} \leq \mathbf{A}' \leq \mathbf{B}' \leq \mathbf{B}$*, then* $\Omega(\mathbf{A}, \mathbf{B}) \geq \Omega(\mathbf{A}', \mathbf{B}')$ *where* $\mathbf{A} = (A_1, \cdots, A_m) \leq (A'_1, \cdots, A'_m) = \mathbf{A}'$ *if* $A_i \leq A'_i$ *for each* i*.*

Definition

Let $\mathcal{L}$ be a bounded lattice. An aggregation function over the lattice $\mathcal{L}$ is a mapping:

$$M : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

such that

i) $M(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$ *and* $M(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$;

ii) M *is increasing with respect to* $\leq$.

Aggregation functions on lattices

Proposition

Let $\delta_{R_1}, \cdots, \delta_{R_m}$ *be restricted lattice dissimilarity functions such that* $\delta_{R_i} : \mathcal{C}_i \times \mathcal{C}_i \to \mathcal{C}_i$. *Let* $M_1, \cdots, M_m$ *be aggregation functions* $M_i : \mathcal{C}_i \times \cdots \times \mathcal{C}_i \to \mathcal{C}_i$ *such that*

(L1) $M_i(x_1, \cdots, x_n) = 1_{\mathcal{L}}$ *iff* $x_i = \vee(\mathcal{C}_i)$ *for each* $i = 1, \cdots, n$

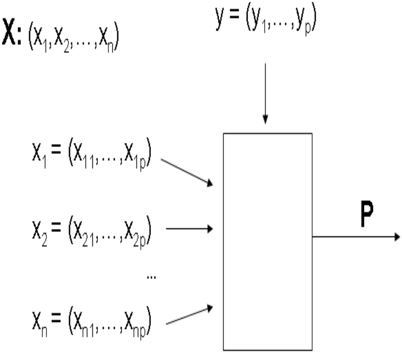
(L2) $M_i(x_1, \cdots, x_n) = 0_{\mathcal{L}}$ *iff* $x_i = \wedge(\mathcal{C}_i)$ *for each* $i = 1, \cdots, n$

Then

$$\Omega(\mathbf{A}, \mathbf{B}) = \left(\underset{i=1}{\overset{n}{M_1}}(\delta_{R_1}(A_1(u_i), B_1(u_i))), \cdots, \underset{i=1}{\overset{n}{M_m}}(\delta_{R_m}(A_m(u_i), B_m(u_i))) \right)$$

is a lattice distance in $\mathcal{FS}(U)^m$.

Penalty functions on a Cartesian product of lattices



Construction method of PFCPL

Theorem

Let $Y = (y_1, \cdots, y_m) \in \mathcal{L}_m$. For each y_i $(i = 1, \cdots, m)$ let's consider the set

$$B_{y_i}(u_j) = y_i \text{ for every } u_j \in U$$

and let $\mathbf{B}_Y = (B_{y_1}, \cdots, B_{y_m}) \in \mathcal{FS}(U)^m$. Consider the aggregation functions $M_1, \cdots, M_m$ given by $M_i : \mathcal{C}_i \times \cdots \times \mathcal{C}_i \to \mathcal{C}_i$ such that each of them, when it is composed with a convex function, gives back a convex function. Let's consider the restricted lattice dissimilarity functions $\delta_R(x, y) = (\delta_{R_1}(x_1, y_1), \cdots, \delta_{R_m}(x_m, y_m))$ such that δ_{R_i} with $i = 1, \cdots, m$ is convex in one variable. Then:

$$P_\Omega : \mathcal{FS}(U)^m \times \mathcal{FS}(U)^m \to \mathcal{L}_m \text{ given by}$$

$$P_\Omega(\mathbf{A}, Y) = \Omega(\mathbf{A}, \mathbf{B}_Y) = \left(\mathop{M_1}_{i=1}^{n} (\delta_{R_1}(A_1(u_i), y_1)), \cdots, \mathop{M_m}_{i=1}^{n} (\delta_{R_m}(A_m(u_i), y_m)) \right)$$

satisfies:

1. $P_\Omega(\mathbf{A}, Y) \geq 0_{\mathcal{L}_m}$;
2. $P_\Omega(\mathbf{A}, Y) = 0_{\mathcal{L}_m}$ *si* $A_k(u_j) = y_k$ *for each* k *and each* j;
3. *Each of the components is convex with respect to the corresponding* y_k.

- H. Bustince, E. Barrenechea, T. Calvo, S. James, G. Beliakov, Consensus in multi-expert decision making problems using penalty functions defined over a Cartesian product of lattices, Information Fusion, In Press, Corrected Proof,

Faithful restricted lattice dissimilarity functions:

$$\delta_R(x,y) = K(d(x,y)) = K(\sum_{i=1}^{m} |x_i - y_i|)$$

with $K : \mathcal{C} \to \mathcal{C}$ a convex function with a unique minimum at $K(0) = 0$.

Construction method of PFCPL

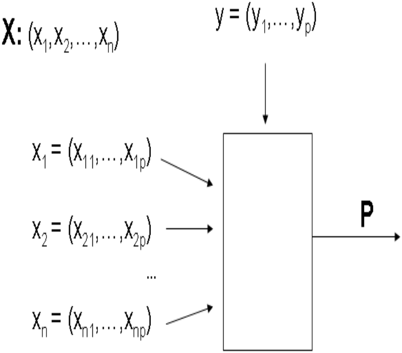
B_{y_q} is the fuzzy set over U such that all memberships are equal to $y_q \in [0,1]$; that is, $B_{y_q}(u_i) = y_q \in [0,1]$ for every $u_i \in U$.

Let $Y = (y_1, \cdots, y_m)$ and $\mathbf{B}_Y = (B_{y_1}, \cdots, B_{y_m}) \in \mathcal{FS}(U)^m$.

we denote by $\mathcal{C}^*$ a finite chain of elements in $[0,1]$ and

$$\mathcal{L}_m^* = \mathcal{C}^* \times \cdots \times \mathcal{C}^*$$

Penalty functions on a Cartesian product of lattices



Construction method of PFCPL

Theorem

Let $K_i : \mathbb{R} \to \mathbb{R}^+$ be convex functions with a unique minimum at $K_i(0) = 0$ $(i = 1, \cdots, m)$, and consider the distance between fuzzy sets given by:

$$\mathcal{D}(A, B) = \sum_{i=1}^{n} |A(u_i) - B(u_i)|$$

where $A, B \in \mathcal{FS}(U)$ and $Cardinal(U) = n$. Then the mapping

$P_\nabla : \mathcal{FS}(U)^m \times \mathcal{L}_m^* \to \mathbb{R}^+$ *given by*

$$P_\nabla(\mathbf{A}, Y) = \mathcal{D}(\mathbf{A}, \mathbf{B}_Y) = \sum_{q=1}^{m} K_q(\mathcal{D}(A_q, B_{y_q})) = \sum_{q=1}^{m} K_q\left(\sum_{p=1}^{n} |A_q(u_p) - y_q|\right)$$

satisfies

1. $P_\nabla(\mathbf{A}, Y) \geq 0$;
2. $P_\nabla(\mathbf{A}, Y) = 0$ *iff* $A_q = y_q$ *for every* $q = 1, \cdots, m$;
3. *is convex in* y_q *for each* $q = 1, \cdots, m$.

- H. Bustince, E. Barrenechea, T. Calvo, S. James, G. Beliakov, Consensus in multi-expert decision making problems using penalty functions defined over a Cartesian product of lattices, Information Fusion, In Press, Corrected Proof,

Construction method of PFCPL

Note that P_∇ is a penalty function over the Cartesian product of lattices $\mathcal{L}_m^{*n+1}$.

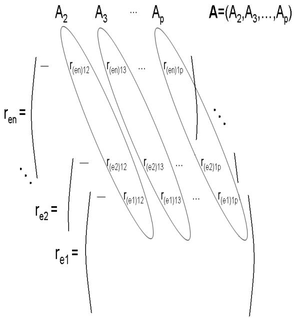
Example

- *If we take* $K_q(x) = x^2$ *for every* $q \in \{1, \cdots, m)$, *then*

$$\boxed{P_\nabla(\mathbf{A}, Y) = \sum_{q=1}^{m} \left(\sum_{p=1}^{n} |A_q(u_p) - y_q| \right)^2}$$

- *If* $K_q(x) = x$ *for every* $q \in \{1, \cdots, m)$, *then*

$$P_\nabla(\mathbf{A}, Y) = \sum_{q=1}^{m} \sum_{p=1}^{n} |A_q(u_p) - y_q|$$

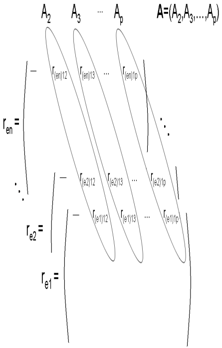


Take q $(q \leq p-1)$ aggregation functions $M_1, \cdots, M_q$.
Let $\mathbb{P}_q$ be the set of variations with repetition of the q aggregation functions taken in groups of $p-1$ elements.
we denote:

$$\mathbf{M}_{\sigma(i)} = \{M_{(\sigma(i),1)}, \cdots, M_{(\sigma(i),p-1)}\}$$

where $M_{(\sigma(i),j)}$ represents the j-th element of $\mathbf{M}_{\sigma(i)}$

GDM. Algorithm



(A) Select P_{∇}.For instance:

$$P_{\nabla}(\mathbf{A}, Y) = \sum_{q=1}^{m} \left(\sum_{p=1}^{n} |A_q(u_p) - y_q| \right)^2$$

(B) For the first row take:

$$\left(r_{((e1)12)}, \cdots, r_{((en)12)}\right), \cdots, \left(r_{((e1)1p)}, \cdots, r_{((en)1p)}\right)$$

where each of these tuples is an element in the Cartesian product of n copies of the considered chain.
(B1) we take the first element of $\mathbb{P}_q$,
$\mathbf{M}_{\sigma(1)} = \{M_{(\sigma(1),1)}, \cdots, M_{(\sigma(1),p-1)}\}$ and calculate:

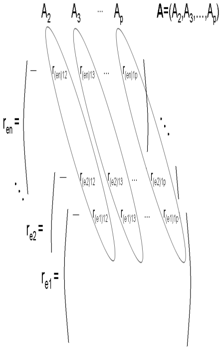
$$M_{(\sigma(1),1)}\left(r_{((e1)12)}, \cdots, r_{((en)12)}\right) = y_{((\sigma(1),1),12)},$$

$$M_{(\sigma(1),2)}\left(r_{((e1)13)}, \cdots, r_{((en)13)}\right) = y_{((\sigma(1),2),13)},$$

$$\cdots$$

$$M_{(\sigma(1),p-1)}\left(r_{((e1)1p)}, \cdots, r_{((en)1p)}\right) = y_{((\sigma(1),p-1),1p)}.$$

GDM. Algorithm



(B2) we calculate::

$$P_{\nabla}(\mathbf{A}, Y) = \sum_{q=1}^{m} \left(\sum_{p=1}^{n} |A_q(u_p) - y_q| \right)^2$$

(B3) We repeat step (B2) for the other elements of $\mathbb{P}_q$.
(B4) we take as solution the element of $\mathbb{P}_q$ which minimizes:

$$P_{\nabla}(\mathbf{A}, Y) = \sum_{q=1}^{m} \left(\sum_{p=1}^{n} |A_q(u_p) - y_q| \right)^2$$

we denote this element by $M_1\sigma(*)$, where 1 means that we are in the first row.

$$r^c = \left(\begin{array}{ccc} - & M_{1(\sigma(*),1)}(r_{((e1)12)}, \cdots, r_{((en)12)}) \cdots\cdots & M_{1(\sigma(*),p-1)}(r_{((e1)1p} \\ & - & \\ & & - \end{array} \right.$$

(C) we repeat step (B) for every row.

GDM. Algorithm

Example

A diagnosis of hypertension has been confirmed to a 45 years old patient. The corresponding higienic-dietetical indications have been given to him, and eventually the doctors have decided to start a pharmacological treatment.
From the following set of alternatives (pharmacological groups):

{*Beta-blockers*, *calcium antagonists*, *diuretical*, *IECA*, *Ara II*, *Alfablockers*}

four doctors have been asked for providing their corresponding preference matrix of one medicament over another for the considered patient. The proposed normalized preference relations are:

Example

$$e_1 = \begin{pmatrix} - & 0{,}8000 & 0{,}8000 & 0{,}5000 & 0{,}5000 & 1 \\ 0{,}2000 & - & 0{,}4000 & 0{,}4167 & 0{,}4167 & 1 \\ 0{,}2000 & 0{,}6000 & - & 0{,}5000 & 0{,}5000 & 1 \\ 0{,}5000 & 0{,}5833 & 0{,}5000 & - & 0{,}5000 & 1 \\ 0{,}5000 & 0{,}5833 & 0{,}5000 & 0{,}5000 & - & 1 \\ 0{,}0000 & 0{,}0000 & 0{,}0000 & 0{,}0000 & 0{,}0000 & - \end{pmatrix} \quad e_2 = \begin{pmatrix} - & 0{,}8000 & 0{,}3636 & 0{,}5000 & 0{,}4000 & 0{,}7500 \\ 0{,}2000 & - & 0{,}5263 & 0{,}2500 & 0{,}3333 & 0{,}8182 \\ 0{,}6364 & 0{,}4737 & - & 0{,}6000 & 0{,}6000 & 0{,}5000 \\ 0{,}5000 & 0{,}7500 & 0{,}4000 & - & 0{,}6923 & 0{,}5000 \\ 0{,}6000 & 0{,}6667 & 0{,}4000 & 0{,}3077 & - & 1 \\ 0{,}2500 & 0{,}1818 & 0{,}5000 & 0{,}5000 & 0{,}0000 & - \end{pmatrix}$$

$$e_3 = \begin{pmatrix} - & 0{,}3636 & 0{,}5000 & 0{,}0000 & 0{,}2727 & 0{,}7000 \\ 0{,}6364 & - & 0{,}5833 & 0{,}0000 & 0{,}2727 & 0{,}7692 \\ 0{,}5000 & 0{,}4167 & - & 0{,}0000 & 0{,}2000 & 0{,}5833 \\ 1 & 1 & 1 & - & 1 & 1 \\ 0{,}7273 & 0{,}7273 & 0{,}8000 & 0{,}0000 & - & 0{,}7273 \\ 0{,}3000 & 0{,}2308 & 0{,}4167 & 0{,}0000 & 0{,}2727 & - \end{pmatrix} \quad e_4 = \begin{pmatrix} - & 0{,}7000 & 0{,}8000 & 0{,}5000 & 0{,}2000 & 1 \\ 0{,}3000 & - & 0{,}4000 & 0{,}3320 & 0{,}4167 & 1 \\ 0{,}2000 & 0{,}6000 & - & 0{,}4000 & 0{,}6000 & 1 \\ 0{,}5000 & 0{,}6680 & 0{,}6000 & - & 0{,}5000 & 1 \\ 0{,}8000 & 0{,}5833 & 0{,}4000 & 0{,}5000 & - & 1 \\ 0{,}0000 & 0{,}0000 & 0{,}0000 & 0{,}0000 & 0{,}0000 & - \end{pmatrix}$$

Example

We take $K_{kj}(x) = x^2$ *for every* k, j *and for every* $x \in [0, 1]$ *and*

M_1 : *arithmetic mean* $M_1(x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i$;

M_2 : *OWA operator associated to the quantifier at least one half with the pair* $(a = 0, b = 0{,}5)$ *whose weighing vector is:* $w = (0{,}5, 0{,}5, 0, 0)$;

M_3 : *OWA operator associated to the quantifier the largest possible amount with the pair* $(a = 0{,}5, b = 1)$ *whose weighing vector is:* $w = (0, 0, 0{,}5, 0{,}5)$;

M_4 : *OWA operator associated to the quantifier most of with the pair* $(a = 0{,}3, b = 0{,}8)$ *whose weighing vector is:* $w = (0, 0{,}4, 0{,}5, 1)$;

M_5 : *minimum;*

M_6 : *geometric mean.*

GDM. Algorithm

Example

In this setting, the collective preference relation is:

$$r^c = \begin{pmatrix} - & 0{,}8000 & 0{,}6159 & 0{,}5000 & 0{,}3432 & 0{,}8625 \\ 0{,}2000 & - & 0{,}4774 & 0{,}2578 & 0{,}3599 & 0{,}8969 \\ 0{,}3841 & 0{,}5226 & - & 0{,}4000 & 0{,}6000 & 0{,}7708 \\ 0{,}5000 & 0{,}6923 & 0{,}5300 & - & 0{,}6731 & 1 \\ 0{,}6568 & 0{,}6402 & 0{,}4000 & 0{,}3269 & - & 1 \\ 0{,}1375 & 0{,}0000 & 0{,}1031 & 0{,}2292 & 0{,}0000 & - \end{pmatrix}$$

which corresponds to the following choice of aggregation functions:

$$\begin{pmatrix} - & M_2 & M_1 & M_2 & M_1 & M_1 \\ M_3 & - & M_1 & M_4 & M_1 & M_1 \\ M_1 & M_1 & - & M_4 & M_2 & M_1 \\ M_3 & M_4 & M_4 & - & M_1 & M_2 \\ M_1 & M_1 & M_3 & M_1 & - & M_2 \\ M_1 & M_1 & M_1 & M_3 & M_3 & - \end{pmatrix}$$

Exploitation phase: voting method

$$x_{best} = \arg \max_{i=1,\cdots,p} \sum_{1 \leq j \neq i \leq p} r^c_{ij}$$

$$x_1 \to 0{,}8000 + 0{,}6159 + 0{,}5000 + 0{,}3432 + 0{,}8625 = 3{,}1216$$
$$x_2 \to 0{,}2000 + 0{,}4774 + 0{,}2578 + 0{,}3599 + 0{,}8969 = 2{,}192$$
$$x_3 \to 0{,}3841 + 0{,}5226 + 0{,}4000 + 0{,}6000 + 0{,}7708 = 2{,}6775$$
$$x_4 \to 0{,}5000 + 0{,}6923 + 0{,}5300 + 0{,}6731 + 1{,}0000 = 3{,}3954$$
$$x_5 \to 0{,}6568 + 0{,}6402 + 0{,}4000 + 0{,}3269 + 1{,}0000 = 3{,}0239$$
$$x_6 \to 0{,}1375 + 0{,}0000 + 0{,}1031 + 0{,}2292 + 0{,}0000 = 0{,}4698$$

In our example, the solution alternative is x_4; that is, IECA

- Bustince, H.; Jurio, A.; Pradera, A.; Mesiar, R.; Beliakov, G., Generalization of the weighted voting method using penalty functions constructed via faithful restricted dissimilarity functions, European Journal of Operational Research, 225(3), (2013), 472-478

GDM. And when it does not work?

If the preference matrices have all their entries close to 0.5, we have that:

(1) The penalty function selects the same aggregation function in every case.

(2) Experts find difficulties to select one preference over another. That is, we are working with imperfect information.

In this setting, Zadeh suggests the use of extensions of fuzzy sets.

$$\begin{pmatrix} - & 0{,}59 & 0{,}61 & 0{,}48 & 0{,}41 & 0{,}49 \\ 0{,}41 & - & 0{,}59 & 0{,}49 & 0{,}61 & 0{,}48 \\ 0{,}39 & 0{,}41 & - & 0{,}40 & 0{,}55 & 0{,}50 \\ 0{,}52 & 0{,}51 & 0{,}60 & - & 0{,}37 & 0{,}58 \\ 0{,}59 & 0{,}39 & 0{,}45 & 0{,}63 & - & 0{,}37 \\ 0{,}51 & 0{,}52 & 0{,}50 & 0{,}42 & 0{,}63 & - \end{pmatrix}$$

Fuzzy sets and L-fuzzy sets

Definition

A fuzzy set A over a referential set U is an object:

$$A = \{(u_i, A(u_i) = \mu_A(u_i)) | u_i \in U\}$$

where $A = \mu_A : U \longrightarrow [0,1]$.

$$A \cup B(u_i) = max(A(u_i), B(u_i))$$
$$A \cap B(u_i) = min(A(u_i), B(u_i))$$

FS

$(FS(U), \cup, \cap)$ is a complete lattice

Definition

Let $(L, \vee, \wedge)$ be a complete lattice. A L-fuzzy set over the referential set U is a mapping $A : U \to L$

$$A \cup B(u_i) = \vee(A(u_i), B(u_i))$$
$$A \cap B(u_i) = \wedge(A(u_i), B(u_i))$$

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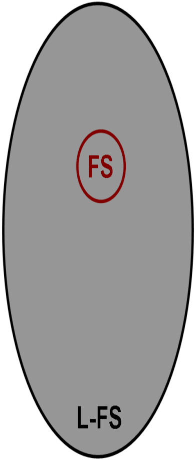

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Let $(L, \vee, \wedge)$ *be a complete lattice. A* L*-fuzzy set over the referential set* U *is a mapping*

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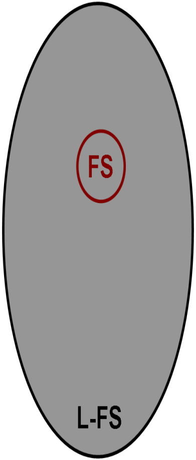
Origin of the extensions

- LA Zadeh, Fuzzy Algorithms, Information and Control, 12(2), (1968)
- M. Nasu, N. Honda, Fuzzy Events Realized by Finite Probabilistic Automata, Information and Control, 12(4), (1969)
- M. Mizumoto, J. Toyoda, K. Tanaka, Some Considerations on Fuzzy Automata, Electronics and Communications in Japan, 52(7), (1969)
- PN, Marinos, Fuzzy Logic and its Application to Switching Systems, IEEE Transactions on Computers, 18(4), (1969)
- ...

- L. A. Zadeh, Quantitative fuzzy semantics, Inform. Sci. 3 (1971) 159-176

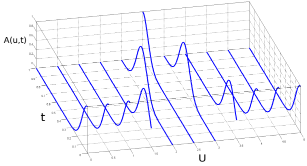
Type-2 fuzzy sets



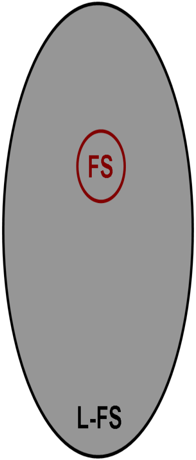
- L. A. Zadeh, Quantitative fuzzy semantics, Inform. Sci. 3 (1971) 159-176

Definition

A type-2 fuzzy set is a mapping:
$$A : U \to FS([0,1])$$



- Type-2 fuzzy sets are a particular case of L-fuzzy sets.
- $T2FS(U) \equiv (FS([0,1]))^U$



Problems:

1. Notation
   - M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type 2, Inform. Control, 31, (1976), 312-340
   - J.M. Mendel, R. John, Type-2 Fuzzy Sets Made Simple, IEEE Transactions on Fuzzy Systems 10(2) (2002) 117–127

$$\int_{u\in U}\int_{t\in J_u} A(u,t)/(u,t) \qquad J_u \subset [0,1]$$

Definition

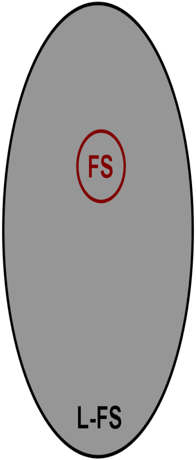
Let $A : U \to FS([0,1])$ be a type 2 fuzzy set. Then A is denoted as

$$\{(u_i, A(u_i,t)) \mid u_i \in U \ ,\ t \in [0,1]\}\ .$$

where $A(u_i,\cdot) : [0,1] \to [0,1]$ is defined as

$$A(u_i,t) = A(u_i)(t)$$

Type-2 fuzzy sets



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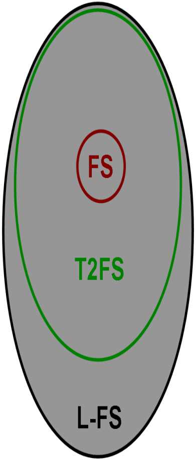


2. Structure
    - M. Mizumoto, K. Tanaka, Some properties of fuzzy sets of type 2, Inform. Control, 31, (1976), 312-340
    - D. Dubois, H. Prade, Operations in a fuzzy-valued logic, Inform. Control, 43(2), (1979) 224-254

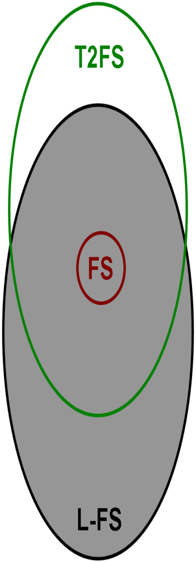
Definition

$$A \cup_{T2} B(u_i) = A(u_i) \cup B(u_i)$$
$$A \cap_{T2} B(u_i) = A(u_i) \cap B(u_i)$$

Proposition

$(T2FS(U), \cup_{T2}, \cap_{T2})$ is a bounded lattice with respect to the order: $A \leq_{T2FS(U)} B$ if and only if $A \cup_{T2} B = B$

Type-2 fuzzy sets



Definition

$$A = \{(u_i, A(u_i, t)) \mid u_i \in U \ , \ t \in [0,1]\}$$
$$B = \{(u_i, B(u_i, t)) \mid u_i \in U \ , \ t \in [0,1]\}$$

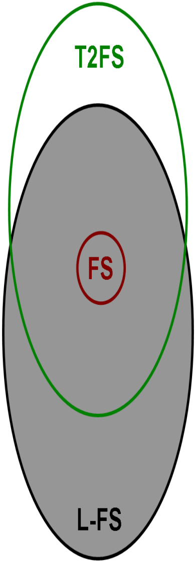
$$\bullet A \sqcap B = \{(u_i, A \sqcap B(u_i, t)) \mid u_i \in U \ , \ t \in [0,1]\}$$
$$A \sqcap B(u_i, t) = \sup_{min(z,w)=t} min(A(u_i, z), B(u_i, w))$$

$$\bullet A \sqcup B = \{(u_i, A \sqcup B(u_i, t)) \mid u_i \in U \ , \ t \in [0,1]\}$$
$$A \sqcup B(u_i, t) = \sup_{max(z,w)=t} min(A(u_i, z), B(u_i, w))$$

$(T2FS(U), \sqcup, \sqcap)$ **is NOT a lattice**

- C. Walker, E. Walker, Type-2 operations on finite chains, Fuzzy Sets and Systems, In Press (2013)

Type-2 fuzzy sets

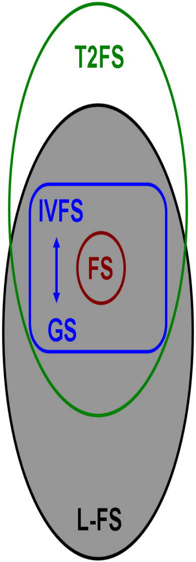


3. Computational efficiency: regression to infinity
4. Applications

   It does not exist yet an application that shows the advantage of using these sets.

   1. **Computing with words**:
      - J.M. Mendel Type-2 fuzzy sets for computing with words Conference Information: IEEE International Conference on Granular Computing, MAY 10-12, 2006 Atlanta, (2006) GA 8-8.
      - J.M. Mendel, Computing with words and its relationships with fuzzistics Information sciences 177(4) (2007) 988-1006
   2. **Perceptual computing**: JM Mendel,
   3. **Control**:
      - H. Hagras, A Hierarchical Type-2 Fuzzy Logic Control Architecture for Autonomous Mobile Robots, IEEE Transactions on Fuzzy Systems 12, (2004) 524–539.

Interval-valued fuzzy sets



Definition

An interval-valued fuzzy set is a mapping:

$$A : U \to L([0,1])$$

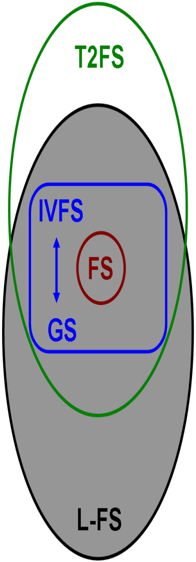
$A(u_i) = [\underline{A}(u_i), \overline{A}(u_i)]$ denotes the membership degree of u_i to A.

- They are a particular case of L-fuzzy sets
- $L([0,1]) = \{\mathbf{x} = [\underline{x}, \overline{x}] | (\underline{x}, \overline{x}) \in [0,1]^2$ and $\underline{x} \leq \overline{x}\}$

1. In 1975 Sambuc: Φ-flou
2. Name of interval-valued fuzzy sets, 80s (Gorzalczany and Turksen)

- J.L. Deng, Introduction to grey system theory, Journal of Grey Systems 1 (1989) 1–24

Interval-valued fuzzy sets



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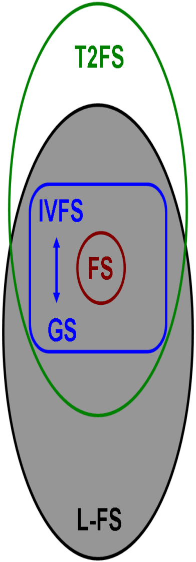
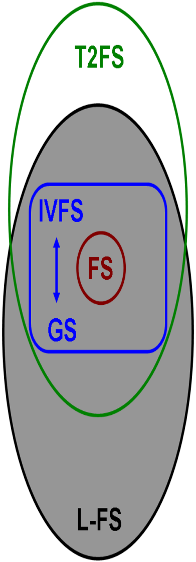


Definition

$$A \cup_{L([0,1])} B(u_i) = [max(\underline{A}(u_i), \underline{B}(u_i)), max(\overline{A}(u_i), \overline{B}(u_i))]$$

$$A \cap_{L([0,1])} B(u_i) = [min(\underline{A}(u_i), \underline{B}(u_i)), min(\overline{A}(u_i), \overline{B}(u_i))]$$

$(IVFS(U), \cup_{L([0,1])}, \cap_{L([0,1])})$ is a complete lattice



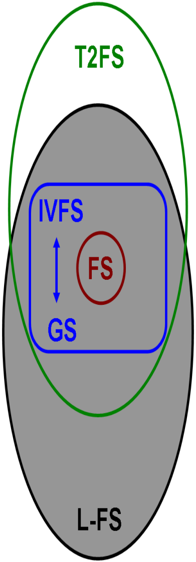
A.- Mathematical interpretation. Theoretical interest.

D. Dubois' paradox:

$$min(A(u_i), 1 - A(u_i)) \leq 0{,}5$$
$$min([\underline{A}(u_i), \overline{A}(u_i)], [1 - \overline{A}(u_i), 1 - \underline{A}(u_i)]) \leq ??$$

- H.Bustince, F.Herrera, J.Montero (Eds.), Fuzzy Sets and Their Extensions: Representation Aggregation and Models, Springer, Berlin, 2007.

B.- The expert does not know the exact value of the membership of the element to the fuzzy set. However, the expert knows that this value is bounded by the bounds of the interval-valued membership to the IVFS.



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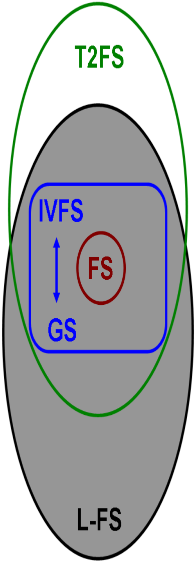
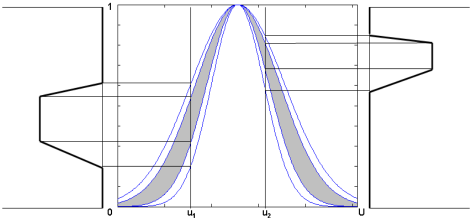


- G. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice Hall, Upper Saddle River, NJ, 1995.

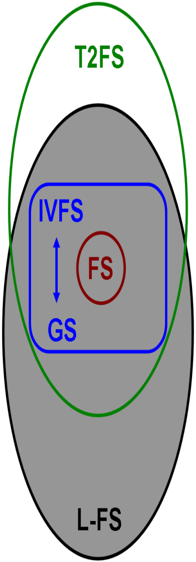


- G. Deschrijver, E.E. Kerre, On the position of intuitionistic fuzzy set theory in the framework of theories modeling imprecision, Information Sciences 177, (2007) 1860-1866
- J.M. Mendel, Advances in type-2 fuzzy sets and systems, Information Sciences 177, (2007) 84-110

2000, Name:

Interval type-2 fuzzy sets

Problems with interval-valued fuzzy sets



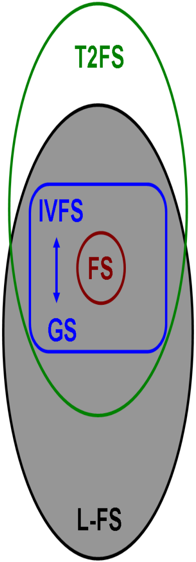
1.- Computational efficiency. Double information used. Numerical measures.

2.- A great number of works are a straight **adaptation** of developments already done for fuzzy sets without taking into account the specific characteristics of intervals.

It is not considered:

1. The length of the intervals and its meaning.
2. ORDER:the usual order between real numbers is linear.IT DOES NOT EXIST A NATURAL LINEAR ORDER BETWEEN INTERVALS.

Problems with interval-valued fuzzy sets



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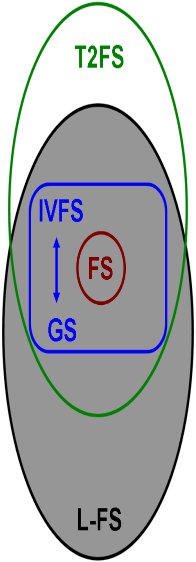

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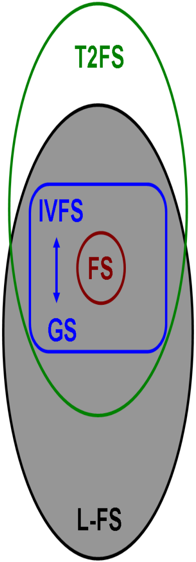

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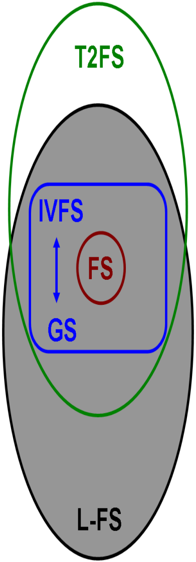
Admissible orders



Proposition

Let $A,\ B\colon [0,1]^2 \to [0,1]$ be two aggregation functions, such that for all $(x,y),\ (u,v) \in K([0,1])$, the equalities $A(x,y) = A(u,v)$ and $B(x,y) = B(u,v)$ can hold only if $(x,y) = (u,v)$. Define the relation $\preceq_{A,B}$ on $L([0,1])$ by

$[x,y] \preceq_{A,B} [u,v]$ if and only if

$$A(x,y) < A(u,v)$$
$$\text{or } A(x,y) = A(u,v) \text{ and } B(x,y) \leq B(u,v). \tag{1}$$

Then $\preceq_{A,B}$ is an admissible order on $L([0,1])$.

- Bustince, H., Fernandez, J., Kolesárová, A., Mesiar, R. Generation of linear orders for intervals by means of aggregation functions. *Fuzzy Sets and Systems*, 220, (2013), pp. 69-77.

Defining linear orders

The linear order $\preceq_{A,B}$ refines the standard partial order $\leq$ on intervals,

$$[x, y] \leq [u, v] \text{ whenever } x \leq u \text{ and } y \leq v$$

That is, $[x, y] \leq [u, v]$ implies $[x, y] \preceq_{A,B} [u, v]$ whichever A and B are.

And we can also recover some orders...

Defining linear orders: the lexicographical order

The lexicographical orders :

$$[x, y] \preceq_{P1} [u, v] \text{ whenever } x < u \text{ or } x = u \text{ and } y \leq v$$

and

$$[x, y] \preceq_{P2} [u, v] \text{ whenever } y < v \text{ or } y = v \text{ and } x \leq u$$

can be recovered by taking

- $A(x, y) = x$ and $B(x, y) = y$, in the first case, or
- $A(x, y) = y$ and $B(x, y) = x$ for the second case.

Defining linear orders: Yager-Xu's order

We can also recover the Yager-Xu's order:

$$[x, y] \preceq_{YX} [u, v]$$

whenever

- $x + y - 1 < u + v - 1$ (score) or
- $x + y - 1 = u + v - 1$ and $x - y + 1 \leq u - v + 1$ (accuracy)

since it can be seen as a linear order $\preceq_{M,G}$ on $L([0, 1])$, where

- *M* is the arithmetic mean and
- *G* is the geometric mean.

- Z.Xu and R.Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. General Syst, 35, (2006) 417-433

Definition

Let $\preceq$ *be an admissible order on* $L([0,1])$ *and* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$, $w_1 + \cdots + w_n = 1$, *a weighting vector. An interval-valued OWA operator associated with* $\preceq$ *and* $\mathbf{w}$ *is a mapping* $IVOWA_{\mathbf{w}}^{\preceq} \colon (L([0,1]))^n \to L([0,1])$ *defined by*

$$IVOWA_{\mathbf{w}}^{\preceq}([a_1, b_1], \ldots, [a_n, b_n]) = \sum_{i=1}^{n} w_i \cdot [a_{(i)}, b_{(i)}],$$

where $[a_{(i)}, b_{(i)}]$, $i = 1, \ldots, n$, *denotes the* ith *greatest interval of the input intervals with respect to the order* $\preceq$.

The above Definition extends the usual definition of OWA operators, as is shown in the next proposition.

Proposition

Let $\preceq$ *be an admissible order on* $L([0,1])$ *and let* $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ *with* $w_1 + \cdots + w_n = 1$ *be a weighting vector. Then*

$$OWA_{\mathbf{w}}(x_1, \ldots, x_n) = IVOWA_{\mathbf{w}}^{\preceq}([x_1, x_1], \ldots, [x_n, x_n]).$$

Interval-valued OWA operators

However, in general the representability of IVOWA operators in the form

$$IVOWA_{\mathbf{w}}^{\preceq}([a_1,b_1],\ldots,[a_n,b_n]) = [OWA_{\mathbf{w}}(a_1,\ldots,a_n), OWA_{\mathbf{w}}(b_1,\ldots,b_n)]$$

does not hold, as is shown in the following example.

Example

Consider the weighting vector $\mathbf{w} = (1,0,0)$ *and the lexicographical order* $\preceq_{Lex1}$. *For the intervals* $\left[\frac{1}{2},\frac{3}{4}\right]$, $\left[\frac{1}{3},\frac{1}{2}\right]$ *and* $\left[\frac{1}{3},1\right]$ *it holds*

$$\left[\frac{1}{3},\frac{1}{2}\right] \preceq_{Lex1} \left[\frac{1}{3},1\right] \preceq_{Lex1} \left[\frac{1}{2},\frac{3}{4}\right].$$

Therefore

$$IVOWA_{\mathbf{w}}^{\preceq_{Lex1}}\left(\left[\frac{1}{2},\frac{3}{4}\right],\left[\frac{1}{3},\frac{1}{2}\right],\left[\frac{1}{3},1\right]\right) = \left[\frac{1}{2},\frac{3}{4}\right],$$

and on the other hand,

$$\left[OWA_{\mathbf{w}}\left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right), OWA_{\mathbf{w}}\left(\frac{3}{4},\frac{1}{2},1\right)\right] = \left[\frac{1}{2},1\right].$$

Definition

Let $F : U \to L([0,1])$ *be an interval–valued fuzzy set, and* $m : 2^U \to [0,1]$ *a fuzzy measure. The (A, B)–Choquet integral* $\mathbf{C}_m^{A,B}(F)$ *is given by*

$$\mathbf{C}_m^{A,B}(F) = \sum_{i=1}^{n} F(u_{\sigma_{A,B}(i)})(m(\left\{u_{\sigma_{A,B}(i)}, \ldots, u_{\sigma_{A,B}(n)}\right\}) - m(\left\{u_{\sigma_{A,B}(i+1)}, \ldots, u_{\sigma_{A,B}(n)}\right\})),$$

where $\sigma_{A,B} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *is a permutation such that* $F(u_{\sigma_{A,B}(1)}) \preceq_{A,B} F(u_{\sigma_{A,B}(2)}) \preceq_{A,B} \cdots \preceq_{A,B} F(u_{\sigma_{A,B}(n)})$.

Remark

(i) *The concept of an interval–valued (A, B)–Choquet integral $\mathbf{C}_m^{A,B}$ extends the standard discrete Choquet integral. Indeed if $F : U \to L([0, 1])$ is singleton–valued, i.e., F is a fuzzy subset of U, then $C_m(F) = \mathbf{C}_m(F) = \mathbf{C}_m^{A,B}(F)$ independently of A, B.*

(ii) *For a fixed $F : U \to L([0, 1])$ such that f_* and f^* are comonotone, i.e., $(f_*(u_i) - f_*(u_j)) \cdot (f^*(u_i) - f^*(u_j)) \geq 0$ for all u_i, $u_j \in U$, for any A, B satisfying our constraints, $\mathbf{C}_m(F) = \mathbf{C}_m^{A,B}(F)$.*

(iii) *In general the integral $\mathbf{C}_m$ cannot be expressed in the form $\mathbf{C}_m^{A,B}$, since representability in terms of the bounds of the considered intervals is not assured.*

- For several couples $(A_1, B_1), (A_2, B_2), \ldots,$ the linear orders $\preceq_{A_1,B_1}$, $\preceq_{A_2,B_2}$, $\ldots,$ may coincide.

- Indeed, then $\mathbf{C}_m^{A_1,B_1} = \mathbf{C}_m^{A_2,B_2}$.

- For instance, we can take $\preceq_{Min,Max} \equiv \preceq_{P_1,P_2} \equiv \preceq_{P_1,B}$, where $P_1, P_2 : [0,1]^2 \to [0,1]$ are projections, $P_1(x,y) = x,\ P_2(x,y) = y$ and $B : [0,1]^2 \to [0,1]$ is an arbitrary cancellative (strictly monotone) aggregation function.

Let us consider the projections P_1 and P_2. The next relationship holds:

Proposition

Let a fuzzy set $F : U \to L([0,1])$ *and a fuzzy measure* $m : 2^U \to [0,1]$ *be fixed. Denote*

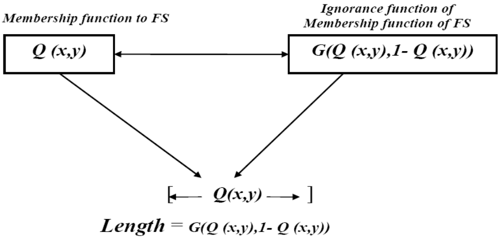
$$\mathbf{C}_m(F) = [\alpha, \beta],\ \mathbf{C}_m^{P_1,P_2}(F) = [a, b],\ \mathbf{C}_m^{P_2,P_1}(F) = [c, d]$$

Then $\alpha = a$ *and* $\beta = d$.

GDM with IVFS. Why?

$$e_i = \begin{pmatrix} - & 0{,}59 & 0{,}61 & 0{,}48 & 0{,}41 & 0{,}49 \\ 0{,}41 & - & 0{,}59 & 0{,}49 & 0{,}61 & 0{,}48 \\ 0{,}39 & 0{,}41 & - & 0{,}40 & 0{,}55 & 0{,}50 \\ 0{,}52 & 0{,}51 & 0{,}60 & - & 0{,}37 & 0{,}58 \\ 0{,}59 & 0{,}39 & 0{,}45 & 0{,}63 & - & 0{,}37 \\ 0{,}51 & 0{,}52 & 0{,}50 & 0{,}42 & 0{,}63 & - \end{pmatrix}$$

Construction of intervals: Ignorance functions



Definition

An ignorance function is a continuous function $G_i : [0, 1]^2 \to [0, 1]$ *such that:*

$G_i1)$ $G_i(x, y) = G_i(y, x)$ *for every* $x, y \in [0, 1]$;

$G_i2)$ $G_i(x, y) = 0$ *if and only if* $x = 1$ *or* $y = 1$;

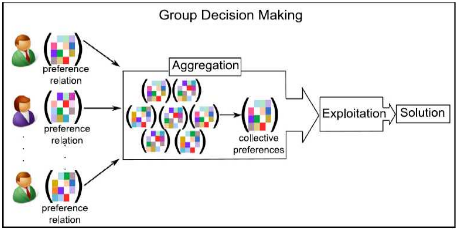
$G_i3)$ *If* $x = 0{,}5$ *and* $y = 0{,}5$, *then* $G_i(x, y) = 1$;

$G_i4)$ G_i *is decreasing in* $[0{,}5, 1]^2$;

$G_i5)$ G_i *is increasing in* $[0, 0{,}5]^2$.

- H. Bustince, M. Pagola, E. Barrenechea, J. Fernandez, P. Melo-Pinto, P. Couto, H.R. Tizhoosh, J. Montero, Ignorance functions. An application to the calculation of the threshold in prostate ultrasound images, Fuzzy Sets and Systems, 161(1) 2010, 20-36

GDM with IVFS



We have n experts and p alternatives, so we have n interval-valued preference matrices:

$$e_{IV_i} = \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & \cdots, & \cdots, & x_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \\ x_p \end{array} & \begin{pmatrix} - & [\underline{e}_{i_{12}}, \overline{e}_{i_{12}}] & [\underline{e}_{i_{13}}, \overline{e}_{i_{13}}] & \cdots, & [\underline{e}_{i_{1p}}, \overline{e}_{i_{1p}}] \\ [\underline{e}_{i_{21}}, \overline{e}_{i_{21}}] & - & [\underline{e}_{i_{23}}, \overline{e}_{i_{23}}] & \cdots, & [\underline{e}_{i_{2p}}, \overline{e}_{i_{2p}}] \\ \cdots & \cdots & \cdots & - & \cdots \\ [\underline{e}_{i_{p1}}, \overline{e}_{i_{p1}}] & [\underline{e}_{i_{p2}}, \overline{e}_{i_{p2}}] & \cdots & \cdots & - \end{pmatrix} \end{array}$$

GDM with IVFS

Aggregation phase:
PROBLEM: we do not know how to define interval-valued penalty functions

1.- Apply the previous algorithm to the lower bounds, using penalty functions;
2.- Apply the previous algorithm to the upper bounds, using penalty functions and the same aggregation functions;
    2.1- If for some position we don't recover an interval, delete the chosen aggregation and repeat the algorithm until we get an interval.

$$r^c = \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & \cdots, & \cdots, & x_p \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ \\ x_p \end{array} & \begin{pmatrix} - & [\underline{r}_{i_{12}}, \overline{r}_{i_{12}}] & [\underline{r}_{i_{13}}, \overline{r}_{i_{13}}] & \cdots, & [\underline{r}_{i_{1p}}, \overline{r}_{i_{1p}}] \\ [\underline{r}_{i_{21}}, \overline{r}_{i_{21}}] & - & [\underline{r}_{i_{23}}, \overline{r}_{i_{23}}] & \cdots, & [\underline{r}_{i_{2p}}, \overline{r}_{i_{2p}}] \\ \cdots & \cdots & \cdots & - & \cdots \\ [\underline{r}_{i_{p1}}, \overline{r}_{i_{p1}}] & [\underline{r}_{i_{p2}}, \overline{r}_{i_{p2}}] & \cdots & \cdots & - \end{pmatrix} \end{array}$$

Exploitation phase:

1.-Select an interval-valued aggregation function Ag to aggregate the elements of each row

$$
\begin{array}{lllll}
Ag\Big([\underline{r}_{i_{12}}, \overline{r}_{i_{12}}] & [\underline{r}_{i_{13}}, \overline{r}_{i_{13}}] & \cdots, & [\underline{r}_{i_{1p}}, \overline{r}_{i_{1p}}] \Big) & = [\underline{r}_1, \overline{r}_1] \\
Ag\Big([\underline{r}_{i_{21}}, \overline{r}_{i_{21}}] & [\underline{r}_{i_{23}}, \overline{r}_{i_{23}}] & \cdots, & [\underline{r}_{i_{2p}}, \overline{r}_{i_{2p}}] \Big) & = [\underline{r}_2, \overline{r}_2] \\
\cdots & & \cdots\cdots & & \cdots \\
Ag\Big([\underline{r}_{i_{p1}}, \overline{r}_{i_{p1}}] & [\underline{r}_{i_{p3}}, \overline{r}_{i_{p3}}] & \cdots, & [\underline{r}_{i_{pp}}, \overline{r}_{i_{pp}}] \Big) & = [\underline{r}_p, \overline{r}_p]
\end{array}
$$

2.- Select a **linear order** and take the biggest as the solution alternative:

$$x_{best} = \arg \max_{1 \leq i \leq p} \left([\underline{r}_i, \overline{r}_i] \right)$$

Bustince, H.; Jurio, A.; Pradera, A.; Mesiar, R.; Beliakov, G., Generalization of the weighted voting method using penalty functions constructed via faithful restricted dissimilarity functions, European Journal of Operational Research, 225(3), (2013), 472-478

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\cdots & \cdots\cdots & & \cdots \\
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PROBLEM: WHICH ORDER DO WE CHOOSE?

- If the application determines the order, then we are done.
- Otherwise, we can choose different linear orders:
  - If with every linear order we get the same solution alternative, we are done.
  - Otherwise, use game theory to measure the relevance of each possible winning alternative in coalitions with the others.

Bustince, H. ; Galar, M. ; Bedregal, B. ; Kolesárová, A. ; Mesiar, R., A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy sets applications , IEEE Transactions on Fuzzy Systems, In Press

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Conclusions

- Choice of the aggregation using penalty functions.
- Methods to build penalty functions on a Cartesian product on lattices.
- Algorithm for the aggregation phase.
- Conditions under which we must use IVFSs.
- Admissible orders for intervals.
- Interval-valued OWA operators.
- Interval-valued Choquet integrals.
- Algorithm in GDM using IVFSs and penalty functions.

Thanks for your attention