E-Fuzzy Groups

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Abstract

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An E-fuzzy group is a lattice-valued algebraic structure, defined on a crisp algebra which is not necessarily a group. The crisp equality is replaced by a particular fuzzy one - denoted by E. Classical group-like properties are formulated as appropriate fuzzy identities - special lattice theoretic formulas. We prove basic features of E-fuzzy groups: properties of the unit and inverses, cancellability, solvability of equations, subgroup properties and others. We also prove that for every cut of an E-fuzzy group, which is a classical subalgebra of the underlying algebra, the quotient structure over the corresponding cut of the fuzzy equality is a classical group.

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- Fuzzy structures and general algebra: Di Nola, Gerla 1987; Kuraoka, Suzuki 2002; Bělohlávek, Vychodil 2002; 2005; 2006.

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- Previous research: Šešelja, Tepavčević 1992; 1993; 1994; 1996; 1997; 2009; Budimirović, Šešelja, Tepavčević 2010; 2013.

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An **algebra** of type $\mathcal{L}$ – $\mathcal{A} = (A, F)$; A is a nonempty set and F is a set of fundamental operations on A.
We use the notions of a **subalgebra**, **term**, **identity**, **congruence relation** on $\mathcal{A}$.

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If $\{\mu_i \mid i \in I\}$ is a family of fuzzy sets on the same domain A, then the **fuzzy intersection** $\mu = \bigcap\{\mu_i \mid i \in I\}$ of this family is a fuzzy set on A, defined by

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A **fuzzy** (binary) **relation** ρ **on** A is a fuzzy set on A^2, i.e., it is a mapping $\rho : A^2 \to L$.

Fuzzy relations on fuzzy sets

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Let $\mu : A \to L$ be a fuzzy set on A and let $\rho : A^2 \to L$ be a fuzzy relation on A. If for all $x, y \in A$, ρ satisfies

$$\rho(x, y) \leqslant \mu(x) \wedge \mu(y),$$

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$$\rho \text{ is } \textbf{reflexive} \text{ if } \rho(x, x) = \mu(x) \text{ for every } x \in A.$$

$$\rho \text{ is } \textbf{symmetric} \text{ if } \ \rho(x, y) = \rho(y, x) \ \text{ for all } \ x, y \in A;$$

$$\rho \text{ is } \textbf{transitive} \text{ if } \rho(x, y) \geqslant \rho(x, z) \wedge \rho(z, y) \text{ for all } \ x, y, z \in A.$$

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A fuzzy equivalence relation ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$,:

$$\text{if } \rho(x,x) \neq 0, \text{ then } \rho(x,x) > \rho(x,y),$$

is called a **fuzzy equality** relation on a fuzzy set μ.

Preliminaries

If $\mathcal{A} = (A, F)$ is an algebra, then a **fuzzy subalgebra** of $\mathcal{A}$ is any mapping $\mu : A \to L$ which is not constantly equal to 0, and which fulfils the following:
For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)),$$

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In particular, if $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is a group, then $\mu : G \to L$ is known to be a **fuzzy subgroup** of $\mathcal{G}$ if for all $x, y \in G$,

$$\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y)\ , \quad \mu(x^{-1}) \geqslant \mu(x), \text{ and } \mu(e) = 1.$$

Let $\mathcal{A} = (A, F)$ be an algebra. A fuzzy relation $\rho : A^2 \to L$ is **compatible** with the operations in F if the following holds: for every n-ary operation $f \in F$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$

$$\bigwedge_{i=1}^{n} \rho(a_i, b_i) \leqslant \rho(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)), \text{ and}$$

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A compatible fuzzy equivalence on μ is a **fuzzy congruence** on this fuzzy subalgebra.
A **fuzzy equality on a fuzzy subalgebra** μ is a fuzzy congruence on μ, such that

$$\rho(x, x) \neq 0 \text{ implies } \rho(x, x) > \rho(x, y).$$

Fuzzy identity

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If $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$ are terms in the language of an algebra $\mathcal{A}$, where variables appearing in these terms are among $x_1, \ldots, x_n$, we say that the expression

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Then, a fuzzy subalgebra μ of $\mathcal{A}$ **satisfies a fuzzy identity** $E(u,v)$ **with respect to fuzzy equality** E^μ on μ, if the following condition is fulfilled for all $a_1,\ldots,a_n \in A$ and the term-operations u^A and v^A on $\mathcal{A}$ corresponding to terms u and v respectively:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E^\mu(u^A(a_1,\ldots,a_n),v^A(a_1,\ldots,a_n)).$$

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Proposition

Let $u = v$ be an identity which holds on an algebra $\mathcal{A}$. If $\mu : A \to L$ is a fuzzy subalgebra on $\mathcal{A}$, and E^{μ} a fuzzy equality on μ, then the fuzzy identity $E(u, v)$ is satisfied on μ with respect to E^{μ}.

E-fuzzy algebra

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Let

$$\bar{\mathcal{A}} = (\mathcal{A}, \mu,\ E^{\mu})$$

be a structure in which $\mathcal{A} = (A, F)$ is an algebra with a set F of operations, $\mu : A \to L$ is a fuzzy subalgebra of $\mathcal{A}$, $E^{\mu} : A^2 \to L$ is a fuzzy equality on μ. Then, we say that $\bar{\mathcal{A}}$ is an ***E*-fuzzy algebra**.
If, in addition, $\mathcal{F}$ is a collection of fuzzy identities, and every fuzzy identity from $\mathcal{F}$ is valid on μ with respect to E^{μ}, then we say that $\bar{\mathcal{A}}$ satisfies all fuzzy identities from $\mathcal{F}$.

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In particular, here we deal with E-fuzzy algebras of the form $\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E^{\mu})$, where $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e), $\mu : G \to L$ is a fuzzy subalgebra of $\mathcal{G}$, and $E^{\mu} : G^2 \to L$ is a fuzzy equality on μ.

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$$E(x \cdot (y \cdot z), (x \cdot y) \cdot z);$$
$$E(x \cdot e, x), \quad E(e \cdot x, x);$$
$$E(x \cdot x^{-1}, e), \quad E(x^{-1} \cdot x, e);$$

i.e., associativity, and properties of neutral and inverse elements, respectively.

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i.e., associativity, and properties of neutral and inverse elements, respectively.

Element e is said to be the **unit** in $\bar{\mathcal{G}}$, and x^{-1} is the **inverse** of element x in $\bar{\mathcal{G}}$. We also say that $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is the **underlying algebra** of E-fuzzy group $\bar{\mathcal{G}}$.

According to the definitions, the fact that μ is a fuzzy subalgebra of $\mathcal{G}$ means that for all $x, y \in G$

- $\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y)$,
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In addition, the requirement that $\bar{\mathcal{G}}$ fulfills the listed group-like fuzzy identities, means that for all x, y, z from G,

(i) $E^{\mu}(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$,

(ii) $E^{\mu}(x \cdot e, x) \geqslant \mu(x)$ and $E^{\mu}(e \cdot x, x) \geqslant \mu(x)$,

(iii) $E^{\mu}(x \cdot x^{-1}, e) \geqslant \mu(x)$ and $E^{\mu}(x^{-1} \cdot x, e) \geqslant \mu(x)$.

Theorem

Let $\bar{\mathcal{G}}' = (\mathcal{G}', \mu,\ E^{\mu})$ *be a fuzzy algebra described above, fulfilling the following:*

(i') $E^{\mu}(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$,

(ii') $E^{\mu}(x \cdot e', x) \geqslant \mu(x)$,

(iii') $E^{\mu}(x \cdot x', e') \geqslant \mu(x)$,

for all x, y, z *from* G. *Then,* $\bar{\mathcal{G}}'$ *is an* E*-fuzzy group.*

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a	a	c	b	a
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E^{μ}	e	a	b	c
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b	0.3	0.3	0.5	0.3
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μ is a fuzzy subalgebra of $\mathcal{G}$ and E^{μ} is a fuzzy equality on μ.

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μ is a fuzzy subalgebra of $\mathcal{G}$ and E^{μ} is a fuzzy equality on μ.
$\bar{\mathcal{G}} = (\mathcal{G}, \mu, \ E^{\mu})$ is an E-fuzzy group.

Theorem

Let $\mathcal{G} = (G, \cdot, ^{-1}, e)$ be a group, $\mu : G \to L$ its fuzzy subgroup, and E^{μ} a fuzzy equality on μ. Then, $\bar{\mathcal{G}} = (\mathcal{G}, \mu, \ E^{\mu})$ is an E-fuzzy group.

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Let $\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E^{\mu})$ be an E-fuzzy group, and $x \in G$ such that $\mu(x) \neq 0$. Then $x \cdot e = e \cdot x = x$, where e is a unit in $\bar{\mathcal{G}}$.

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The unit e of an E-fuzzy group $\bar{\mathcal{G}} = (\mathcal{G}, \mu, \ E^{\mu})$ is a unique idempotent element in $\bar{\mathcal{G}}$ among those $x \in G$ for which $\mu(x) > 0$.

Theorem

Let $\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E^{\mu})$ *be an E-fuzzy group and* $x \in G$*, such that* $\mu(x) > 0$*. Then* $(x^{-1})^{-1} = x$*.*

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$$E^{\mu}(a \cdot x, b) \quad \text{and} \quad E^{\mu}(y \cdot a, b)$$

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Elements x_0 and y_0 are **solutions** of equations $E^{\mu}(a \cdot x, b)$ and $E^{\mu}(y \cdot a, b)$, respectively. If $\mu(x_0) = 0$ (analogously $\mu(y_0) = 0$), then obviously x_0 (y_0) is a solution of the corresponding equation; we say that it is a **trivial solution**.

Theorem

Let $\bar{\mathcal{G}} = (\mathcal{G}, \mu, \; E^{\mu})$ be an E-fuzzy group. Then, fuzzy equations

$$(i) \;\; E^{\mu}(a \cdot x, b) \;\; \textit{and} \;\; (ii) \;\; E^{\mu}(y \cdot a, b)$$

have nontrivial solutions for arbitrary $a, b \in G$, such that $\mu(a) \wedge \mu(b) \neq 0$.

E-fuzzy subgroup

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Let $\nu : G \to L$ be a nonempty fuzzy subset of a fuzzy set $\mu : G \to L$, E^{μ} a fuzzy relation on μ, and $E^{\nu} : G^2 \to L$ a fuzzy relations on G. We say that E^{ν} is a **restriction** of E^{μ} to ν if

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Lemma

Let $\nu : G \to L$ be a nonempty fuzzy subset of $\mu : G \to L$, and E^μ a fuzzy relation on μ. Then a restriction E^ν of E^μ to ν is a fuzzy relation on ν.

Proposition

If $E^\mu : A^2 \to L$ is a fuzzy equality on $\mu : A \to L$, then the restriction E^ν of E^μ to a nonempty fuzzy subset ν of μ is a fuzzy equivalence on ν. In addition, if μ and ν are fuzzy subalgebras of an algebra $\mathcal{A} = (A, F)$, and E^μ is compatible with operations in F, then also E^ν is compatible.

Let $\bar{\mathcal{G}}^\mu = (\mathcal{G}, \mu, \ E^\mu)$ and $\bar{\mathcal{G}}^\nu = (\mathcal{G}, \nu, \ E^\nu)$ be fuzzy groups over the same algebra $G = (G, \cdot, ^{-1}, e)$. We say that $\bar{\mathcal{G}}^\nu$ is an ***E*-fuzzy subgroup** of E-fuzzy group $\bar{\mathcal{G}}^\mu$, if ν is a fuzzy subset of μ and E^ν is a restriction of E^μ to ν.

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Theorem

Let $\bar{\mathcal{G}}^{\mu} = (\mathcal{G}, \mu,\ E^{\mu})$ be an E-fuzzy group and $E^1 : G^2 \to L$ a fuzzy relation on G, such that $E^1 \leqslant E^{\mu}$. Let E^1 fulfils all properties of a fuzzy equality except reflexivity. In addition, let E^1 satisfies also the following condition:

$$E^1(x, y) = E^{\mu}(x, y) \wedge E^1(x, x) \wedge E^1(y, y).$$

Now, let $\nu : G \to L$ be defined by $\nu(x) := E^1(x, x)$, for every $x \in G$. Then, $\bar{\mathcal{G}}^{\nu} = (\mathcal{G}, \nu,\ E^1)$ is an E-fuzzy subgroup of E-fuzzy group $\bar{\mathcal{G}}^{\mu}$.

Theorem

Let $\bar{\mathcal{G}}^{\mu} = (\mathcal{G}, \mu, E^{\mu})$ be an E-fuzzy group, $\nu : G \to L$ a nonempty fuzzy subset of μ, and E^{ν} a restriction of E^{μ} to ν. Then the structure $\bar{\mathcal{G}}^{\nu} = (\mathcal{G}, \nu, E^{\nu})$ is an E-fuzzy subgroup of $\bar{\mathcal{G}}^{\mu}$ if and only if it is an E-fuzzy algebra.

Theorem

$\{\bar{\mathcal{G}}^{\mu_i} = (\mathcal{G},\ \mu_i,\ E^{\mu_i}) \mid i \in I\}$ be a nonempty family of E-fuzzy subgroups of an E-fuzzy group $\bar{\mathcal{G}}^{\mu} = (\mathcal{G},\ \mu,\ E^{\mu})$, where $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is a given algebra. Further, for all $x, y \in G$, such that $x \neq y$, and $\bigwedge_{i \in I} \mu_i(x) > 0$, let

$$E^{\mu}(x, y) \wedge \bigwedge_{i \in I} \mu_i(x) \wedge \bigwedge_{i \in I} \mu_i(y) < \bigwedge_{i \in I} \mu_i(x).$$

Finally, let $\delta = \bigcap_{i \in I} \mu_i$ and let E^{δ} be the restriction of E^{μ} to δ. Then the structure $\bar{\mathcal{G}}^{\delta} = (\mathcal{G},\ \delta,\ E^{\delta})$, is an E-fuzzy subgroup of E-fuzzy group $\bar{\mathcal{G}}$.

Cut properties

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Theorem

Let $\bar{\mathcal{G}}^{\mu} = (\mathcal{G}, \mu,\ E^{\mu})$ be an E-fuzzy algebra. Then, $\bar{\mathcal{G}}^{\mu}$ is an E-fuzzy group if and only if for every $p \in L$, the cut μ_p is a subalgebra of $\mathcal{G}$, the cut relation E_p^{μ} is a congruence on μ_p, and the quotient structure μ_p/E_p^{μ} is a group.

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Theorem

Let $\bar{\mathcal{G}}^\mu = (\mathcal{G}, \mu, \ E^\mu)$ be an E-fuzzy group, such that $\mu(x) \neq 0$ for every $x \in G$, and let E^μ fulfils the following:

$$\text{for all } x, y \in G \text{ such that } x \neq y, \ E^\mu(x, y) < \bigwedge_{z \in G} \mu(z).$$

Then, the underlying algebra $\mathcal{G}$ of $\overline{\mathcal{G}}$ is a group.

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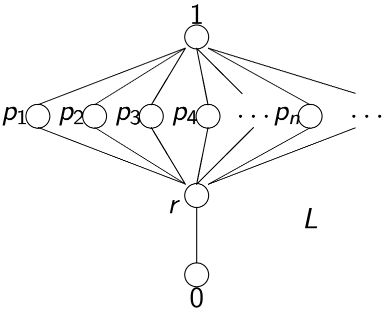

$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

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E^μ	**0**	**1**	**2**	**3**	**4**	**5**	$\dots$
0	1	0	r	0	r	0	$\cdots$
1	0	p_1	0	r	0	r	$\cdots$
2	r	0	p_2	0	r	0	$\cdots$
3	0	r	0	p_3	0	r	$\cdots$
4	r	0	r	0	p_4	0	$\cdots$
5	0	r	0	r	0	p_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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2	r	0	p_2	0	r	0	...
3	0	r	0	p_3	0	r	...
4	r	0	r	0	p_4	0	...
5	0	r	0	r	0	p_5	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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$$\begin{array}{c|cc} \oplus & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{n} \\ \mathbf{n} & \mathbf{n} & \mathbf{0} \end{array}\;; \qquad \begin{array}{c|cc} E^{\mu}_{p_n} & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & 1 & 0 \\ \mathbf{n} & 0 & 1 \end{array}\,.$$

Cut subalgebras:

μ_1 – the trivial one-element subalgebra $\{\mathbf{0}\}$.
For every $p_n \in L$, $\mu_{p_n} = \{\mathbf{0}, \mathbf{n}\}$.

$$\begin{array}{c|cc} \oplus & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{n} \\ \mathbf{n} & \mathbf{n} & \mathbf{0} \end{array}\;; \qquad \begin{array}{c|cc} E^{\mu}_{p_n} & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & 1 & 0 \\ \mathbf{n} & 0 & 1 \end{array}\,.$$

For every $p_n \in L$, the quotient structure $\mu_{p_n}/E^{\mu}_{p_n}$ is a two-element group, isomorphic to μ_{p_n}.

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Thank you for your attention!