Extreme value theorems on a non-additive probability space

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Extreme value theory is important statistical discipline used in many sectors. For example: meteorology, hydrology, finance, traffic prediction, management strategy, biomedical processing, ...

The extreme value theory is built on two basic theorems, which describe the extreme value distribution. The first extreme value theorem is Fisher-Tippet, Gnedenko theorem (1928, 1943) and the second extreme value theorem is Balkema-Haan, Pickands theorem (1974, 1975).

The aim of this work is prove validity of these theorems on a non-additive probability space.

First extreme value theorem

Let $X_1, X_2, \ldots$ be independent identically distribution (iid) real random variables with distibution function $F : \mathbb{R} \to \mathbb{R}$, such that

$$F(x) = P(X < x).$$

First extreme value theorem

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We define the maximum as

$$M_1 = X_1,$$

$$M_n = max\{X_1, X_2, ..., X_n\}, \quad for\ n \geq 2.$$

First extreme value theorem

Theorem(Fisher-Tippett,1928; Gnedenko,1943)

Let $X_1, X_2, ...X_n$ be a sequence of iid random variables. If there exist normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ and some non-degenerate distribution function H such that

$$\lim_{n\to\infty} P\left(\frac{M_n - b_n}{a_n}\right) = H(x),$$

for $\forall x \in \mathbb{R}$, then H belongs to the type of one of the following three types of standard extreme value distributions:

1. Gumbel
2. Fréchet
3. Weibull

Extreme value distributions

1. Gumbel

$$H(x) = \exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right), \quad x \in \mathbb{R}$$

2. Fréchet

$$H(x) = \begin{cases} 0 & \text{pre } x \leq \mu \\ \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right) & \text{pre } x > \mu, \alpha > 0 \end{cases}$$

3. Weibull

$$H(x) = \begin{cases} \exp\left(-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{-\alpha}\right) & \text{pre } x < \mu, \alpha < 0 \\ 1 & \text{pre } x \geq \mu \end{cases}$$

Second extreme value theorem

Let $X_1, X_2, \ldots$ are independent identically distribution (iid) real random variables with distribution function F.
Consider the distribution of X conditionally on exceeding some high threshold w:

$$
\begin{aligned}
F_w(x) &= P(X - w < x | X \geq w) \\
&= \frac{P(w \leq X < x + w)}{P(X \geq w)} \\
&= \frac{F(x+w) - F(w)}{1 - F(w)}
\end{aligned}
$$

for $0 < x < \omega(F)$, where $\omega(F) = \sup\{x; F(x) < 1\}$.

- A point $\omega(F)$ is called survival function (or tail of the distribution function F).
- Function F_w is called excess distribution.

Second extreme value theorem

Definition (Pareto distribution)

Random variable x has generalized Pareto distribution (GPD) if its distribution function is of the form

$$G_{\alpha,\beta}(x) = \begin{cases} 1 - \left(1 + \alpha \frac{x}{\beta}\right)^{-1/\alpha} & \text{if } \alpha \neq 0, \\ 1 - \exp\left(-\frac{x}{\beta}\right) & \text{if } \alpha = 0, \end{cases}$$

where $x \in \langle 0, \infty)$ if $\alpha \geq 0$ and $x \in \langle 0, -\beta/\alpha \rangle$ for $\alpha < 0$.

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where $x \in \langle 0, \infty)$ if $\alpha \geq 0$ and $x \in \langle 0, -\beta/\alpha\rangle$ for $\alpha < 0$.

Theorem(Balkema,de Haan and Pickands 1974/75)

Function F_w is an excess distribution function if and only if we can find a positive measurable function β for every $\alpha > 0$ such that

$$\lim_{w \to \omega(F)} \sup_{0 \leq x \leq \omega(F) - w} \left|F_w(x) - G_{\alpha,\beta}(x)\right| = 0.$$

Non-additive probability space - basic definition and notions

Definition

Let Ω be a nonempty set and $\mathcal{S}$ be a σ-algebra of subsets of Ω. A mapping $\mu : \mathcal{S} \to [0, 1]$ is called a continous probability, if the following conditions hold:

(i) $\mu(\Omega) = 1, \mu(\emptyset) = 0,$

(ii) $A_i \nearrow A \Rightarrow \mu(A_i) \nearrow \mu(A),$

(iii) $A_i \searrow A \Rightarrow \mu(A_i) \searrow \mu(A),$

$\forall A_i, A \in \mathcal{S}$ $(i = 1, 2, \ldots)$.

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$\forall A_i, A \in \mathcal{S}$ $(i = 1, 2, \ldots)$.

This measure is non-additive probability measure and probability space $(\Omega, \mathcal{S}, \mu)$ is a non-additive probability space.

Non-additive probability space - basic definition and notions

Definition

A mapping $\xi : \Omega \to \mathbb{R}$ is called a random variable, if it is measurable, i. e. $\xi^{-1}(I) \in S$ for every interval of real numbers I.

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Definition

Let $\xi : \Omega \to R$ be a random variable. Then the function $\dot{F} : R \to \langle 0, 1 \rangle$ defined by

$$\dot{F}(x) = \mu(\xi^{-1}((-\infty, x))), \quad x \in \mathbb{R}$$

is called a non-additive distribution function.

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Proposition

Non-additive distribution function is a distribution function, i. e.

- $\dot{F}$ is non - decreasing,
- $\dot{F}$ is left continuous in any point $x \in R$,
- $\lim_{n \to \infty} \dot{F}(x) = 1, \qquad \lim_{n \to -\infty} \dot{F}(x) = 0.$

Non-additive probability space - basic definition and notions

- If $\dot{F} : \mathbb{R} \to \langle 0, 1 \rangle$ is a distribution function, then there exits exactly one probability measure $\lambda_{\dot{F}} : \mathcal{B}(\mathbb{R}) \to \langle 0, 1 \rangle$ defined on the σ-algebra of all Borel subsets of R such that

$$\lambda_{\dot{F}}(\langle a, b)) = \dot{F}(b) - \dot{F}(a)$$

for any $a, b \in R, a \leq b$.

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  for any $a, b \in R, a \leq b$.
- The corresponding integral with respect to $\lambda_{\dot{F}}$:

$$\int_R f d\lambda_{\dot{F}} = \int_{-\infty}^{\infty} f(x) d\dot{F}(x).$$

Non-additive probability space - basic definition and notions

Definition

Let $\xi : \Omega \to R$ be a random variable on $(\Omega, \mathcal{S}, \mu)$, let $\dot{F}$ its non-additive distribution function. We say that ξ is integrable, if there exists

$$\int_{-\infty}^{\infty} td\dot{F}(t) = E(\xi).$$

We say that ξ is square integrable if there exists

$$\int_{-\infty}^{\infty} t^2 d\dot{F}(t).$$

In this case we define the dispersion $D(\xi)$ by the formula

$$D(\xi) = \int_{-\infty}^{\infty} t^2 d\dot{F}(t) - \left(\int_{-\infty}^{\infty} td\dot{F}(t) \right)^2 =$$

$$= \int_{-\infty}^{\infty} (t - E(\xi))^2 \, d\dot{F}(t),$$

Non-additive probability space - indepedence

- In the classical case two random variables are independent if

$$P(\xi^{-1}(A) \cap \eta^{-1}(B)) = P(\xi^{-1}(A)).P(\eta^{-1}(B)).$$

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Definition

Let $(\Omega, \mathcal{S}, \mu)$ be non-additive probability space. Let maps $\xi, \eta : \Omega \to \mathbb{R}$ be random variables and functions $\dot{F}$, $\dot{G}$ be their distribution functions and $\lambda_{\dot{F}}$, $\lambda_{\dot{G}}$ be their probability measures.
Let map $T : \Omega \to \mathbb{R}^2$, $T(\omega) = (\xi(\omega), \eta(\omega))$. We say that ξ, η are independent, if

$$\lambda_T(C) = \lambda_{\dot{F}} \times \lambda_{\dot{G}}(C),$$

for any $C \in \mathcal{B}(\mathbb{R}^2)$.

Kolmogorov's construction

Theorem (Riečan, 2013)

Let $(\xi_n)_n$ be a sequence of independent random variables in $(\Omega, \mathcal{S}, \mu)$, $T_n = (\xi_1, ..., \xi_n), n = 1, 2, ...,$

$$\mu_{T_n} : \mathcal{B}(R^n) \to [0, 1],$$

$$\mu_{T_n}(A) = \mu(T_n^{-1}(A)),$$

$A \in \mathcal{B}(R^n), n = 1, 2, ...$
Then for any $n \in N$, and any $A \in \mathcal{B}(R^n)$

$$\mu_{T_{n+1}}(A \times R) = \mu_{T_n}(A).$$

Kolmogorov's construction

Theorem (Riečan, 2013)

Let $(\xi_n)_n$ be a sequence of independent random variables on $(\Omega, \mathcal{S}, \mu)$,
$\Pi_n : R^N \to R^n$, $\Pi_n((x_i)_{i=1}^{\infty}) = (x_1, ..., x_n)$,
$\mathcal{C}$ be the family of all sets of the form $\Pi_n^{-1}(A)$, for some $n \in N, A \in \mathcal{B}(R^n)$
$\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$.

Then there exists a probability measure

$$P : \sigma(\mathcal{C}) \to [0, 1]$$

such that

$$P(\Pi_n^{-1}(A)) = \mu_{T_n}(A)$$

for any $n \in N, A \in \mathcal{B}(R^n)$.

Kolmogorov's construction

Theorem (Riečan, 2013)

Let $(\xi_n)_n$ be a sequence of independent random variables on the space $(\Omega, \mathcal{S}, \mu)$. Let

$$(R^N, \sigma(\mathcal{C}), P)$$

be the probability space constructed in previous theorem. Define

$$f_n : R^N \to R^n$$

by the formula

$$f_n((x_i)_{i=1}^{\infty}) = x_n,$$

$n = 1, 2, ...$ Then $(f_n)_n$ is a sequence of independent random variables in the space $(R^N, \sigma(\mathcal{C}), P)$.

Kolmogorov's construction - summary

- sequence $(\xi_n)_n$ be independent random variables in our non-additive space $(\Omega, \mathcal{S}, \mu)$, where μ is a continuous probability
- sequence $(f_n)_n$ be independent random variables in the probability space $(R^N, \sigma(\mathcal{C}), P)$ with a σ-additive probability
- the convergence of $(\xi_n)_n$ corresponds to convergence of $(f_n)_n$

Convergence in distribution

Theorem (Riečan, 2013)

Let $(\xi_n)_n$ be a sequence of independent random variables on $(\Omega, \mathcal{S}, \mu)$. Let $(R^N, \sigma(\mathcal{C}), P)$ the corresponding probability space $(\Omega, \mathcal{S}, \mu)$. Let $(f_n)_n$ the sequence of random variables on $(R^N, \sigma(\mathcal{C}), P)$ stated in previous theorem. Let $g_n : R^n \to R$ be a Borel measurable functions $(n = 1, 2, ...)$. Then

$$\lim_{n\to\infty} \mu(\{\omega \in \Omega; g_n(\xi_1(\omega), ..., \xi_n(\omega)) < x\}) = F(x)$$

if and only if

$$\lim_{n\to\infty} P(\{u \in R^N; g_n(f_1(u), ..., f_n(u)) < x\}) = F(x).$$

Convergence in distribution

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The convergence follows by the equality

$$\mu(\{\omega \in \Omega; g_n(\xi_1(\omega), ..., \xi_n(\omega)) < x\}) = P(\{u \in R^N; g_n(f_1(u), ..., f_n(u)) < x\})$$

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- Let $(\xi_n)_n$ be a sequence of independent identically distributed on a non-additive probability space $(\Omega, \mathcal{S}, \mu)$ with non-additive distribution function $\dot{F}$.

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First extreme value theorem on a non-additive probability space

Fisher-Tippett, Gnedenko theorem

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$$\lim_{n\to\infty} \mu\left(\left\{\omega \in \Omega; \frac{1}{a_n}\left(\dot{M}_n(\omega) - b_n\right) < x\right\}\right) = H(x),$$

for $x \in \mathbb{R}$. Then H belongs to the type of one of the following three types of standard extreme value distributions:

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3. Weibull

Second extreme value theorem on a non-additive probability space

Definition

Let threshold $w > 0$. We define excess distribution $\dot{F}_w$ such that

$$\dot{F}_w(x) = \frac{\dot{F}(x+w) - \dot{F}(w)}{1 - \dot{F}(w)},$$

for $0 < x < \omega\left(\dot{F}\right)$, where $\omega\left(\dot{F}\right) = \sup\left\{x; \dot{F}(x) < 1\right\}$.

Balkema,de Haan and Pickands theorem

Function $\dot{F}_w$ is an excess distribution function if and only if we can find a positive measurable function β for every $\alpha > 0$ such that

$$\lim_{w \to \omega\left(\dot{F}\right)} \sup_{0 \le x \le \omega\left(\dot{F}\right) - w} \left| \dot{F}_w(x) - G_{\alpha,\beta_{(w)}}(x) \right| = 0.$$

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Thank you.