Some new construction methods of additive generators of copulas

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Outline

Multivariate Archimedean copulas

Overview of known construction methods

New construction methods

Archimedean copulas

Theorem (Moynihan 1978)

A function $C\colon [0,1]^2 \to [0,1]$ *is an Archimedean copula if and only if there is a convex strictly decreasing function* $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$*, so that*

$$C(x,y) = f^{(-1)}(f(x) + f(y)),$$

where the pseudo-inverse $f^{(-1)}\colon [0,\infty] \to [0,1]$ *is given by* $f^{(-1)}(u) = f^{-1}(\min(u, f(0)))$.

We denote by $\mathcal{F}_2$ the class of all additive generators f of binary copulas.

n-ary Archimedean copulas

Theorem (McNeil and Nešlehová 2009)

Let $f\colon [0,1] \to [0,\infty]$ *be a continuous strictly decreasing function such that* $f(1) = 0$ *(i.e., an additive generator of a continuous Archimedean t-norm). Then the function* $C\colon [0,1]^n \to [0,1]$ *given by*

$$C(x_1, \ldots, x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right).$$

is an n-ary copula if and only if the function $g\colon [-\infty, 0] \to [0,1]$ *given by* $g(u) = f^{(-1)}(-u)$ *is* $(n-2)$*-times differentiable with non-negative derivatives* $g', \ldots, g^{(n-2)}$ *on* $]-\infty, 0[$*, and* $g^{(n-2)}$ *is convex.*

We denote by $\mathcal{F}_n$ the class of all additive generators f generating n-ary copulas, and by $\mathcal{F}_\infty$ all *universal* additive generators.

n-ary Archimedean copulas

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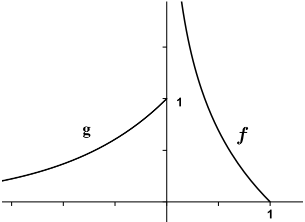

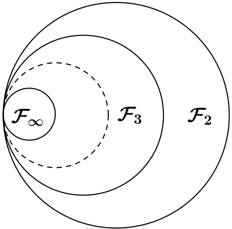

Examples of universal generators

- $f_\Pi(x) = -\log x$ generates the product copula Π
- $f(x) = \frac{1}{x} - 1$ is a generator of Ali-Mikhail-Haq copula

$$C(x_1, \ldots, x_n) = \frac{1}{\sum_{i=1}^{n} \frac{1}{x_i} - (n-1)}$$

Mainstreams in construction

- Solutions of some problem. For example Frank, Plackett, Clayton and Gumbel copulas.
- Ad hoc. For example Yager copulas (subfamily of Yager t-norms)
- Aggregation functions preserving the classes of additive generators (of binary copula) or of their pseudo-inverses.
- Construction of additive generator of copulas (binary,n-ary,universal) from some a-priori given function.
  - ... from some a-priori given generator.

Constructions from a given generator <1>

Proposition (Klement,Mesiar and Pap 2005)

Let $\varphi\colon [0,1] \to [0,1]$ be a concave automorphism (strictly increasing, not necessarily a bijection; Sempi and Durante 2005). Then for any $f \in \mathcal{F}_2$ also

$$f \circ \varphi \in \mathcal{F}_2.$$

Example

Consider $f_\Pi(x) = -\log x$ and

$$\varphi(x) = a + (1-a)x, \;\; a \in]0,1[.$$

Then $f_\Pi \circ \varphi(x) = -\log(a+(1-a)x), \;\; x \in [0,1]$, and the corresponding copula is given by
$C(x,y) = \max\left(0, \frac{(a+(1-a)x)(a+(1-a)y)-a}{1-a}\right).$

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Proposition (Bacigál, Juráňová and Mesiar 2010)

Let $\varphi\colon [0,1] \to [0,1]$ *be an automorphism of* $[0,1]$ *such that its inverse* $\varphi^{-1}\colon [0,1] \to [0,1]$ *is absolutely monotone on* $]0,1[$ *(i.e.,* $(\varphi^{-1})^{(k)}(x) \geq 0$ *for any* $k \in N$ *and* $x \in]0,1[$*). Then for any* $f \in \mathcal{F}_\infty$ *also*

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Constructions from a given generator <2>

Proposition (Bacigál, Juráňová and Mesiar 2010)

Let $\eta\colon [0,\infty] \to [0,\infty]$ *be a convex automorphism of* $[0,\infty]$. *Then for any* $f \in \mathcal{F}_2$ *also*

$$\eta \circ f \in \mathcal{F}_2$$

Example

- $\eta(u) = u^{\lambda}$ *with* $f(x) = -\log(x)$ *leads to Gumbel family.*
- $\eta(u) = \lambda^{u} - 1$, $\lambda \in]1,\infty[$, *and* $\eta(u) = \lambda^{-u} - 1$, $\lambda \in]0,1]$ *gives what was proposed in Junker and May (2005).*

Proposition

Let $n \in \{2,3,\ldots\}$. *Let* $\eta\colon [0,\infty] \to [0,\infty]$ *be an automorphism such that its inverse* $\eta^{-1}\colon [0,\infty] \to [0,\infty]$ *has* $(n-2)$ *derivatives (all derivatives) on* $]0,\infty[$, $(\eta^{-1})^{(k)}(x) \geq 0$ *for all* $x \in]0,\infty[$ *and* $k \in \{1,\ldots,n-2\}$ *(*$k \in N$*) so that* $(\eta^{-1})^{(n-2)}$ *is a convex function. Then for any* $f \in \mathcal{F}_n$ *(any* $f \in \mathcal{F}_\infty$*) also*

$$\eta \circ f \in \mathcal{F}_n \quad (\eta \circ f \in \mathcal{F}_\infty)$$

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$$\eta \circ f \in \mathcal{F}_n \quad (\eta \circ f \in \mathcal{F}_\infty)$$

Constructions from a given generator <3>

Proposition (Jágr, Komorníková and Mesiar 2010)

Let $f \in \mathcal{F}_n$, $n \in \{2, 3, \ldots\} \cup \{+\infty\}$. Then $(f_\lambda)_{\lambda \in]0,1]} \subset \mathcal{F}_n$, where $f_\lambda \colon [0, 1] \to [0, \infty]$ is given by

$$f_\lambda(x) = f(\lambda x) - f(\lambda)$$

The parametric family $(f_\lambda)_{\lambda \in]0,1]}$ is non-trivial (i.e., its members generates different copulas for different parameters) if and only if f does not belong to the Clayton family of additive generators.

Constructions from a given generator <4>

Theorem (McNeil & Nešlehová 2009)

For every $f \in \mathcal{F}_n$, the function $F\colon]-\infty,\infty[\to [0,1]$ given by

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 - \sum_{k=0}^{n-2} \frac{x^k g^{(k)}(-x)}{k!} - \frac{x^{n-1} g_-^{(n-1)}(-x)}{(n-1)!} & \text{otherwise} \end{cases}$$

is a distribution function of a positive random variable X (called also positive distance function), where $g_-^{(n-1)}$ is the left-derivative of order $n-1$.

Due to (Williamson 1956), if F is a positive distance function, then, for a fixed $n \in N$, the inverse transformation is given by

$$g(x) = \int_{-x}^{\infty} \left(1 + \frac{x}{t}\right)^{n-1} dF(t),$$

where $x \in]-\infty, 0]$, $g(-\infty) = 0$.

Constructions from a given generator <4>

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Constructions from a given generator <4>

Due to the two transformations, one can construct new additive generators of (n-dimensional) copulas as follows:

- take, for an arbitrary $m \in \{2, 3, \ldots\}$, an additive generator $f \in \mathcal{F}_m$
- introduce a positive distance function F
- possibly modify F into a new positive distance function $\tilde{F}$ (e.g. $\tilde{F}(x) = F(x - a)$ for a fixed constant $a \in]0, \infty[$)
- apply the Williamson transform to $\tilde{F}$, considering a fixed $n \in \{2, 3, \ldots\}$, obtaining a function $\tilde{g} \colon [-\infty, 0] \to [0, 1]$
- $\tilde{f}$ linked to $\tilde{g}$ is an additive generator from $\mathcal{F}_n$

Constructions from a given generator <4>

Example

Consider $f_W \in \mathcal{F}_2$ with $g(x) = \max(0, x+1)$. Then a positive distance function is given by

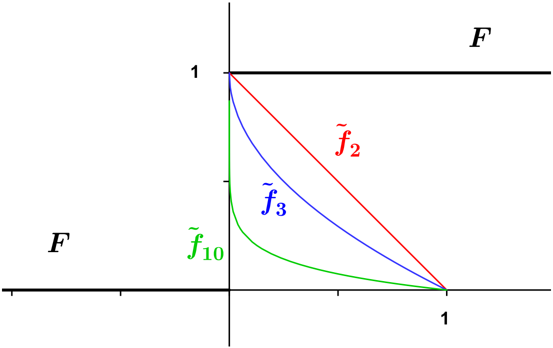
$$F(x) = 1 - g(-x) - xg'_-(-x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases},$$

which is the Dirac distribution function focused in point $x_0 = 1$. For an arbitrary $n \in \{2, 3, \ldots\}$, the Williamson transform defines

$$\tilde{g}(x) = \int_{-x}^{\infty} \left(1 + \frac{x}{t}\right)^{n-1} dF(t) = (1+x)^{n-1}, \quad x \in]-\infty, 0].$$

The related additive generator $\tilde{f}(x) = 1 - x^{\frac{1}{n-1}}$ belongs to $\mathcal{F}_n$. Observe that $\tilde{f}$ generates a non-strict Clayton copula with parameter $\lambda = \frac{1}{n-1}$ (the weakest n-dimensional Archimedean copula).

Constructions from a given generator <4>



Constructions from a given function [1]

Theorem

Let $h\colon [a,b] \to [-\infty,\infty]$ be a strictly decreasing convex continuous function. Then for any non-trivial bounded $[c,d] \subseteq [a,b]$ (if $h(b) = -\infty$ then $[c,d] \subset [a,b[$) the function $f_{c,d}\colon [0,1] \to [0,\infty]$ given by

$$f_{c,d}(x) = h(c + x(d-c)) - h(d)$$

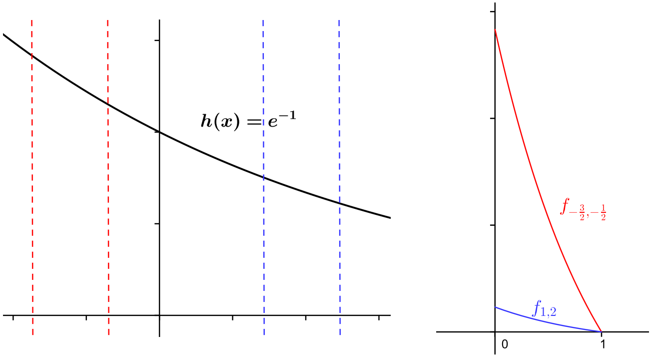
is an additive generator from $\mathcal{F}_2$.

Constructions from a given function [1]

Example

Consider $h(x) = e^{-x}$. Then for any $c, d \in]-\infty, \infty[$, $c < d$,

$$f_{c,d}(x) = e^{-(c+x(d-c))} - e^{-d} = e^{-c}\left(e^{-x(d-c)} - e^{-(d-c)}\right),$$



which generates the same binary copula as $f_\lambda(x) = e^{-\lambda x} - e^{-\lambda}$ with $\lambda = d - c > 0$.

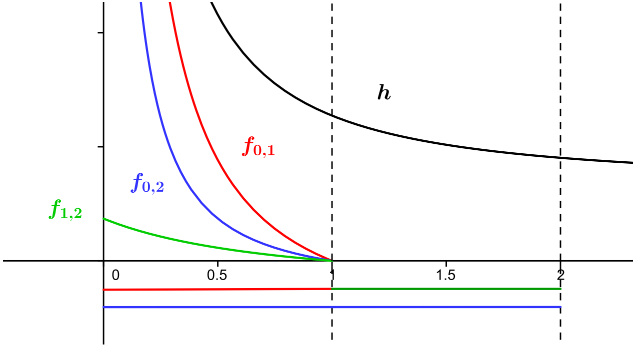
Constructions from a given function [1]

Example

Consider $h(x) = \frac{1}{\arctan x}$*. Then for any* $[c, d] \subset [0, \infty[$,

$$f_{c,d}(x) = \frac{1}{\arctan(c + x(d - c))} - \frac{1}{\arctan d}$$

is an additive generator from $\mathcal{F}_2$*.*



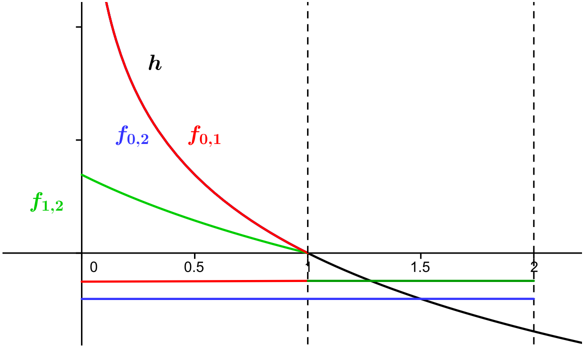
Constructions from a given function [1]

Example

Consider $h(x) = -\log x$. Then for any $0 \le c < d < \infty$,

$$f_{c,d}(x) = -\log(c + x(d-c)) - \log d = -log(a + (1-a)x),$$

where $a = \frac{c}{d} \in [0,1[$, is an additive generator from $\mathcal{F}_2$.



Constructions from a given function [1]

Theorem

Let $h\colon [a,b] \to [-\infty,\infty]$ satisfy the same constraints. Then:

- *If for $n \in \{3,4\ldots\}$, the inverse function h^{-1} has $(n-2)$ derivatives on $]h(b),h(a)[$ so that $(h^{-1})^{(k)}(x)\cdot(-1)^k \geq 0$ for all $k \in \{1,\ldots,n-2\}$ and $x \in]h(b),h(a)[$, and $(h^{-1})^{(n-2)}(-1)^k$ is convex, then for any bounded interval $[c,d] \subset [a,b]$ (if $h(b) = -\infty$ then $d < b$), the function $f_{c,d}$ is an additive generator from $\mathcal{F}_n$.*
- *If the inverse function h^{-1} is totally monotone on $]h(b),h(a)[$, then $f_{c,d}$ belongs to $\mathcal{F}_\infty$.*

Example

Define $h(x) = -x^{0.4}$. Obviously $(h^{-1})^{(4)}(u) = -\frac{15}{16}(-u)^{-1.5}$ is not convex, thus for $0 \leq c < d < \infty$, function $f_{c,d}(x) = d^{0.4} - (c+(d-c)x)^{0.4}$ generates a 3-dimensional copula but not a 4-dimensional copula.
Observe that $f_{0,d}(x) = d^{0.4}(1-x^{0.4})$ generates the Clayton copula with parameter -0.4.

Constructions from a given function [1]

Theorem

Let $h\colon [a,b] \to [-\infty,\infty]$ *satisfy the same constraints. Then:*

- *If for* $n \in \{3,4\ldots\}$, *the inverse function* h^{-1} *has* $(n-2)$ *derivatives on* $]h(b),h(a)[$ *so that* $(h^{-1})^{(k)}(x)\cdot(-1)^k \geq 0$ *for all* $k \in \{1,\ldots,n-2\}$ *and* $x \in]h(b),h(a)[$, *and* $(h^{-1})^{(n-2)}(-1)^k$ *is convex, then for any bounded interval* $[c,d] \subset [a,b]$ *(if* $h(b) = -\infty$ *then* $d < b$*), the function* $f_{c,d}$ *is an additive generator from* $\mathcal{F}_n$.
- *If the inverse function* h^{-1} *is totally monotone on* $]h(b),h(a)[$, *then* $f_{c,d}$ *belongs to* $\mathcal{F}_\infty$.

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Constructions from a given function [2]

Theorem

Let $h\colon [a,b] \to [-\infty,\infty]$ satisfy the same constraints. Let $\varphi\colon [\alpha,\beta] \to [a,b]$ be a concave increasing bijection. Then also the function $h \circ \varphi\colon [\alpha,\beta] \to [-\infty,\infty]$ satisfies the constraints, i.e., for any bounded interval $[\gamma,\delta] \subseteq [\alpha,\beta]$ the function

$$f_{\gamma,\delta}(x) = h\big(\varphi(\gamma + (\delta-\gamma)x)\big) - h\big(\varphi(\delta)\big)$$

is an additive generator from $\mathcal{F}_2$.

Example

Let again $h(x) = -x^{0.4}$ and introduce $\varphi(x) = \sqrt{x}$ (concave increasing bijection).

- *due to <1>, $f_{c,d} \circ \varphi_{|[0,1]}(x) = d^{0.4} - (c + (d-c)\sqrt{x})^{0.4}$,*
- *due to [1], $h \circ \varphi(x) = -x^{0.2}$, we have*

$$f_{\gamma,\delta}(x) = \delta^{0.2} - (\gamma + (\delta-\gamma)x)^{0.2}$$

Note that both $f_{c,d} \circ \varphi_{|[0,1]}$ and $f_{\gamma,\delta}$ are additive generators from $\mathcal{F}_2$.

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Note that both $f_{c,d} \circ \varphi_{|[0,1]}$ and $f_{\gamma,\delta}$ are additive generators from $\mathcal{F}_2$.

Construction by gluing two generators

Theorem

Let $f_1, f_2 \in \mathcal{F}_2$ and $k \in]0,1[$ be given. Define a function $f\colon [0,1] \to [0,\infty]$, denoted also by $f = f_1 *_k f_2$, by

$$f(x) = \begin{cases} \frac{f_1(x)}{f_1(k)} & \text{if } x \in [0,k], \\ \frac{f_2(x)}{f_2(k)} & \text{otherwise} \end{cases} \qquad \textit{whenever } \frac{f'_{1-}(k)}{f_1(k)} \le \frac{f'_{2-}(k)}{f_2(k)}$$

$$f(x) = \begin{cases} \frac{f_2(x)}{f_2(k)} & \text{if } x \in [0,k], \\ \frac{f_1(x)}{f_1(k)} & \text{otherwise.} \end{cases} \qquad \textit{otherwise}$$

Then $f \in \mathcal{F}_2$.

Note that due to the Williamson transform, this approach can be extended for any dimension n.

Construction by gluing two generators

Example

Consider $f_W(x) = 1 - x$, $f_\Pi(x) = -\log x$. For any fixed $k \in]0,1[$, $\frac{f_W'(k)}{f_W(k)} = \frac{-1}{1-k} \geq \frac{1}{k\log k} = \frac{f_\Pi'(k)}{f_\Pi(k)}$. Therefore, $f_k = f_W *_k f_\Pi$ is given by

$$f_k(x) = \begin{cases} \log_k(x) & \text{if } x \in [0,k], \\ \frac{1-x}{1-k} & \text{otherwise.} \end{cases}$$

The corresponding Archimedean copula $C_k \in \mathcal{C}_2$ is given by

$$C(x) = \begin{cases} xy & \text{if } (x,y) \in [0,k]^2, \\ x+y-1 & \text{if } x+y \geq k+1, \\ x \cdot k^{\frac{1-y}{1-k}} & \text{if } x \leq k < y, \\ y \cdot k^{\frac{1-x}{1-k}} & \text{if } y \leq k < x, \\ k^{\frac{2-x-y}{1-k}} & \text{otherwise.} \end{cases}$$

The family $(C_k)_{k\in]0,1[}$ is continuous and strictly increasing in parameter k, with limit members $C_0 = W$ and $C_1 = \Pi$.

Construction by gluing two generators

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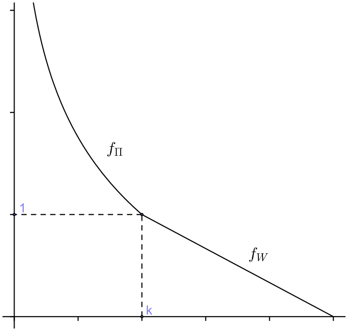

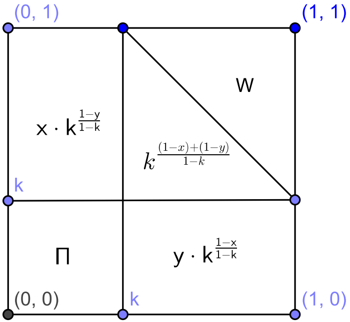

Conclusions

- We have reviewed some construction methods known in the literature for additive generators of copulas (2-dimensional, n-dimensional, for any dimension), including a method based on the Williamson transform.
- While these methods are based on an a priori knowledge of some additive generators, we have introduced a rather general construction method based on a given special real function h, and yielding 2-parameter families of additive generators.
- Moreover, we have introduced a parametric family of methods gluing two additive generators from $\mathcal{F}_2$ into a new additive generator from $\mathcal{F}_2$.