Fuzzy concept-forming operators on heterogeneous structures

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Outline of presentation

1. an introduction to Fuzzy Formal Concept Analysis
2. a new extension called heterogeneous formal context
3. a relationship between related approaches

Formal concept analysis

- method of relational data analysis (Rudolf Wille, TU Darmstadt)
- used in information retrieval, knowledge discovery, preprocessing data, social networks
- **input:** objects (rows) $\times$ attributes (columns) table

	a_1	a_2	a_3
o_1	1	1	1
o_2	1	0	1
o_3	0	1	1

- **output:**
  - set of concepts (closed pair – subset of objects and subset of attributes)
  - attribute dependencies

holds $\{a_2\} \to \{a_3\}$

$$\{a_1, a_2\} \to \{a_3\}$$

does not hold $\{a_1\} \to \{a_2\}$

Fuzzy extensions

- limitations of binary data tables
- how to deal with more complex information?
- some fruitful answers using **fuzzification**:

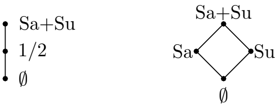
The extension title	*Authors*
***L*-fuzzy concept lattices**	*Burusco, Fuentes-González*
one-sided concept lattices	*Bělohlávek et al., Yahia et al., Krajči*
multi-adjoint concept lattices	*Medina, Ojeda-Aciego, Ruiz-Calviño*
generalized concept lattices	*Krajči*

- characteristics:
  - fuzzy relation in data tables
  - homogeneous set of truth values for objects
  - homogeneous set of truth values for attributes

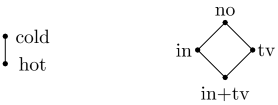
Our new extension

The previous approaches do not cover the following consideration:

- set of objects B: **people going to stay together** (different types)



- set of attributes A: **cottage conditions** (different types)



- the table values R: **degrees of accepted discomfort** (different types)

Formalization

Let A and B be non-empty sets (attributes and objects).

Let $\mathcal{C} = ((C_a, \leq_{C_a}) : a \in A)$, $\mathcal{D} = ((D_b, \leq_{D_b}) : b \in B)$ be systems of complete lattices.

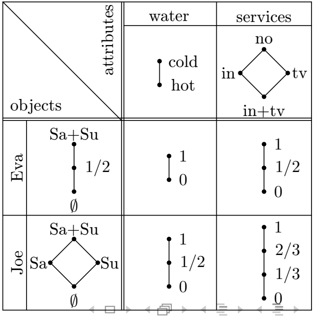
Let $\mathcal{P} = ((P_{a,b}, \leq_{P_{a,b}}) : a \in A, b \in B)$ be a system of posets.

R be a function from $A \times B$ such that $R(a, b) \in P_{a,b}$ for all $a \in A$ and $b \in B$.

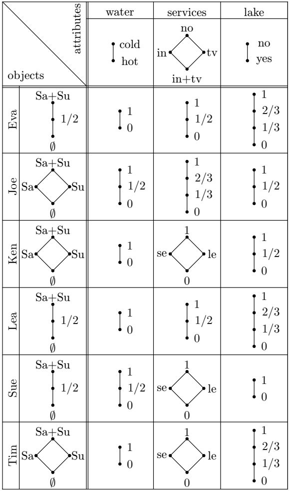
$\odot = (\bullet_{a,b} : a \in A, b \in B)$ be a system of operations, $\bullet_{a,b}$: $C_a \times D_b \to P_{a,b}$.

Then we call the tuple $\langle A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot \rangle$ a *heterogeneous formal context*.

objects \ attributes		a_1	a_2
		C_{a_1}	C_{a_2}
b_1	D_{b_1}	$P(a_1, b_1)$	$P(a_2, b_1)$
b_2	D_{b_2}	$P(a_1, b_2)$	$P(a_2, b_2)$



- natural claims about some additional level of generalization?
    - enables the formulation of preferences
    - enables an interpretation of concept-forming operators
    - enables an interpretation of concepts
    - preserves the basic theorem on concept lattices



Interpretation of concept-forming operators

Let G be a set of functions $g : g(b) \in D_b$ for $b \in B$,

- e.g., $g(\text{Eva}) \in \{\emptyset, 1/2, \text{Sa+Su}\}$, $g(\text{Joe}) \in \{\emptyset, \text{Sa}, \text{Su}, \text{Sa+Su}\}$, $g(\text{Ken}) \in \{\emptyset, 1/2, \text{Sa+Su}\}$.

Let F be a set of functions $f : f(a) \in C_a$ for $a \in A$,

- e.g., $f(\text{water}) \in \{\text{hot}, \text{cold}\}$, $f(\text{services}) \in \{\text{in+tv}, \text{in}, \text{tv}, \text{no}\}$, $f(\text{lake}) \in \{\text{yes}, \text{no}\}$.

$\nearrow : G \to F$
(the number of days $\to$ the degrees of cottage conditions)

$$(\nearrow(g))(a) = \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a,b)\}.$$

- **the worst conditions** for a specific number of days

$\swarrow : F \to G$
(the degrees of cottage conditions $\to$ the number of days)

$$(\swarrow(f))(b) = \sup\{d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a,b)\}.$$

- **the maximum number of days** by the appointed cottage conditions

Mappings $\nearrow$ and $\swarrow$ form a Galois connection.

Interpretation of concepts

A pair $\langle g, f \rangle$ such that $\nearrow(g) = f$ and $\swarrow(f) = g$.

- 5 concepts obtained for 2 people and 2 attributes
- 8 concepts obtained for 3 people and 3 attributes
- 9 concepts obtained for 6 people and 3 attributes

extents			intents		
Eva	Joe	Ken	water	services	lake
∅	∅	∅	cold	no	no
∅	Sa	∅	cold	no	yes
1/2	∅	1/2	cold	in	no
1/2	Sa	1/2	cold	in	yes
Sa+Su	∅	1/2	cold	in+tv	no
Sa+Su	Sa	Sa+Su	cold	in+tv	yes
1/2	Sa+Su	1/2	hot	in	yes
Sa+Su	Sa+Su	Sa+Su	hot	in+tv	yes

Interpretation of the concept:

- the worst acceptable cottage conditions by appointed extent

Basic theorem on heterogeneous concept lattices

Antoni, Krajči, Krídlo, Macek, Pisková. On heterogenenous formal contexts. **Fuzzy sets and systems**. 234 (2014) 22–33.

1 A heterogeneous concept lattice $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ is a complete lattice in which

$$\bigwedge_{i\in I}\langle g_i, f_i\rangle = \left\langle \bigwedge_{i\in I} g_i, \nearrow\left(\swarrow\left(\bigvee_{i\in I} f_i\right)\right)\right\rangle$$

and

$$\bigvee_{i\in I}\langle g_i, f_i\rangle = \left\langle \swarrow\left(\nearrow\left(\bigvee_{i\in I} g_i\right)\right), \bigwedge_{i\in I} f_i\right\rangle.$$

2 For each $a \in A$ and $b \in B$, let $P_{a,b}$ have the least element $0_{P_{a,b}}$ such that $0_{C_a} \bullet_{a,b} d = c \bullet_{a,b} 0_{D_b} = 0_{P_{a,b}}$ for all $c \in C_a$, $d \in D_b$. Then a complete lattice L is isomorphic to $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ if and only if there are mappings $\alpha : \bigcup_{a\in A}(\{a\} \times C_a) \to L$ and $\beta : \bigcup_{b\in B}(\{b\} \times D_b) \to L$ such that:

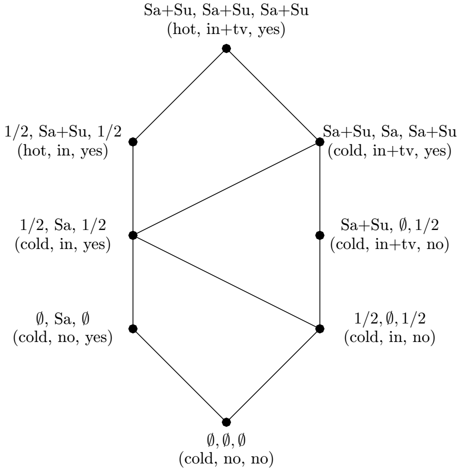
- 1a) α does not increase in the second argument (for a fixed first argument);
- 1b) β does not decrease in the second argument (for a fixed first argument);
- 2a) $\mathrm{Rng}(\alpha)$ is inf-dense in L;
- 2b) $\mathrm{Rng}(\beta)$ is sup-dense in L; and
- 3) For every $a \in A$, $b \in B$, $c \in C_a$ and $d \in D_b$,

$$\alpha(a, c) \geq \beta(b, d) \quad \text{if and only if} \quad c \bullet_{a,b} d \leq R(a, b).$$

Heterogeneous concept lattice

$\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$).

- $\langle(1/2, \text{Sa}, 1/2), (\text{cold}, \text{in}, \text{yes})\rangle \leq \langle(\text{Sa+Su}, \text{Sa+Su}, \text{Sa+Su}), (\text{hot}, \text{in+tv}, \text{yes})\rangle$



Galois connectional formal context

- proposed by J. Pócs, Note on generating fuzzy concept lattices via Galois connections, **Information Sciences, 2012**
- **idea:** diversify all objects, diversify all attributes, every field of table is Galois connection

B	nonempty set of objects
A	nonempty set of attributes
$\mathcal{C} = ((C_a, \leq_{C_a}) : a \in A)$	system of complete lattices
$\mathcal{D} = ((D_b, \leq_{D_b}) : b \in B)$	system of complete lattices
$\mathcal{G} = ((\phi_{a,b}, \psi_{a,b}) : a \in A, b \in B)$	system of (antitone) Galois connection
$(\phi_{a,b}, \psi_{a,b})$	a Galois connection from $(C_a, \leq_{C_a})$ to $(D_b, \leq_{D_b})$

Galois connection $(\uparrow, \downarrow)$:

$\uparrow : \prod_{b \in B} D_b \to \prod_{a \in A} C_a$

$$(\uparrow(g))(a) = \bigwedge_{b \in B} \psi_{a,b}(g(b)).$$

$\downarrow : \prod_{a \in A} C_a \to \prod_{b \in B} D_b$

$$(\downarrow(f))(b) = \bigwedge_{a \in A} \phi_{a,b}(f(a)).$$

Comparison

our heterogeneous approach	**Pócs connectional approach**
can be expressed by connectional *(using G-ideal)*	can be expressed by heterogeneous *(construction of* $\bullet_{a,b}$*)*
longterm and shortterm preferences	all information in Galois connections
metadata and data distinguished	metadata and data mixed
interpretation on an example	more difficult to interpret

Multi-adjoint formal context

- proposed by Medina, Ojeda-Aciego, Ruiz-Calviño, Formal Concept Analysis via multi-adjoint concept lattices, **Fuzzy sets and systems, 2009**
- **idea:** every object is associated with particular adjoint triple (inspiration for our extension)

B	nonempty set of objects
A	nonempty set of attributes
$(C, \leq_C)$	one complete lattice for objects
$(D, \leq_D)$	one complete lattice for attributes
$(P, \leq_P)$	one poset for table values
$R : A \times B \to P$	P-fuzzy relation
$\bullet_i : C \times D \to P,\ i \in \{1, \ldots, n\}$	conjunction in adjoint triple
$(\bullet_i, \to_{1_i}, \to_{2_i}),\ i \in \{1, \ldots, n\}$	system of adjoint triples
$\sigma : B \to \{1, \ldots, n\}$	objects associated with adjoint triples

Galois connection $(\uparrow^\sigma, \downarrow^\sigma)$:

$$\uparrow^\sigma : D^B \to C^A \qquad (\uparrow^\sigma(g))(a) = \bigwedge_{b \in B} R_{a,b} \to_{1_{\sigma(b)}} g(b).$$

$$\downarrow^\sigma : C^A \to D^B \qquad (\downarrow^\sigma(f))(b) = \bigwedge_{a \in A} R_{a,b} \to_{2_{\sigma(b)}} f(a).$$

Comparison

our heterogeneous approach	**Multi-adjoint approach**
using diverse lattice for diverse $b \in B$ and diverse lattice for diverse $a \in A$	using the same lattice[1] for all $b \in B$ and the same lattice for all $a \in A$
diversification of fuzzy operations for every object-attribute pair	diversification of fuzzy operations for every object (or every attribute)
concept-forming operators based on conjunction	concept-forming operators based on implications
a cottage example	a journal submission example

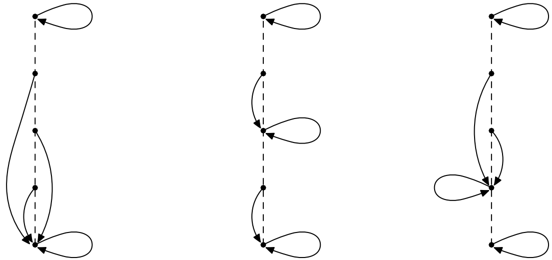
[1]: For generalization of such consideration see Medina, Ojeda-Aciego: On multi-adjoint concept lattices based on heterogeneous conjunctors, Fuzzy Sets and Systems, 2012

Conclusion

- full diversification of objects, attributes and table values
- focus on group preferences
- comparison with related approaches

Future inspiration:

- multilattices by Medina, Ojeda-Aciego, Ruiz-Calviño
- lattices with hedges by Bělohlávek et al.



Thank you for your attention

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objects \ attributes		water	services	lake
		cold, hot	no, in, tv, in+tv	no, yes
Eva	Sa+Su, 1/2, ∅	1, 0	1, 1/2, 0	1, 2/3, 1/3, 0
Joe	Sa+Su, Sa, Su, ∅	1, 1/2, 0	1, 2/3, 1/3, 0	1, 1/2, 0
Ken	Sa+Su, Sa, Su, ∅	1, 0	1, se, le, 0	1, 1/2, 0
Lea	Sa+Su, 1/2, ∅	1, 0	1, 1/2, 0	1, 2/3, 1/3, 0
Sue	Sa+Su, 1/2, ∅	1, 1/2, 0	1, se, le, 0	1, 0
Tim	Sa+Su, Sa, Su, ∅	1, 0	1, se, le, 0	1, 2/3, 1/3, 0