On estimation of approximation error on fuzzy sets by means of fuzzy valued integral

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The aim

- To define an L-fuzzy valued norm by using the L-fuzzy valued integral over an L-set with respect to an L-fuzzy valued measure μ.
- To describe the space $\mathscr{L}_1(E,\Sigma,\mu)$ of L-fuzzy integrable over a measurable L-set $E \in \Sigma$ real valued functions.
- To show possible applications of the introduced L-fuzzy valued norm in approximation theory:
  - approximation error estimation for a given function
  - approximation error estimation for a class of functions

L-fuzzy valued integral

We define an L-fuzzy valued integral

$$\int_E f \ d\mu,$$

where

- E is a measurable L-set, i.e. $E \in \Sigma$,
- $f : X \to \mathbb{R}$ is a non-negative measurable function with respect to σ-algebra of crisp sets Φ,
- μ is an L-fuzzy valued measure of L-sets.

L-fuzzy real numbers

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton.

Definition

An L-fuzzy real number is a function $z : \mathbb{R} \to L$ such that

- z is non-increasing;
- $\bigwedge\limits_{x} z(x) = 0_L$, $\bigvee\limits_{x} z(x) = 1_L$;
- z is left semi-continuous, i.e. $\bigwedge\limits_{t<x} z(t) = z(x)$.

$\mathbb{R}(L)$ - the set of all L-fuzzy real numbers (*the L-fuzzy real line*).
An L-fuzzy number z is called *non-negative* if $z(0) = 1_L$.
$\mathbb{R}_+(L)$ - the set of all non-negative L-fuzzy real numbers.

L-fuzzy real numbers

Operations with L-fuzzy real numbers such as addition $\oplus$ and multiplication by a real positive number are defined as follows:

$$(z_1 \oplus z_2)(t) = \bigvee_{\tau} \{z_1(\tau) \wedge z_2(t-\tau)\},\ (zr)(t) = z(\frac{t}{r}).$$

The supremum and the infimum of a set of non-negative L-fuzzy numbers $F \subset \mathbb{R}_+(L)$ are defined by the formulas:

$$(Inf\ \ F)(t) = \bigwedge \{z(t) \mid z \in F\},\ \ t \in \mathbb{R},$$

$$Sup\ \ F = Inf\{z \mid z \in \mathbb{R}(L), z \geq z' \text{ for all } z' \in F\}.$$

The schema of construction of an L-fuzzy valued measure

$\Phi \subset \mathbf{2}^X$, Φ - sigmaalgebra
$\nu : \Phi \to [0, +\infty[$, ν - finite measure
$\Downarrow$

By using fuzzy sets $A(M, \alpha)$ we construct the T_M-semiring $\wp = \{A(M, \alpha) | M \in \Phi \text{ and } \alpha \in L\}$ and define L-fuzzy valued elementary measure $m : \wp \to \mathbb{R}_+(L)$

$$m(A(M, \alpha)) = z(\nu(M), \alpha)$$

$\Downarrow$

On the next step we get a T_M-tribe Σ of measurable fuzzy sets and extend elementary measure m to the L-fuzzy valued measure μ defined on the T_M-tribe Σ.

L-fuzzy valued integral

By analogy with the classical case we define an L-fuzzy valued integral stepwise, first considering the case of simple non-negative measurable functions (for short SNMF):

$$\int\limits_E (\sum_{i=1}^{n} c_i \chi_{C_i}) \ d\mu = \bigoplus_{i=1}^{n} (c_i \ \mu(C_i \wedge E)),$$

whenever
$c_i \in \mathbb{R}_+, C_i \in \Phi, \chi_{C_i}$ is the characteristic function, $i \in \{1, ..., n\}$, and $C_1, ..., C_n$ are pairwise disjoint sets.
Then considering the case for non-negative measurable function f (for short NMF):

$$\int\limits_E f \ d\mu = Sup\{\int\limits_E g \ d\mu \,|\, g \leq f \text{ and } g \text{ is SNMF}\}.$$

L-fuzzy valued integral

For $\mathbb{I}_f = \int\limits_E f \ d\mu$ due to properties of the supremum of a set of L-fuzzy numbers, we have

- $\mathbb{I}_f$ is non-increasing,
- $\bigvee\limits_t \mathbb{I}_f(t) = 1_L$,
- $\mathbb{I}_f$ is left semi-continuous, i.e. $\bigwedge\limits_{t<t_0} \mathbb{I}_f(t) = \mathbb{I}_f(t_0)$.

Definition

We say that a non-negative measurable function f is L-fuzzy integrable iff

$$\bigwedge_t \mathbb{I}_f(t) = 0_L.$$

It holds when f is integrable on the set $SuppE$ with respect to ν.

Properties of an L-fuzzy valued integral

- $r \in \mathbb{R}_+ \Rightarrow \int\limits_E rf d\mu = r \int\limits_E f d\mu$
- $f_1 \le f_2 \Rightarrow \int\limits_E f_1 d\mu \le \int\limits_E f_2 d\mu$
- $E_1 \subset E_2 \Rightarrow \int\limits_{E_1} f \ d\mu \le \int\limits_{E_2} f \ d\mu$
- $(E_k)_{k\in\mathbb{N}} : E_k \le E_{k+1}$ and $\bigvee\limits_{k\in\mathbb{N}} E_k = E \Rightarrow$
$$\int\limits_E f d\mu = Sup\{\int\limits_{E_k} f d\mu \mid k \in \mathbb{N}\}$$
- $(f_n)_{n\in\mathbb{N}} : f_n \le f_{n+1}$ and $\lim\limits_{n\to\infty} f_n = f \Rightarrow$
$$\int\limits_E f \ d\mu = Sup\{\int\limits_E f_n \ d\mu \mid n \in \mathbb{N}\}$$
- $\int\limits_E (f_1 + f_2) d\mu = \int\limits_E f_1 d\mu \oplus \int\limits_E f_2 d\mu$
- $E_1 \wedge E_2 = \emptyset \Rightarrow \int\limits_{E_1 \vee E_2} f d\mu = \int\limits_{E_1} f d\mu \oplus \int\limits_{E_2} f d\mu$

L-fuzzy valued norm

For a given linear space Y by the analogy with the classical case we consider the concept of a norm taking values in $R_+(L)$ as following:

Definition

An L-fuzzy valued norm on a linear space Y is a function $\|\cdot\| : Y \to \mathbb{R}_+(L)$ with the following properties:
for all $r \in \mathbb{R}$ and all $y, y_1, y_2 \in Y$ it holds

- $\|y\| = z(0, 1_L) \Leftrightarrow y = 0_Y$,
- $\|ry\| = |r|\|y\|$,
- $\|y_1 + y_2\| \leq \|y_1\| \oplus \|y_2\|$.

Space $\mathscr{L}_1(E,\Sigma,\mu)$

We denote by $\mathscr{L}_1(E,\Sigma,\mu)$ the space of all L-fuzzy integrable over E real valued functions. We consider $\mathscr{L}_1(E,\Sigma,\mu)$ as a space equipped with the L-fuzzy valued norm defined as follows:

$$\|f\|_\mu = \int_E |f| \; d\mu,$$

where μ is an L-fuzzy valued measure and $E \in \Sigma$.
The function

$$\|\cdot\|_\mu : \mathscr{L}_1(E,\Sigma,\mu) \to \mathbb{R}_+(L)$$

satisfies the conditions of an L-fuzzy valued norm.

Approximation error

Let us suppose that $E \in \Sigma$ and $f \in \mathscr{L}_1(suppE, \Phi, \nu)$. We consider a method of approximation described by

$$\mathscr{A} : \mathscr{L}_1(suppE, \Phi, \nu) \to \mathscr{U},$$

where $\mathscr{U}$ is a space of functions used for approximation. Usually, it is finite-dimensional. For example, it could be a space of polynomials or splines.

Approximation error

Definition

The error of approximation $\mathscr{A}$ of a function f on an L-fuzzy set E is defined as follows:

$$e(f, \mathscr{A}, E) = \|f - \mathscr{A}f\|_\mu.$$

Notice that the error of approximation

$$e(f, \mathscr{A}, E) = \int_E |f - \mathscr{A}f| d\mu$$

is an L-fuzzy real number.

Numerical example

$L = [0,1]$, $X = [0,1]$, ν is the Lebesgue measure.
We consider the errors of approximation of the function (the Runge example)

$$f = \frac{1}{1+25x^2}$$

by two interpolation methods on two different L-sets.

Numerical example - Interpolation methods

We consider two methods of interpolation with respect to the uniform mesh on $[0, 1]$:

- approximation $\mathscr{A}_1$ by the Lagrange interpolation polynomial of degree 10,
- approximation $\mathscr{A}_2$ by the interpolation natural cubic spline with respect to the same mesh.

Numerical example - L-sets

Approximations are analyzed on two different L-sets E_1 and E_2:

$$E_1(x) = \begin{cases} 1, & x \in [0, 0.2], \\ 1.25(1-x), & x \in [0.2, 1], \end{cases}$$

$$E_2(x) = \begin{cases} 1.25x, & x \in [0, 0.8], \\ 1, & x \in [0.8, 1]. \end{cases}$$

Let us note that

$$\mu(E_1) = \mu(E_2) \text{ and } Supp(E_1) = Supp(E_2).$$

Numerical example

The errors $e(f, \mathscr{A}_j, E_i)$ of approximation of the function f on the L-set E_i by the method $\mathscr{A}_j$, $i = 1, 2$, $j = 1, 2$, are presented in the following table. Take into account that in the table we use the notation:

$$e(f, \mathscr{A}_j, E_i)(t) = \alpha.$$

Numerical example - Approximation error on L-set E_1

$e(f,\mathscr{A}_1,E_1)$		$e(f,\mathscr{A}_2,E_1)$	
t	α	t	α
$6.3508\cdot 10^{-4}$	0	$28.2092\cdot 10^{-4}$	0
$6.2584\cdot 10^{-4}$	0.1	$28.1501\cdot 10^{-4}$	0.1
$6.2388\cdot 10^{-4}$	0.2	$28.1325\cdot 10^{-4}$	0.2
$6.2302\cdot 10^{-4}$	0.3	$28.1258\cdot 10^{-4}$	0.3
$6.2245\cdot 10^{-4}$	0.4	$28.1216\cdot 10^{-4}$	0.4
$6.2193\cdot 10^{-4}$	0.5	$28.1087\cdot 10^{-4}$	0.5
$6.2141\cdot 10^{-4}$	0.6	$28.0879\cdot 10^{-4}$	0.6
$6.2079\cdot 10^{-4}$	0.7	$28.0013\cdot 10^{-4}$	0.7
$6.1984\cdot 10^{-4}$	0.8	$27.8377\cdot 10^{-4}$	0.8
$6.1775\cdot 10^{-4}$	0.9	$27.4130\cdot 10^{-4}$	0.9
$6.0979\cdot 10^{-4}$	1.0	$25.6889\cdot 10^{-4}$	1.0

Numerical example - Approximation error on L-set E_2

$e(f, \mathscr{A}_1, E_2)$	
t	α
$6.3508 \cdot 10^{-4}$	0
$0.9613 \cdot 10^{-4}$	0.1
$0.3556 \cdot 10^{-4}$	0.2
$0.2124 \cdot 10^{-4}$	0.3
$0.1658 \cdot 10^{-4}$	0.4
$0.1467 \cdot 10^{-4}$	0.5
$0.1381 \cdot 10^{-4}$	0.6
$0.1334 \cdot 10^{-4}$	0.7
$0.1296 \cdot 10^{-4}$	0.8
$0.1251 \cdot 10^{-4}$	0.9
$0.1173 \cdot 10^{-4}$	1.0

$e(f, \mathscr{A}_2, E_2)$	
t	α
$28.2092 \cdot 10^{-4}$	0
$10.4002 \cdot 10^{-4}$	0.1
$4.3225 \cdot 10^{-4}$	0.2
$1.6781 \cdot 10^{-4}$	0.3
$0.6252 \cdot 10^{-4}$	0.4
$0.2681 \cdot 10^{-4}$	0.5
$0.1353 \cdot 10^{-4}$	0.6
$0.1059 \cdot 10^{-4}$	0.7
$0.0953 \cdot 10^{-4}$	0.8
$0.0860 \cdot 10^{-4}$	0.9
$0.0816 \cdot 10^{-4}$	1.0

Numerical example - crisp case

Let us note that

$$suppE_1 = suppE_2 = [0,1]$$

and

$$e(f, \mathscr{A}_1, [0,1]) = 6.3509 \cdot 10^{-4},$$

$$e(f, \mathscr{A}_2, [0,1]) = 28.2092 \cdot 10^{-4}.$$

Numerical example

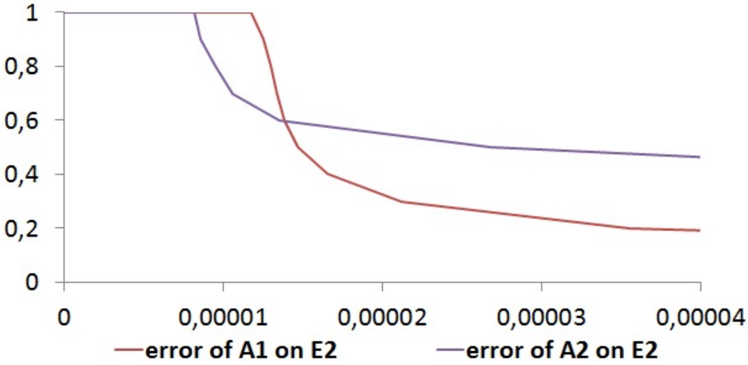


Figure: The graphs of the errors $\alpha = e(f, \mathscr{A}_1, E_2)(t)$ and $\alpha = e(f, \mathscr{A}_2, E_2)(t)$.

Conditions on functions

Now we consider functions f that satisfy the following conditions:

- there exists $(n-1)$ derivative $f^{(n-1)}$ and it is absolutely continuous on $[0,1]$,
- $|f^{(n)}|$ is integrable on $[0,1]$.

Conditions on approximation

We do the following assumptions regarding the choice of approximation method $\mathscr{A}$:

- for all $p \in P_{n-1}$ we have $\mathscr{A}p = p$, where P_{n-1} is a class of all polynomials with degree not greater then $(n-1)$;
- approximation $\mathscr{A}$ is linear;
- for $r(x) = \int\limits_0^1 g(u)\ h(x,u)\ du$ it holds

$$(\mathscr{A}\,r)(x) = \int\limits_0^1 g(u)\ (\mathscr{A}\,h(x,u))du,$$

where approximation $\mathscr{A}$ is applied only to argument x of function h.

Approximation error - Integral representation

$$f(x) - (\mathscr{A}f)(x) = \int\limits_0^1 f^{(n)}(u) U_{n-1}(x,u)\ du.$$

$$U_{n-1}(x,u) = \frac{\varphi_{n-1}(x,u) - \mathscr{A}\,\varphi_{n-1}(x,u)}{(n-1)!}.$$

$$\varphi_{n-1}(x,u) = (x-u)_+^{n-1} = \begin{cases} (x-u)^{n-1}, & x \geq u; \\ 0, & u > x. \end{cases} .$$

Approximation error for classes of functions

$f \in KW_1^n$	$f \in KW_\infty^n$
$\int_0^1 \lvert f^{(n)}(u) \rvert \, du \le K$	$\sup_{u \in [0,1]} \lvert f^{(n)}(u) \rvert \le K$

Approximation error

$$e(KW_r^n, \mathscr{A}, E) = Sup\{\|f - \mathscr{A}f\|_\mu \mid f \in KW_r^n\}.$$

Approximation error for classes of functions

$$e(KW_1^n, \mathscr{A}, E) \leq K \int\limits_E \sup_{u \in [0,1]} |U_{n-1}(x,u)| \; d\mu.$$

$$e(KW_\infty^n, \mathscr{A}, E) \leq K \int\limits_E (\int\limits_0^1 |U_{n-1}(x,u)| \; du) \; d\mu.$$

Numerical example

We examine the approximation $\mathscr{A}$ by a polygons (i.e. first degree spline) with respect to the uniform mesh $\{x_0, x_1, ..., x_{10}\}$ on $[0,1]$ over L-set E defined as follows:

$$E(x) = \begin{cases} 1 - x, & x \in [0,1], \\ 0, & otherwise. \end{cases}$$

Numerical example - Class KW^1_∞

Approximation error for class KW^1_∞ is bounded as follows

$$e(KW^1_\infty, \mathscr{A}, E) \leq K \int_E \left(\int_0^1 |U_0(x,u)| \; du \right) \; d\mu.$$

Denoting

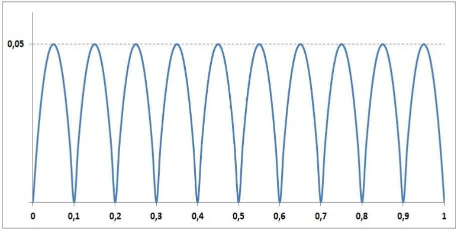
$$z_\infty = \int_E \left(\int_0^1 |U_0(x,u)| \; du \right) \; d\mu$$

we obtain

$$e(KW^1_\infty, \mathscr{A}, E) \leq K \; z_\infty.$$

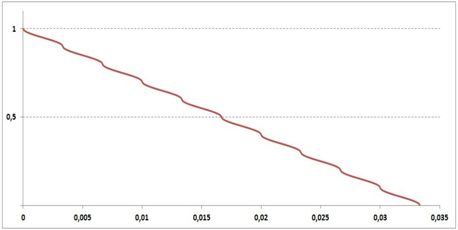
Numerical example - Class KW^1_∞

$$\int_0^1 |U_0(x,u)|\ du = 2h\left\{\frac{x}{h}\right\}\left(1-\left\{\frac{x}{h}\right\}\right)$$



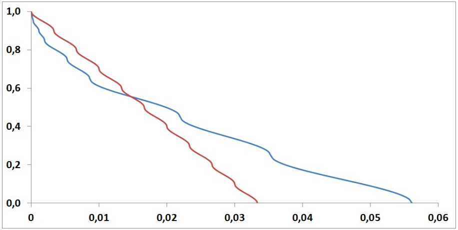
Numerical example - Class KW^1_∞

$$z_\infty^{-1}(\alpha) = \frac{h^2}{3}\left(\left[\frac{1-\alpha}{h}\right] + \left\{\frac{1-\alpha}{h}\right\}^2 \left(3 - 2\left\{\frac{1-\alpha}{h}\right\}\right)\right).$$



Numerical example - Class KW^1_∞ - not uniform mesh

$$z_\infty^{-1}(\alpha) = \sum_{j=1}^{i-1} \frac{h_j^2}{3} + \frac{h_i^2}{3}\left(\frac{1-\alpha-x_{i-1}}{h}\right)^2\left(3-2\left(\frac{1-\alpha-x_{i-1}}{h}\right)\right).$$



Thank you for attention!!!

Appendix

L-fuzzy valued measure

Definition

Let Σ be a T_M-tribe. A function $\mu : \Sigma \to \mathbb{R}_+(L)$ is called an L-fuzzy valued measure if it satisfies the following conditions:

- $\mu(\emptyset) = z(0, 1_L)$;
- μ is T_M-valuation, i.e. for all $A, B \in \Sigma$ it holds $\mu(A \wedge B) \oplus \mu(A \vee B) = \mu(A) \oplus \mu(B)$;
- μ is left T_M-continuous, i.e. $\bigvee\limits_{n \in \mathbb{N}} \mu(A_n) = \mu(A)$, where $(A_n)_{n \in N} \subset \Sigma$, $\bigvee\limits_{n \in \mathbb{N}} A_n = A \in \Sigma$.

Calculation method for L-fuzzy valued integral (case when $L = [0, 1]$)

The main idea of the method is based on the following reasoning.

- The fuzzy set we want to integrate over can be viewed as a non-negative function.
- Let us assume that this function is measurable with respect to σ-algebra F. It is known that every non-negative measurable function can be presented as a limit of a non-decreasing sequence of SNMF.
- Obviously, every fuzzy set that is SNMF can be presented as a union of T_M-disjoint fuzzy sets from the class $\wp$. And integral over element from the class $\wp$ can be easily calculated.

Integration over $E = E(M, a) \in \wp$

For all $E(M,\alpha) \in \wp$ it holds

$$\int\limits_{E(M,\alpha)} f d\mu = z(\int\limits_{M} f \ d\nu, \alpha).$$

Integration over E - SNMF

If E is SNMF then $E(\mathbb{R}) = \{\alpha_1, ..., \alpha_n\}$. We assume that

$$\alpha_1 > \alpha_2 > ... > \alpha_n$$

$$\int_E f \ d\mu = \begin{cases} 1, \ t \leq \int\limits_{E^{\alpha_1}} f d\nu \\ ... \\ \alpha_i, \int\limits_{E^{\alpha_i}} f d\nu < t \leq \int\limits_{E^{\alpha_{i+1}}} f d\nu \\ ... \\ \alpha_n, \int\limits_{E^{\alpha_{n-1}}} f d\nu < t \leq \int\limits_{E^{\alpha_n}} f d\nu \\ 0, \ otherwise \end{cases}$$

Integration over E - NMF

As was already mentioned every NMF can be presented as a limit of a non-decreasing sequence of SNMF.
$E = \bigvee_n E_n$ where $(E_n)_{n \in \mathbb{N}}$ is non-decreasing sequence.
Denoting $I = \int_E f \ d\mu$ and $I_n = \int_{E_n} f \ d\mu$ we get

$$I = Sup\{I_n \mid n \in \mathbb{N}\}.$$

From the last equality we can get approximate value of integral by fixing n. Obviously, integral accuracy in this case will be dependent on the n value.

Space $\mathscr{L}_p(E,\Sigma,\mu)$

We consider Space $\mathscr{L}_p(E,\Sigma,\mu)$ where $1 \leq p \leq \infty$ with the norm $\|\cdot\|_p$ defined as follows:

$$\|f\|_p = \left(\int_0^1 |f|^p dx\right)^{\frac{1}{p}}, \text{ where } 1 \leq p < \infty ,$$

and

$$\|f\|_p = \sup_{x\in[0,1]} |f(x)| , \text{ where } p = \infty .$$