Strong Law of Large Numbers on the Kôpka D-posets

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Definition

The structure $(D, \leq, \ominus, 0, 1)$ *is called a D-poset if the relation* $\leq$ *is a partial ordering on* D, 0 *is the smallest and* 1 *is the largest element on* D *and*

1. $b \ominus a$ *is defined iff* $a \leq b$,
2. *if* $a \leq b$ *then* $b \ominus a \leq b$ *and* $b \ominus (b \ominus a) = a$,
3. $a \leq b \leq c \Rightarrow c \ominus b \leq c \ominus a, (c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Definition

Let $(D, \leq, \ominus, 0, 1)$ be a D-poset. It is called the Kôpka D-poset, if there is a binary operation $* : D \times D \to D$, which is commutative, associative and has the following properties:

1. $a * 1 = a, \forall a \in D$;
2. $a \leq b \Rightarrow a * c \leq b * c, \forall a, b, c \in D$;
3. $a \ominus (a * b) \leq 1 \ominus b, \forall a, b \in D$.

Definition

D-poset is called σ-complete if every subset of countable elements has the supremum and the infimum.

Definition

Kôpka D-poset $(D, \ominus, *, 0, 1)$ is called continuous if the following holds:

$$a_n \nearrow a \Rightarrow b * a_n \nearrow b * a, \forall a_n, a, b \in D.$$

Definition

A state on a D-poset D is any mapping $m : D \to [0,1]$ satisfying the following properties:

1. $m(1) = 1,\ m(0) = 0;$
2. $a_n \nearrow a \Rightarrow m(a_n) \nearrow m(a), \forall a_n, a \in D;$
3. $a_n \searrow a \Rightarrow m(a_n) \searrow m(a), \forall a_n, a \in D.$

The state m is called additive, if

$$a, b \in D, a \leq b \Rightarrow m(b) = m(a) + m(b \ominus a).$$

Remark

If D is a D-lattice, then the state m is additive if

$$m(a \vee b) = m(a) + m(b)$$

whenever $a \wedge b = 0,\ a, b \in D.$

The Basic Notions

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If D is a σ-complete D-lattice, then m is called σ-additive, if

$$m\left(\bigvee_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$$

whenever $a_i \wedge a_j = 0 \ (i \neq j), \ a_i \in D \ (i = 1, 2, \ldots)$.

Proposition

Let D be a σ-complete D-lattice, $m : D \to [0,1]$ be a state. If the state m is additive, then it is σ-additive.

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Let $\mathcal{J} = \{(-\infty, t); t \in \mathbb{R}\}$*. An observable on* D *is any mapping* $x : \mathcal{J} \to D$ *satisfying the following conditions:*

1. $A_n \nearrow \mathbb{R} \Rightarrow x(A_n) \nearrow 1$,
2. $A_n \searrow \emptyset \Rightarrow x(A_n) \searrow 0$,
3. $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$.

Definition

Let $x : \mathcal{J} \to D$ *be an observable on a* σ*-complete* D*-poset* D *and* $a, b \in \mathbb{R}$*. Then*

$$x([a,b)) = x((-\infty, b)) \ominus x((-\infty, a)).$$

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Observables

Definition

Let $x : \mathcal{J}^* \to D$ *be an observable in a D-poset* D, α, β, t *be real numbers,* $\alpha > 0$. *Then*

$$(\alpha x + \beta)((-\infty, t)) = x\left((-\infty, \frac{t-\beta}{\alpha})\right)$$

and

$$(\alpha x + \beta)([-t, t)) = x\left([-\frac{t+\beta}{\alpha}, \frac{t-\beta}{\alpha})\right).$$

A Construction of the Distribution Function

Theorem

Let $x : \mathcal{J} \to D$ be an observable, $m : D \to [0,1]$ be a state. Define a mapping $F : \mathbb{R} \to [0,1]$ by the formula

$$F(t) = m(x((-\infty, t))).$$

Then F is a distribution function.

There exists exactly one probability measure $\lambda_F : \mathcal{B}(\mathbb{R}) \to [0,1]$ such that

$$\lambda_F([a,b)) = F(b) - F(a)$$

for any $a, b \in \mathbb{R}, a < b$.

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Definition

An observable $x : \mathcal{J} \to D$ is integrable, if there exists

$$E(x) = \int_{\mathbb{R}} t dF(t),$$

where F is the distribution function of x. It is square integrable, if there exists the dispersion

$$\sigma^2(x) = \int_{\mathbb{R}} t^2 dF(t) - E(x)^2.$$

Independent random variables

$\xi, \eta : \Omega \to \mathbb{R}$:

$$P(\xi^{-1}(A) \cap \eta^{-1}(B)) = P(\xi^{-1}(A)) \cdot P(\eta^{-1}(B))$$

$\forall A, B \in \mathcal{B}(\mathbb{R})$

Product of measures

There exists exactly one probability measure

$$\lambda_{F_1} \times \lambda_{F_2} : \mathcal{B}(\mathbb{R}^2) \to [0,1]$$

such that

$$\lambda_{F_1} \times \lambda_{F_2}(A \times B) = \lambda_{F_1}(A) \cdot \lambda_{F_2}(B)$$

for any $A, B \in \mathcal{B}(\mathbb{R})$.

Independence in a classic probability space $(\Omega, \mathcal{S}, P)$

Probability distribution of the sum $\xi + \eta$

$$P(\{\omega; \xi(\omega) + \eta(\omega) < t\}), t \in \mathbb{R}.$$

$$T: \ \Omega \to \mathbb{R}^2, \ \ T(\omega) = (\xi(\omega), \eta(\omega)),$$

$$g: \ \mathbb{R}^2 \to \mathbb{R}, \ \ g(u, v) = u + v.$$

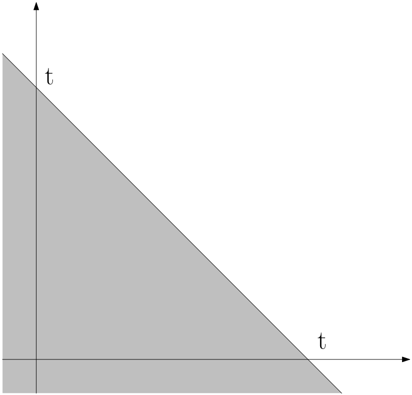
$$\xi(\omega) + \eta(\omega) < t,$$

$$\Leftrightarrow$$

$$\omega \in T^{-1}(g^{-1}((-\infty, t))).$$

$$P(\{\omega, \xi(\omega) + \eta(\omega) < t\}) = P(T^{-1}(g^{-1}((-\infty, t)))).$$

Independence in a classic probability space $(\Omega, \mathcal{S}, P)$



$$g^{-1}((-\infty, t)) = \Delta_t = \{(u, v) \in \mathbb{R}^2; u + v < t\}$$

Independence in a classic probability space $(\Omega, \mathcal{S}, P)$

Theorem

Let $\xi, \eta : \Omega \to \mathbb{R}$ *be independent random variables,* $\Delta_t = \{(u, v) \in \mathbb{R}^2; u + v < t\}, t \in \mathbb{R}, T = (\xi, \eta) : \Omega \to \mathbb{R}^2$. *Then*

$$P(T^{-1}(\Delta_t)) = \lambda_{F_1} \times \lambda_{F_2}(\Delta_t)$$

for any $t \in \mathbb{R}$.

Definition

Let D be a D-poset, $x_1, \ldots, x_n : \mathcal{J}^* \to D$ be observables,
$\Delta_t^n = \{(u_1, \ldots, u_n) \in \mathbb{R}^n; u_1 + \ldots + u_n < t\}, \mathcal{M}_n = \{\Delta_t^n; t \in \mathbb{R}\}$.
The observables are called to be independent, if there exists a mapping $h_n : \mathcal{M}_n \to D$ with the following properties:

1. $t \nearrow t_0 \Rightarrow h_n(\Delta_t^n) \nearrow h_n(\Delta_{t_0}^n)$,
2. $t \nearrow \infty \Rightarrow h_n(\Delta_t^n) \nearrow 1$,
3. $t \searrow -\infty \Rightarrow h_n(\Delta_t^n) \searrow 0$,
4. $m(h_n(\Delta_t^n)) = \lambda_{F_1} \times \ldots \times \lambda_{F_n}(\Delta_t^n), t \in \mathbb{R}$.

Theorem

Let D be a σ-complete continuous Kôpka D-poset, $m : D \to [0,1]$ be an additive state, $x, y : \mathcal{J}^* \to D$ be observables and let

$$m(x(A) * y(B)) = m(x(A)) \cdot m(y(B))$$

$\forall A, B \in \mathcal{J}^*$. Then x, y are independent in the sense of the previous Definition.

Riečan, Lašová, 2010

$$\alpha_{t,n}^2 = \{(i,j); \frac{1}{2^n}(i+j) < t\}$$

$$h_2(\Delta_t^2) = \bigvee_{n=1}^{\infty} \bigvee_{(i,j)\in\alpha_{t,n}^2} x\left([\frac{i-1}{2^n}, \frac{i}{2^n})\right) * y\left([\frac{j-1}{2^n}, \frac{j}{2^n})\right)$$

Theorem

Define $y_n : \mathcal{J}^* \to D$ *by the equality* $y_n((-\infty, t)) = h_n(\Delta_t^n)$. *Then* y_n *is an observable.*

Definition

Let $x_1, \ldots, x_n : \mathcal{J}^* \to D$ *be independent observables. Then the observable* $y_n : \mathcal{J}^* \to D$ *defined in previous Theorem is called the sum of observables* $x_1, \ldots, x_n$, $y_n = \sum_{i=1}^n x_i$, *i.e.*

$$(\sum_{i=1}^{n} x_i)((-\infty, t)) = h_n(\Delta_t^n), t \in \mathbb{R}.$$

Definition

Let $(x_n)_{n=1}^{\infty}$ *be a sequence of observables on a* σ*-complete D-poset* D *with a state* m*. We say that this sequence converges in measure* m *to* 0 *if for each* $0 < \epsilon \in \mathbb{R}$

$$\lim_{n\to\infty} m\left(x_n((-\epsilon,\epsilon))\right) = 1$$

and that it converges to zero m*-almost everywhere if*

$$\lim_{l\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\left(\bigwedge_{n=k}^{k+i} x_n((-\frac{1}{l},\frac{1}{l}))\right) = 1.$$

Kolmogorov's construction

Probability space $(\mathbb{R}^{\mathbb{N}}, \sigma(\mathcal{C}), P)$

$$\mathcal{C} = \{A \subset \mathbb{R}^{\mathbb{N}}; A = \pi_n^{-1}(B), B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}\}$$

$$P_n(A \times \mathbb{R}) = P_{n-1}(A), A \in \mathcal{B}(\mathbb{R}^{n-1}), n \in N$$

$$\pi_n((u_i)_{i=1}^{\infty}) = (u_1, u_2, \ldots, u_n).$$

By the Kolmogorov consistence theorem there exists a probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\pi_n^{-1}(B)) = P_n(B) = \lambda_{F_1} \times \ldots \times \lambda_{F_n}(B)$$

for any $B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}$.

"Translation formulas"

Random variable $\xi_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$

$$\xi_n((u_i)_{i=1}^{\infty}) = u_n$$

$$P(\xi_1 + \ldots + \xi_n < t) = m\left((\sum_{i=1}^{n} x_i)((-\infty, t)) \right)$$

$$P(a \leq \xi_1 + \ldots + \xi_n < b) = m\left((\sum_{i=1}^{n} x_i)([a, b)) \right)$$

Theorem (Central limit theorem)

Let D be a σ-complete D-poset, $(x_n)_{n=1}^{\infty}$ be an independent sequence of equally distributed square integrable observables, $E(x_n) = a,\ \sigma^2(x_n) = \sigma^2,\ (n = 1, 2, \ldots)$. Then for any $t \in \mathbb{R}$

$$\lim_{n\to\infty} m\left(\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - a}{\frac{\sigma}{\sqrt{n}}}((-\infty, t))\right) = \Phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{x^2}{2}}\,dx.$$

Theorem (Weak law of large numbers)

Let D be a D-poset with an additive state $m : D \to [0,1]$, let $(x_n)_{n=1}^{\infty}$ be an independent sequence of integrable observables having the same probability distribution, $E(x_n) = a,\ (n = 1, 2, \ldots)$. Then the sequence

$$\frac{\sum_{i=1}^{n} x_i}{n} - a$$

converges in measure m to 0.

Theorem (Strong law of large numbers)

Let D be a σ-complete D-poset, $(x_n)_{n=1}^{\infty}$ be a sequence of independent, square integrable observables, such that $\frac{\sum_{n=1}^{\infty}\sigma^2(x_n)}{n^2} < \infty$. Then the sequence of observables

$$\frac{x_1 - E(x_1) + x_2 - E(x_2) + \ldots + x_n - E(x_n)}{n} \quad (n = 1, 2, 3, \ldots)$$

converges m-almost everywhere to 0.

Idea of the proof

$$\eta_n = \frac{\xi_1 - E(\xi_1) + \ldots + \xi_n - E(\xi_n)}{n},$$

$$y_n = \frac{\sum_{i=1}^{n} x_i - (E(x_i) + \ldots + E(x_n))}{n}$$

$$P\left(\bigcap_{n=k}^{k+i} \eta_n^{-1}\left([-\frac{1}{l}, \frac{1}{l})\right)\right) \leq m\left(\bigwedge_{n=k}^{k+i} y_n\left([-\frac{1}{l}, \frac{1}{l})\right)\right)$$

(Riečan – Neubrunn, 1997)