Observables on σ-lattice effect algebras

Sylvia Pulmannová

Mathematical Institute, Slovak Academy of Sciences

Bratislava, Slovakia[a]

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Introduction

Prototype of effect algebras – the set of quantum effects – self-adjoint operators A, $0 \leq A \leq I$.

Quantum effects – sharp and unsharp properties of physical systems.

Special subclasses of effect algebras:
– quantum logics – orthomodular posets and lattices,
– MV-algebras – algebraic bases for many-valued logic.

Lattice ordered effect algebras – a common generalization of MV-algebras and orthomodular lattices.

Effect algebras

An *effect algebra* – a partial algebraic structure $(E; \oplus, 0, 1)$,
$\oplus$ – a partial binary operation;
$0, 1$ – constants:

(E1) $a \oplus b = b \oplus a$;

(E2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(E3) for every $a \in E$ there is a unique $a^\perp$ such that $a \oplus a^\perp = 1$;

(E4) if $a \oplus 1$ is defined, then $a = 0$.

ordering, summation

- orthogonality: $a \perp b \Leftrightarrow a \oplus b$ is defined;
- partial ordering: $a \leq b \Leftrightarrow \exists c \in E : a \oplus c = b$;
- $\forall a \in E$, $0 \leq a \leq 1$, $a \perp b \Leftrightarrow a \leq b^{\perp}$;
- subtraction: $c = b \ominus a \Leftrightarrow a \oplus c = b$;
- $\oplus$-sum: $a_1, a_2, \ldots, a_n \in E$,
  $a_1 \oplus a_2 \oplus \ldots \oplus a_n := (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$;
- $(a_i)_{i \in I} \subseteq E$, $\mathcal{F}(I) := \{F \subseteq I, F - \text{finite}\}$,
  $\bigoplus_{i \in I} a_i := \bigvee_{F \in \mathcal{F}(I)} \oplus_{i \in F} a_i$, (if exists).

E is *σ-orthocomplete* iff $\oplus_{i \in I} a_i$ exists provided I is countable, and $\oplus_{i \in F} a_i$ exists $\forall F \in \mathcal{F}(I)$.

states, observables

• A *state* on E: $m : E \to [0,1] \subseteq \mathbb{R}$,
(1) $m(1) = 1$,
(2) $m(a \oplus b) = m(a) + m(b)$;
σ-additive: $m(\bigoplus_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$.

• An *observable* on E: $\xi : \mathcal{A} \to E$ (where $(\Omega, \mathcal{A})$– measurable space – the *value space* of ξ,
(1) $\xi(\Omega) = 1$,
(2) $\xi(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} \xi(A_i)$,
$\forall (A_i)_{i=1}^{\infty} \subseteq \mathcal{A}$, $A_i \cap A_j = 0, i \neq j$.
• $m \circ \xi : \mathcal{A} \to [0,1] \subseteq \mathbb{R}$ – a probability measure – the *distribution of ξ in m*.

sharp and unsharp observables

• $a \in E$: a is *sharp* iff $a \wedge a^{\perp} = 0$, otherwise a is *unsharp*.
• An observable $\xi : \mathcal{A} \to E$ is *sharp* iff $ran(\xi) := \{\xi(A) : A \in \mathcal{A}\}$ consists of sharp elements, otherwise ξ is *unsharp*.
• An observable ξ is *real* iff its value space $(\Omega, \mathcal{A}) \subseteq (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
Expectation of a real observable ξ in a state m:

$$m(\xi) = \int_{\mathbb{R}} t m \circ \xi(dt).$$

compatibility, blocks

• $a, b \in E$ are *compatible* iff $\exists a_1, b_1, c \in E$:
$\exists a_1 \oplus b_1 \oplus c$, $a = a_1 \oplus c, b = b_1 \oplus c$.

• E – a *lattice effect algebra* (LEA) iff $(E; \leq)$ is a lattice.

• A LEA is covered by maximal sets of pairwise compatible elements called *blocks*, which are MV-algebras.

• A LEA is an MV-algebra iff any two elements in E are compatible.

• A LEA is σ-orthocomplete (a σ-LEA) iff it is a σ-lattice.
Blocks in a σ-LEA are σ-MV algebras.

observables on LEAs

• E – a σ-orthocomplete LEA, $S_h(E)$ – the set of all sharp elements in E is a σ-OML.

• $\mathcal{S}_\sigma(E)$ – the set of all σ-additive states on E, assume $\mathcal{S}_\sigma(E) \neq \emptyset$.

M – a block of E, then $\mathcal{S}_\sigma(E) \subseteq \mathcal{S}_\sigma(M)$.

Theorem 1. *The range of every observable on E is contained in a block of E.*

• ξ – $(Z, \mathcal{F})$-observable with range in a block M, $\mathcal{P}(\xi) := \{m \circ \xi : m \in \mathcal{S}_\sigma(E)\}$ – a set of probability measure on $(Z, \mathcal{F})$.

L-S theorem

Theorem 2. *For every σ-MV algebra M there exists a tribe $\mathcal{T}$ of fuzzy sets on a set X and an MV-σ-homomorphism h from $\mathcal{T}$ onto M.*

• $(X, \mathcal{T}, h)$ – *standard Loomis-Sikorski representation of M.*

• A *tribe of fuzzy sets* on a set $X \neq \emptyset$ is a nonempty system $\mathcal{T} \subseteq [0,1]^X$ such that
(T1) $1_X \in \mathcal{T}$;
(T2) $f \in \mathcal{T} \implies 1_X - f \in \mathcal{T}$;
(T3) $(f_n)_{n=1}^{\infty} \subseteq \mathcal{T} \implies \min(\sum_{n=1}^{\infty} f_n, 1) \in \mathcal{T}$.

• $\mathcal{T}$ is a σ-MV algebra closed under pointwise suprema of sequences of its elements.

• $S_h(\mathcal{T})$ – sharp elements in $\mathcal{T}$ – a σ-algebra of sets.

weak Markov kernel

$(Z, \mathcal{F}), (Y, \mathcal{G})$ – measurable spaces,
$\mathcal{P}$ – a family of probability measures on $(Z, \mathcal{F})$.
• $\nu : Z \times \mathcal{G} \to [0, 1]$ is a *weak Markov kernel (WMK) w.r.* $\mathcal{P}$ if:
(i) $z \mapsto \nu(z, G)$ is $\mathcal{F}$-measurable;
(ii) $\forall G \in \mathcal{G}$, $0 \leq \nu(z, G) \leq 1$ $\mathcal{P}$–a.e.;
(iii) $\nu(z, Y) = 1, \nu(z, \emptyset) = 0$ $\mathcal{P}$ –a.e.;
(iv) if $(G_n)_n$ is a sequence in $\mathcal{G}$ such that $G_n \cap G_m = \emptyset (n \neq m)$, then

$$\nu(z, \bigcup_n G_n) = \sum_n \nu(z, G_n),\ \mathcal{P} - \text{a.e.}.$$

smearing

• $(Z, \mathcal{F})$, $(Y, \mathcal{G})$ – measurable spaces;
• ξ – a $(Z, \mathcal{F})$-observable;
• $\nu : Z \times \mathcal{G} \to [0, 1]$ – a WMK w.r. $\mathcal{P}(\xi)$. Then:
$\psi : \mathcal{P}(\xi) \to \mathcal{P}(Y, \mathcal{G})$,

$$\psi(m \circ \xi)(G) := \int_Z \nu(z, G) m \circ \xi(dz)$$

is called the *smearing of* ξ *w.r.* ν.
• If there is an $(Y, \mathcal{G})$-observable η on E such that
$m(\eta(G)) = \psi(m \circ \xi)(G)$, $G \in \mathcal{G}$,
we say that η *is defined by the smearing of* ξ *w.r.* ν.

basic observable

E – a σ-LEA, ξ – $(\Omega, \mathcal{A})$-observable, $ran(\xi) \in M$;
$(X, \mathcal{T}, h)$ – standard LS for M.
$\forall A \in \mathcal{A}$, $\xi(A) = h(f_A)$, $f_A \in \mathcal{T}$ is $S_h(\mathcal{T})$-measurable and unique up to h-null sets.

- $\nu : X \times \mathcal{A} \to [0,1]$, $\nu(x, A) := f_A(x)$ – a WMK w.r. $\{m \circ h : m \in \mathcal{S}_\sigma(M)\}$.
- $m(\xi(A)) = m(h(f_A)) = \int_X f_A(x) P(dx)$,
- $P := m \circ h / S_h(\mathcal{T})$,
- $h/S_h(\mathcal{T}) : S_h(\mathcal{T}) \to S_h(M)$ – a sharp observable on E, so-called *basic observable*.

Theorem 3. *Every observable on a σ-LEA is defined by a smearing of a basic observable.*

spectral measures

$\forall a \in E, \exists M : a \in M$, by LS, $a = h(f_a), f_a \in \mathcal{T}$.

• $\Lambda_a : \mathcal{B}(\mathbb{R}) \to E$, $\Lambda_a(B) = h(f_a^{-1}(B))$ – a sharp real observable.

• $a \mapsto \Lambda_a$ is one-to-one and does not depend on the block M containing a.

$\Longrightarrow$

elements of E are in one-to-one correspondence with a subclass of sharp real observables on E.

• $m(a) = \int_X f_a(x) m \circ h(dx) = \int_0^1 t m \circ h(f^{-1}(dt)) = \int_0^1 t m(\Lambda_a)(dt)$.

sharp real observables

Theorem 4. *A real observable ξ on E is sharp iff there is a block M and a measurable function $f : X \to [0,1]$ such that $\xi(B) = h(f^{-1}(B)), B \in \mathcal{B}(\mathbb{R})$. Moreover, $\xi = \Lambda_a, a \in M$ iff $f \in \mathcal{T}$.*

- M– a block of E, $(X, \mathcal{T}, h)$ – LS representation of M,
- ξ – $(\Omega, \mathcal{A})$ -observable with $ran(\xi) \subseteq M$,
- $\tilde{\xi} : A \mapsto \Lambda_{\xi(A)}, A \in \mathcal{A}$.

$\forall A \in \mathcal{A}$, $\tilde{\xi}(A) = \Lambda_{\xi(A)}$ – a real observable on the σ-OML $S_h(E)$ of sharp elements of E.

properties of $\tilde{\xi}$

Theorem 5. *For every observable* ξ*, the mapping* $\tilde{\xi} : A \mapsto \Lambda_{\xi(A)}$ *has the following properties:*

(GO1) $\tilde{\xi}(\Omega) = \Lambda_1$*, where* Λ_1 *is the (unique) observable on* $Sh(E)$ *with* $\Lambda_1(\{1\}) = 1$*;*

(GO2) *if* $A_1, \ldots, A_n$ *are pairwise disjoint elements of* $\mathcal{A}$*, then* $\tilde{\xi}(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \tilde{\xi}(A_i)$*, where the latter sum is given by the functional calculus for compatible observables;*

(GO3) *for any* $A_n, A \in \mathcal{A}$*,* $n \in \mathbb{N}$ *such that* $A_n \nearrow A$*, we have* $\tilde{\xi}(A_n) \to \tilde{\xi}(A)$ *everywhere.*

generalized observables

Definition 6. *Let* $(\Omega, \mathcal{A})$ *be a measurable space. A mapping* Ξ *from* $\mathcal{A}$ *to a compatible set of real observables on* $S_h(E)$ *with properties* (GO1), (GO2), (GO3) will be called an $(\Omega, \mathcal{A})$-*generalized observable* on E.
If $\Xi = \tilde{\xi}$ for an observable ξ, we say that Ξ is *associated* with ξ.
Define

$$P^m_\Xi(A) = \int_{\mathbb{R}} t m \circ \Xi(A)(dt), A \in \mathcal{A}, m \in \mathcal{S}_\sigma(E).$$

- $A \mapsto P^m_\Xi(A)$ is a probability measure on $(\Omega, \mathcal{A})$.
- $P_{\tilde{\xi}}(A) = m(\xi(A))$.

smearing of gen. obs.

$(Z, \mathcal{F})$, $(Y, \mathcal{G})$ – measurable spaces,
Ξ – $(Z, \mathcal{F})$-generalized observable on E,
$\mathcal{P}(\Xi) := \{P_\Xi^m : m \in \mathcal{S}_\sigma(E)\}$,
$\nu : Z \times \mathcal{G} \to [0, 1]$ – WMK w. r. $\mathcal{P}(\Xi)$.
• $\psi : \mathcal{P}(\Xi) \to \mathcal{P}(Y, \mathcal{G})$,

$$\psi(P_\Xi^m)(G) := \int_Z \nu(z, G) P_\Xi^m(dz)$$

will be called the *smearing* of Ξ w.r. to ν.
• If there is a $(Y, \mathcal{G})$-generalized observable Θ on E with $P_\Theta^m(G) = \psi(P_\Xi^m)(G), G \in \mathcal{G}$, we say that Θ is *defined by the smearing of* Ξ *with respect to* ν.

properties of smearings

• The smearing of $\tilde{\xi}$ coincides with the smearing of ξ.

• Every generalized observable Ξ is defined by a smearing of a basic observable h corresponding to a block M.

• If there is a faithful σ-additive state on E, then the system of generalized observables is closed under smearings.
A state m is *faithful* iff $m(a) = 0 \implies a = 0$.

comparison LEA - E(H)

$\mathcal{E}(H)$ – Hilbert space effects (operators $A : H \to H$, $0 \leq A \leq I$);

• observables on $\mathcal{E}(H)$ – *POVM*s with ranges in $\mathcal{E}(H)$;

• sharp observables on $\mathcal{E}(H)$ – *PVM*s with ranges in $\mathcal{P}(H)$ (projections);

• on $\mathcal{E}(H)$, every generalized observable is associated with an observable;

• every smearing of an observable is again an observable.

• $\mathcal{E}(H)$ is not a lattice, but is covered by MV-algebras – maximal sets of commuting effects;

• a POVM ξ is a smearing of a PVM $\eta \Leftrightarrow ran(\xi)$ is commutative;

• there are POVMs with noncommutative ranges.

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