Copulas and integrals

Radko Mesiar

Faculty of Civil Engineering
Slovak University of Technology
Radlinského 11, 813 68 Bratislava, Slovakia
radko.mesiar@stuba.sk

The stochastic dependence structure of bivariate random variables captured by copulas can be applied in the integration approaches. Indeed, a copula $C : [0,1]^2 \to [0,1]$ can express the connection between function values and measure values through an integral $I_{C,m}$ by $I_{C,m}(a\,1_A) = C(a,\, m(A))$, where m is the (monotone) measure and $a \in [0,1]$ is a constant. A class of copula–based universal integrals (they are well defined for any measurable space $(X, \mathcal{A})$, any normed monotone measure $m : \mathcal{A} \to [0,1]$ and any $\mathcal{A}$–measurable function $f : X \to [0,1]$) was introduced in [1] by

$$I_{C,m}(f) = P_C\left(\left\{(x,y) \in [0,1]^2 \mid y \le m(f \ge x)\right\}\right),$$

where P_C is the probability measure on Borel subsets of $[0,1]^2$ induced by C. As special instances recall that $C = \mathit{\Pi}$ yields the Choquet integral, while $C = Min$ yields the Sugeno integral. If X is finite, $X = \{1,\dots,n\}$, and $\mathcal{A} = 2^X$, then we have two equivalent formulae exploited, e.g., in multicriteria decision support,

$$I_{C,m}(f) = \sum_{i=1}^{n}\left(C(f_{(i)}, m(A_i)) - C(f_{(i-1)}, m(A_i))\right) =$$

$$= \sum_{i=1}^{n}\left(C(f_{(i)}, m(A_i)) - C(f_{(i)}, m(A_{i+1}))\right),$$

where $(f_{(1)},\dots,f_{(n)})$ is a nondecreasing permutation of $(f(1),\dots,f(n))$, $A_{(i)} = \left\{j \mid f(j) \ge f_{(i)}\right\}$, with conventions $f_{(0)} = 0$ and $A_{n+1} = \emptyset$.

Looking on integrals $I_{C,\cdot}$ as functionals, one can introduce an axiomatic approach, too. For example, comonotone additivity is a genuine property of the Choquet integral [4], while the comonotone maxitivity and the min–homogeneity characterize the Sugeno integral [2]. We give a general axiomatic characterization of discrete copula–based universal integrals. A substantial role of the comonotone modularity will be shown, linking these integrals and OMA operators [3].

Acknowledgment. The research summarized in this paper was supported by the Grant APVV–0073–10.

References

1. Klement, E.P., Mesiar, R. and Pap, E.: A universal integral as a common frame for Choquet and Sugeno integral. IEEE Transactions on Fuzzy Systems **18**, pp. 178–187 (2010)

2. Marichal, J.–L.: An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework. IEEE Trans. Fuzzy Syst. **9(1)**, pp. 164 – 172 (2001)
3. Mesiar, R., Mesiarová – Zemánková, A.: The ordered modular averages. IEEE Transactions on Fuzzy Systems **19**, pp. 42 – 50 (2011)
4. Schmeidler, D.: Integral representation without additivity. Proc. Amer. Math. Soc. **97**, pp. 255–261 (1986)