Derived graphs come from an adjunction

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Graphs

A *graph* is a quadruple $G = (V, D, s, t, \lambda)$, where

- D is the *set of darts of* G
- V is the *set of vertices of* G
- $s, t \colon D \to V$ are the *source and target maps*, respectively.
- $\lambda \colon D \to E$ is a mapping such that $\lambda \circ \lambda = \mathrm{id}_D$.
- $s \circ \lambda = t$.

The mapping λ is called the *dart-reversing involution* of G.

Graphs

All the data in a graph (V, D, s, t, λ) can be expressed graphically by a commutative diagram:

$$
\begin{tikzcd}
D \arrow[rr, bend left, "\mathrm{id}_D"] \arrow[dr, "\lambda"] \arrow[ddr, bend right, "s"'] & & D \arrow[ddl, bend left, "s"] \\
& D \arrow[ur, "\lambda"'] \arrow[d, "t"] & \\
& V &
\end{tikzcd}
\tag{1}
$$

Morphisms of graphs

A *morphism of graphs* $f\colon G \to H$ is a pair of mappings (f^V, f^D), where

- $f^V\colon V(G) \to V(H)$
- $f^D\colon D(G) \to D(H)$
- for every dart $d \in D(G)$

$$s(f^D(d)) = f^V(s(d))$$
$$t(f^D(d)) = f^V(t(d))$$
$$\lambda(f^D(d)) = f^D(\lambda(d))$$

Clearly, graphs equipped with morphisms form a category, denoted by **Graph**.

Voltage graphs

A *voltage graph* is a triple (G, Γ, α), where

- G is a graph
- Γ is a group
- $\alpha\colon D(G) \to \Gamma$ is a mapping such that

$$\alpha(\lambda(d)) = (\alpha(d))^{-1}$$

The mapping α is called a Γ-*voltage on* G.

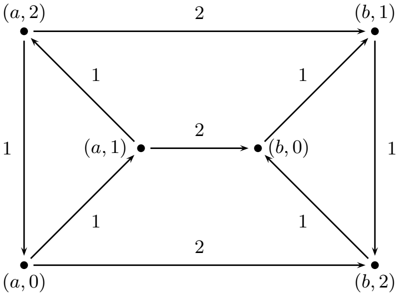
Derived graph

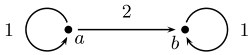
Definition

[Gro74] Let (G, Γ, α) be a voltage graph. There is a *derived* Γ*-voltage graph of* (G, Γ, α), denoted by $(G^{\alpha}, \Gamma, \alpha')$

- $V(G^{\alpha}) = V(G) \times \Gamma$
- $D(G^{\alpha}) = D(G) \times \Gamma$
- $s(d, x) = (s(d), x)$
- $t(d, x) = (t(d), x.\alpha(d))$
- $\lambda(d, x) = (\lambda(d), x.\alpha(d))$
- $\alpha(d, x) = \alpha(d)$

An example; the group is $\mathbb{Z}_3$





- There is always a projection map: $((a, x) \mapsto a)\colon G^\alpha \to G$
- The projection map is a very nice surjection (a *covering*).

Morphisms of voltage graphs

A morphism of voltage graphs $(G, \Gamma, \alpha) \to (G', \Gamma', \alpha')$ is a pair (f, h), where

- $f\colon G \to G'$ is a morphism of graphs
- $h\colon \Gamma \to \Gamma'$ is a morphism of groups such that,
- for all $d \in D(G)$, $h(\alpha(d)) = \alpha'(f^D(d))$.

$$\begin{tikzcd} D(G) \arrow[r, "f^D"] \arrow[d, "\alpha"'] & D(G') \arrow[d, "\alpha'"] \\ \Gamma \arrow[r, "h"'] & \Gamma' \end{tikzcd}$$

- The category of voltage graphs is denoted by **Volt**.

Group labeled graphs

A *group labeled graph* is a triple (G, Γ, β), where G is a graph, Γ is a group and $\beta\colon V(G) \to \Gamma$ is a mapping, called a Γ-*labeling on* G.

Morphisms of group labeled graphs

A morphism of group labeled graphs $(G, \Gamma, \beta) \to (G', \Gamma', \beta')$ is a pair (f, h), where

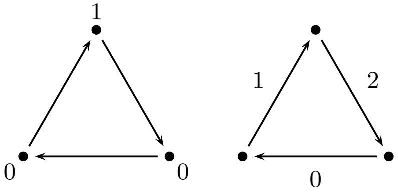
- $f\colon G \to G'$ is a morphism of graphs
- $h\colon \Gamma \to \Gamma'$ is a morphism of groups
- for all $v \in V(G)$, $h(\beta(v)) = \beta'(f^V(v))$.

$$\begin{tikzcd} V(G) \arrow[r, "f^V"] \arrow[d, "\beta"'] & V(G') \arrow[d, "\beta'"] \\ \Gamma \arrow[r, "h"'] & \Gamma' \end{tikzcd}$$

- The category of group labeled graphs is denoted by **Lab**.

From group labeled graphs to voltage graphs

For every group labeled graph $L(G, \Gamma, \beta)$, there is a voltage graph $L(G, \Gamma, \beta) = (G, \Gamma, \alpha)$, with the voltage α given by the rule $\alpha(d) = \beta(s(d))^{-1}\beta(t(d))$.



L is a functor **Lab** $\to$ **Volt**.

Main results

- There is an adjunction

$$\begin{tikzcd} \mathbf{Lab} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \mathbf{Volt} \arrow[ll, bend left, "R"] \end{tikzcd}$$

  between a category **Volt** of voltage graphs and a category **Lab** of group labeled graphs.
- For every voltage graph (G, Γ, α), $LR(G, \Gamma, \alpha)$ is the derived voltage graph.
- The canonical projection $LR(G, \Gamma, \alpha) \to (G, \Gamma, \alpha)$ is the counit of the $L \dashv R$ adjunction.

Where does the adjunction come from?

The ℓ functor

- For every group Γ, $\ell(\Gamma)$ is the graph that
  - has a single vertex,
  - the elements of Γ are the darts of $\ell(\Gamma)$,
  - the λ map is the group inverse.
- $\ell\colon \mathbf{Grp} \to \mathbf{Graph}$ is a functor from the category of group to the category of graphs.

Where does the adjunction come from?

Volt as a comma category

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Volt as a comma category

- A voltage graph is a morphism $\alpha\colon G \to \ell(\Gamma)$.
- A morphism of voltage graphs is then a commutative square in **Graph**

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\alpha"'] & G' \arrow[d, "\alpha'"] \\ \ell(\Gamma) \arrow[r, "\ell(h)"'] & \ell(\Gamma') \end{tikzcd}$$

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$$
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G \arrow[r, "f"] \arrow[d, "\alpha"'] & G' \arrow[d, "\alpha'"] \\
\ell(\Gamma) \arrow[r, "\ell(h)"'] & \ell(\Gamma')
\end{tikzcd}
$$

- So **Volt** is just the *comma category* **Graph** $\downarrow \ell$.

Where does the adjunction come from?

The $\mathring{K}$ functor

- For every group Γ, $\mathring{K}(\Gamma)$ is the graph such that
  - the elements of Γ are the vertices of $\mathring{K}(\Gamma)$,
  - there is exactly one dart between each pair of vertices (including loops).
- $\mathring{K}\colon \mathbf{Grp} \to \mathbf{Graph}$ is a functor from the category of group to the category of graphs.

Where does the adjunction come from?

Lab as a comma category

- A group-labeled graph is a morphism $\beta\colon G \to \mathring{K}(\Gamma)$.

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- A morphism of group-labeled graphs is then a commutative square in **Graph**

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\beta"'] & G' \arrow[d, "\beta'"] \\ \mathring{K}(\Gamma) \arrow[r, "\mathring{K}(h)"'] & \mathring{K}(\Gamma') \end{tikzcd}$$

- So **Lab** is just the *comma category* **Graph** $\downarrow \mathring{K}$.

Where does the adjunction come from?

The left adjoint as a post-composition

- For every group Γ, there is a morphism

$$\mathring{K}(\Gamma) \xrightarrow{q_\Gamma} \ell(\Gamma)$$

given by the rule $q_\Gamma^D(x, y) = y^{-1}x$ on darts.

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The left adjoint as a post-composition

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  given by the rule $q_\Gamma^D(x, y) = y^{-1}x$ on darts.
- This is a natural transformation $q\colon \mathring{K} \to \ell$, because

$$\begin{tikzcd} \mathring{K}(\Gamma) \arrow[r, "\mathring{K}(h)"] \arrow[d, "q_\Gamma"'] & \mathring{K}(\Gamma') \arrow[d, "q_{\Gamma'}"] \\ \ell(\Gamma) \arrow[r, "\ell(h)"'] & \ell(\Gamma') \end{tikzcd}$$

  commutes, for every morphism of groups $h\colon \Gamma \to \Gamma'$.

Where does the adjunction come from?

The left adjoint as a post-composition with q_Γ

$$
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\beta"'] \arrow[dr, phantom, "(2.1)"] & G' \arrow[d, "\beta'"'] \\
\mathring{K}(\Gamma) \arrow[r, "K(h)"'] \arrow[d, "q_\Gamma"] \arrow[dr, phantom, "(2.2)"] & \mathring{K}(\Gamma') \arrow[d, "q_{\Gamma'}"] \\
\ell(\Gamma) \arrow[r, "\ell(h)"'] & \ell(\Gamma')
\end{tikzcd}
\tag{2}
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\end{tikzcd}
\tag{2}
$$

$$L(G, \Gamma, \beta) \simeq (G, \Gamma, q_\Gamma \circ \beta)$$

Where does the adjunction come from?

The right adjoint as a pullback

For every voltage graph (G, Γ, α), we have
$R(G, \Gamma, \alpha) \simeq (G \times_{\ell(\Gamma)} \mathring{K}(\Gamma), \Gamma, q_\Gamma^*(\alpha))$

$$\begin{tikzcd}
G \times_{\ell(\Gamma)} \mathring{K}(\Gamma) \arrow[r, "\alpha^*(q_\Gamma)"] \arrow[d, "q_\Gamma^*(\alpha)"'] & G \arrow[d, "\alpha"] \\
\mathring{K}(\Gamma) \arrow[r, "q_\Gamma"'] & \ell(\Gamma)
\end{tikzcd} \tag{3}$$

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- Derived graph of (G, Γ, α) arises as a pullback of α along q_Γ.

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- Derived graph of (G, Γ, α) arises as a pullback of α along q_Γ.
- The top arrow is the canonical projection.

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The right adjoint as a pullback

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$$\begin{tikzcd} G \times_{\ell(\Gamma)} \mathring{K}(\Gamma) \arrow[r, "\alpha^*(q_\Gamma)"] \arrow[d, "q_\Gamma^*(\alpha)"'] & G \arrow[d, "\alpha"] \\ \mathring{K}(\Gamma) \arrow[r, "q_\Gamma"'] & \ell(\Gamma) \end{tikzcd} \tag{3}$$

- Derived graph of (G, Γ, α) arises as a pullback of α along q_Γ.
- The top arrow is the canonical projection.
- The canonical projection is a fibration of graphs, because q_Γ is a fibration and the square is a pullback, see [BV02].

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Fibrations of graphs.
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