Colimits of effect algebras via a reflection

Dedicated to Sylvia Pulmannová and Anatolij Dvurečenskij

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Effect Algebras

Foulis and Bennett 1994; Kôpka and Chovanec 1994; Giuntini and Greuling 1989

An *effect algebra* is a partial algebra $(E; +, 0, 1)$ satisfying the following conditions.

(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in E$ there is a unique $a^{\perp} \in E$ such that $a + a^{\perp} = 1$.

(E4) If $a + 1$ exists, then $a = 0$

Basic Relationships

Let E be an effect algebra.

- *Cancellativity:* $a + b = a + c \Rightarrow b = c$.
- *Partial difference:* If $a + b = c$ then we write $a = c - b$. The operation $-$ is well defined and $a^{\perp} = 1 - a$.
- *Poset:* Write $b \leq c$ iff $\exists a : a + b = c$; $(E, \leq)$ is then a bounded poset.

Classes of Effect Algebras

The class of effect algebras includes

- orthomodular lattices
- MV-algebras
- Boolean algebras.

Congruences

Definition

Let A be an effect algebra, let $\sim \subseteq A \times A$ be a relation such that the following conditions are satisfied.

- $\sim$ is an equivalence.
- If $x_1 \sim x_2$ and $y_1 \sim y_2$ and $x_1 + y_1$ exists and $x_2 + y_2$ exists, then $x_1 + y_1 \sim x_2 + y_2$.
- If $x_1 \sim x_2$, then $x_1^\perp \sim x_2^\perp$.

Then we say that $\sim$ is *a congruence* on A.

Morphisms

Definition

Let E, F be effect algebras, let $f\colon A \to B$. We say that f is a *morphism of effect algebras* iff

- $f(1) = 1$ and
- for all $x, y \in A$ such that $x + y$ exists in A, $f(x) + f(y)$ exists in B and $f(x + y) = f(x) + f(y)$

The problem with congruences

Whenever $f : A \to B$ is a morphism of effect algebras, its kernel $\Theta_f \subseteq A \times A$, given by

$$x \Theta_f y \iff f(x) = f(y)$$

is a congruence.

The problem with congruences

In non-partial algebras:

- there is a quotient algebra A/Θ
- and a mapping

$$x \mapsto [x]_\Theta$$

  that takes every element to its equivalence class.

The problem with congruences

- For effect algebras, the quotient A/Θ might be non-associative.
- The problem is *existence* of sums, not their value.

Proposed solutions

- We may try to come up with sufficient conditions on Θ under which A/Θ is an effect algebra.

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  - Weaker: if $x_1 \Theta x_2$ and $x_1 + y_1$ exists, then there is $y_2 \Theta y_1$ such that $x_2 \Theta y_2$ exists.
  - Ideals: we might consider only *quotients by ideals*.
- This area was explored around the year 2000 by Gudder, Chevalier, Pulmannová, GJ in several papers.

Congruences, from a categorical viewpoint

- A congruence on an algebra A can be characterized as a *subalgebra of* $A \times A$ that (as a relation) is an equivalence:

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- gives us, in a canonical way, two mappings

$$\begin{tikzcd} \Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \end{tikzcd}$$

Congruences, from a categorical viewpoint

If we compose p_1, p_2 with the quotient map $q(x) = [x]_\Theta$

$$\begin{tikzcd}\Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "q"] & A/\Theta\end{tikzcd}$$

we obtain an equality

$$q \circ p_1 = q \circ p_2$$

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- whenever $f \circ p_1 = f \circ p_2$,
- there is a unique arrow u making the diagram

$$\begin{tikzcd}
\Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "q"] \arrow[dr, "f"'] & A/\Theta \arrow[d, dotted, "u"] \\
 & & B
\end{tikzcd}$$

commute.

Congruences, from a categorical viewpoint

That means, $q\colon A \to A/\Theta$ is *a coequalizer* of the pair

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Problem

Does the category of effect algebras have coequalizers?

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Problem

Does the category of effect algebras have coequalizers?

Or, more generally:

Problem

Does the category of effect algebras have all (small) colimits?

The answer

- Yes [Jacobs and Mandemaker 2012].

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- Yes [Jacobs and Mandemaker 2012].
- There is a *coreflection*

$$\begin{tikzcd}
\mathbf{EA} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{BCM} \arrow[l, bend left, "R"]
\end{tikzcd}$$

between effect algebras **EA** and *barred commutative monoids* **BCM**.

Definition

Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{D}$. A *coreflection* of an object $X \in \mathcal{D}$ is a choice of an object $G(X) \in \mathcal{C}$ and an arrow $\epsilon_X \colon G(X) \to X$ such that for every other arrow $f : A \to X$ with $A \in \mathcal{C}$ there is a unique $u \colon A \to G(X)$ such that

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If every object has a coreflection, $\mathcal{C}$ is a *coreflective subcategory* of $\mathcal{D}$.

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- consider the diagram as $\mathcal{D}$-valued and compute the (co)limit in $\mathcal{D}$;
- coreflect the result into $\mathcal{C}$;
- this is the (co)limit in $\mathcal{C}$.

- **BCM** is a quasivariety, so it has all colimits.

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- Therefore, **EA** is cocomplete.

Non-surjective quotients

Moreover, Jacobs and Mandemaker have constructed an example of a non-surjective coequalizer

$$\begin{tikzcd} B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A \arrow[r, "q"] & Q \end{tikzcd}$$

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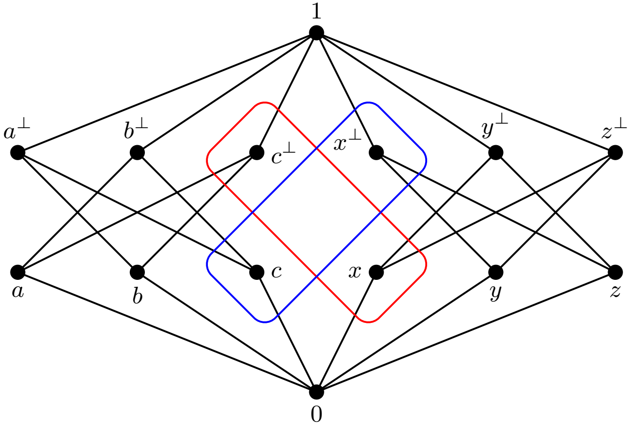
and one can easily adapt their example to show that there are "non-surjective" quotient maps:

$$\begin{tikzcd} \Theta \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[r, "q"] & A/\Theta \end{tikzcd}$$

A congruence



The quotient is isomorphic to the Boolean algebra 2^4.

The problem with this approach

It is difficult to compute colimits in **BCM**.

Test spaces

Idea: representation of a quantum logic by the set of all decompositions of unit.

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Names:

- D. Foulis, C. Randall
- P. Frazer-Lock
- A. Wilce
- A. Pulmannová, S. Gudder, A. Dvurečenskij

Finite multisets

- A *finite multiset over a set X* is a mapping
$$\mathbf{t} \colon X \to \mathbb{N}$$
with finite support.
- The set of all finite multisets over a set X is denoted by $M(X)$.
- $(M(X), +, 0)$ is a monoid (free commutative monoid generated by X).
- $M(X)$ is partially ordered (pointwise).

Finite multiset systems

Objects

A *finite multiset system* $\mathcal{X}$ is a pair $(V(\mathcal{X}), T(\mathcal{X}))$, where $V(\mathcal{X})$ is a set called *the set of points of* $\mathcal{X}$ and $T(\mathcal{X})$ is a set of finite multisets over $V(\mathcal{X})$, called *tests of* $\mathcal{X}$.

Finite multiset systems

Morphisms

A morphism of finite multisets systems $f\colon \mathcal{X} \to \mathcal{Y}$ is a mapping $f\colon V(\mathcal{X}) \to V(\mathcal{Y})$ such that the *pushforward of a test* $t \in T(\mathcal{X})$ is a test of $\mathcal{Y}$:

$$f_*(t)(y) = \sum_{x \in f^{-1}(y)} t(x),$$

Finite multiset systems

Theorem

The category of finite multiset systems is complete and cocomplete.

Finite multiset systems

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The category of finite multiset systems is complete and cocomplete.

Importantly, the limits and colimits are *easy to compute*: there is an obviously defined functor **FinMultSys** → **Set** that *creates all limits and colimits*.

A reflection

Theorem

There is a reflection

$$\begin{tikzcd}
\mathbf{FinMultSys} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{EA} \arrow[l, bend left, "R"]
\end{tikzcd}$$

The R functor

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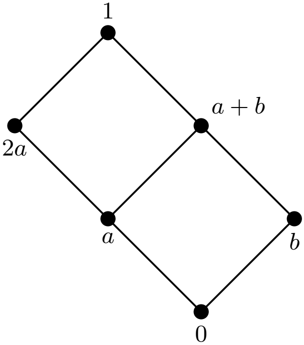
\end{tikzcd}$$

If A is an effect algebra, then $R(A)$ is the set of all *decompositions of unit* (called the *tests of* A).

Example



0	a	b	$2a$	$a+b$	1
0	2	1	0	0	0
0	0	1	1	0	0
5	1	0	0	1	0
3	0	0	0	0	1

Events

- Let $\mathcal{X}$ be a finite multiset system.
- An *event* of $\mathcal{X}$ is a $V(\mathcal{X})$-based multiset

$$a\colon V(\mathcal{X}) \to \mathbb{N}$$

such that there is a test $t \in T(\mathcal{X})$ such that $a \leq t$.

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- Two events a, a' are *perspective* (in symbols $a \sim a'$) if the share a common complement:

  $$a + b \in T(\mathcal{X}) \qquad b + a' \in T(\mathcal{X})$$

Algebraic finite multiset systems

Definition

A finite multiset system is $\mathcal{X}$ *algebraic* if

- The set of tests of $\mathcal{X}$ is order-convex, that means

$$t_1, t_2 \in T(\mathcal{X}) \text{ and } t_1 \leq s \leq t_2 \implies s \in T(\mathcal{X}).$$

- If $a \sim a'$ and a is a complement of c, then a' is a complement of c. In other words,

$$\left.\begin{array}{rl} a' + b & \in T(\mathcal{X}) \\ b + a & \in T(\mathcal{X}) \\ a + c & \in T(\mathcal{X}) \end{array}\right\} \implies a' + c \in T(\mathcal{X})$$

Algebraic finite multiset systems

Theorem

There is a reflection

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\mathbf{FinMultSys} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathbf{AlgFinMultSys} \arrow[ll, bend left]
\end{tikzcd}$$

From effect algebras to algebraic finite multiset systems

For every effect algebra A, the finite multiset system consisting of all decompositions of unit of A is algebraic.

From algebraic finite multiset systems to effect algebras

- Let $\mathcal{X}$ be an algebraic multiset system.
- The relation of perspectivity $\sim$ on the set of all events $E(\mathcal{X})$ is an equivalence relation.
- Perspectivity behaves well with respect to sums of events:

$$a_1 \sim a_2 \text{ and } b_1 \sim b_2 \implies a_1 + b_1 \sim a_2 + b_2$$

- $E(\mathcal{X})/\sim$ forms an effect algebra.

Another reflection

Theorem

There is a reflection

$$\begin{tikzcd}\mathbf{AlgFinMultSys} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathbf{EA} \arrow[ll, bend left]\end{tikzcd}$$

Composed reflection

$$\begin{tikzcd}
\mathbf{FinMultSys} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{AlgFinMultSys} \arrow[l, bend left] \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathbf{EA} \arrow[l, bend left]
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A recipe for constructing a quotient

- We have an effect algebra E and a congruence Θ on E.
- We want to compute E/Θ in the category **EA**.

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- We start with the finite multiset system of decompositions of unit in E.
- We pushforward all tests from the set E to the set of equivalence classes of Θ along the quotient map.
- We add new tests until we get an algebraic finite multiset system.
- This algebraic finite multiset system gives you the quotient E/Θ in the category **EA**.