Tensor product of effect algebras as a Kan extension

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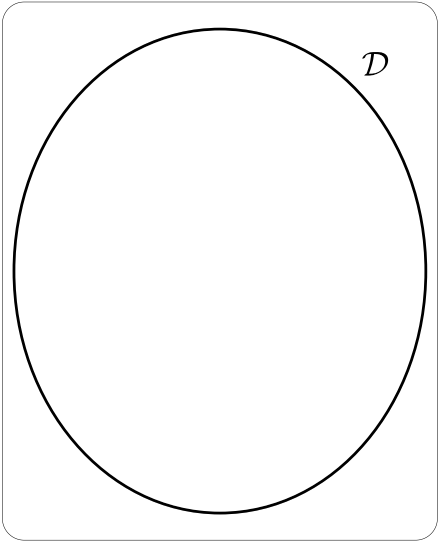
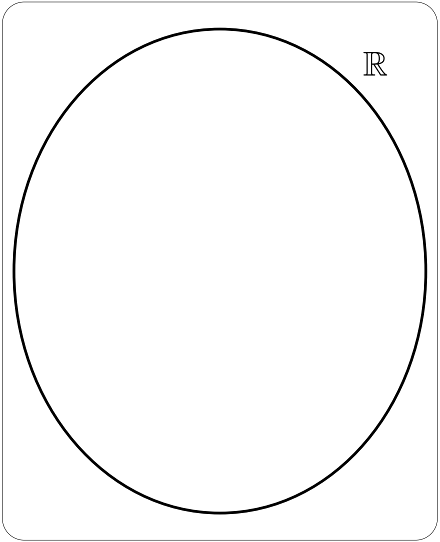
A category

- Objects
- Morphisms
- Composition
- Identity



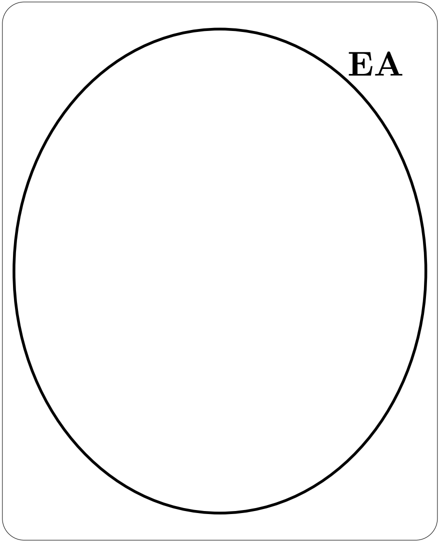
The category (poset) of real numbers

- Objects: real numbers
- Morphisms: $a \leq b$
- Composition:
  $(b \leq c) \circ (a \leq b) \implies (a \leq c)$
- Identity: $a \leq a$



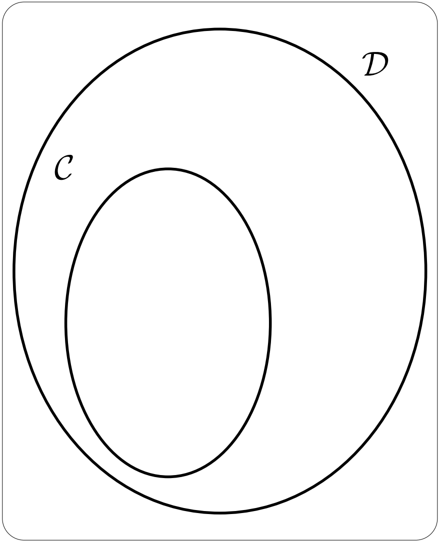
Effect algebras

- Commutative, cancellative, positive, unital partial abelian monoids.
- Independently introduced in 1990's by researchers from Italy, Slovakia, US.
- They generalize (in a useful way) MV-algebras, orthomodular lattices, Boolean algebras.
- 839 citations of the basic paper by Foulis and Bennett

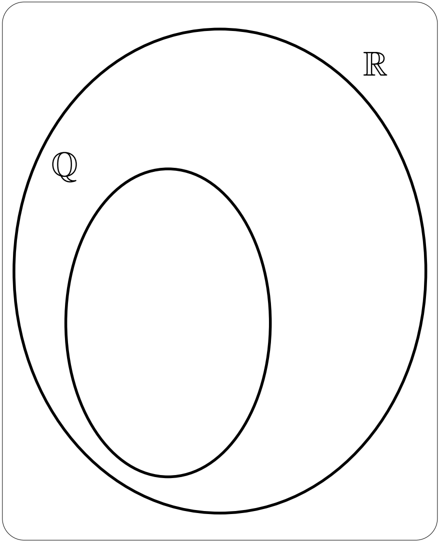


A category with a nice subcategory

We consider in this talk only full subcategories, this makes things easier.

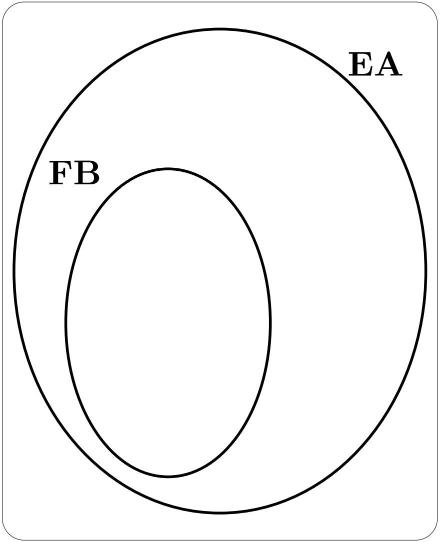


Everybody likes $\mathbb{Q}$

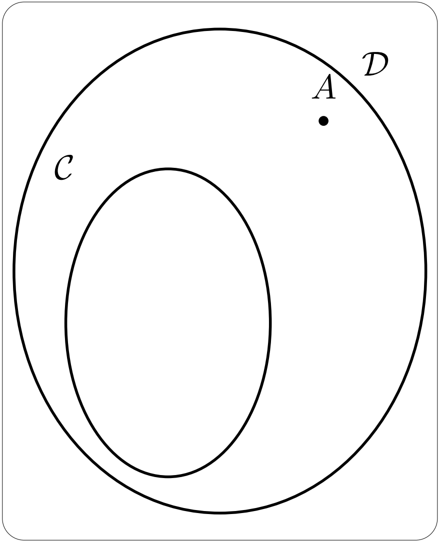


Everybody likes finite Boolean algebras

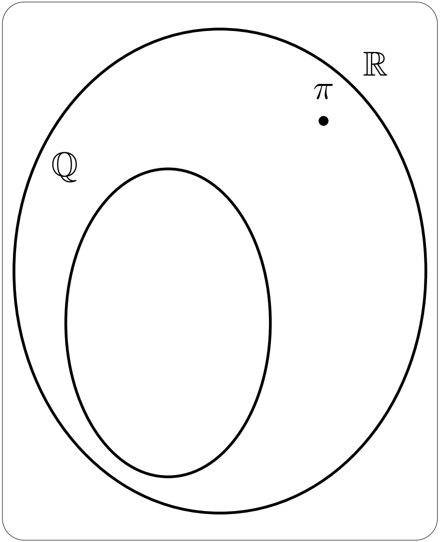
We shall write $2^{[n]}$ for the powerset of $\{1, \ldots, n\}$.



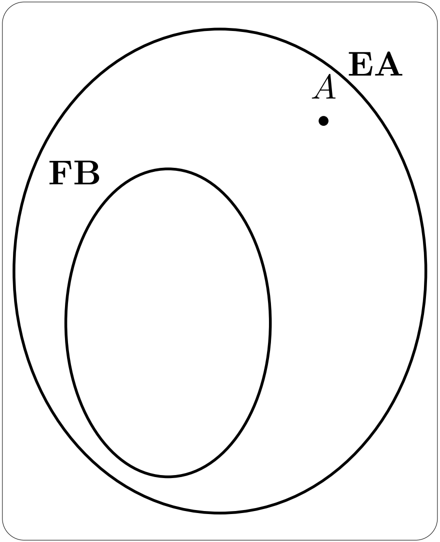
An object A in $\mathcal{D}$



Erm, let's say... π

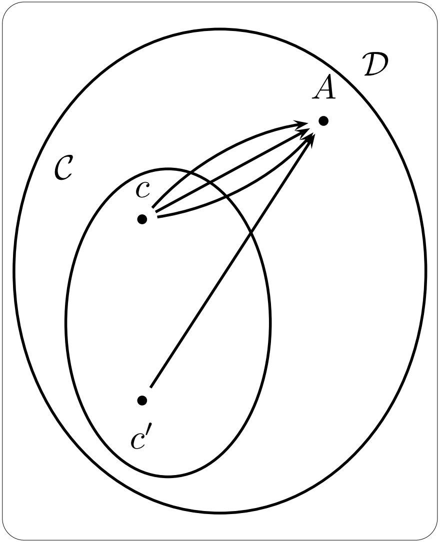


Erm, let's say... A

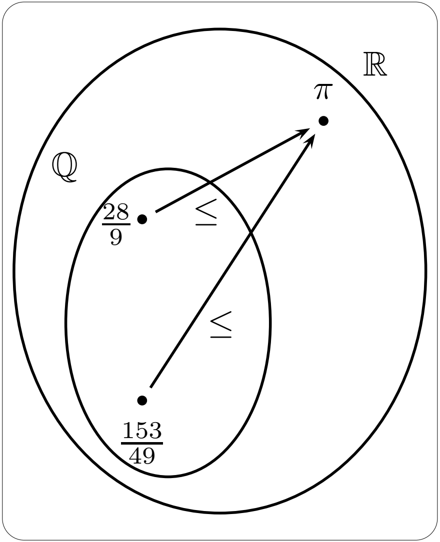


Arrows from $\mathcal{C}$ to A

- We shall look at the arrows from objects of $\mathcal{C}$ to A.
- There can be many arrows from c to A, in general.

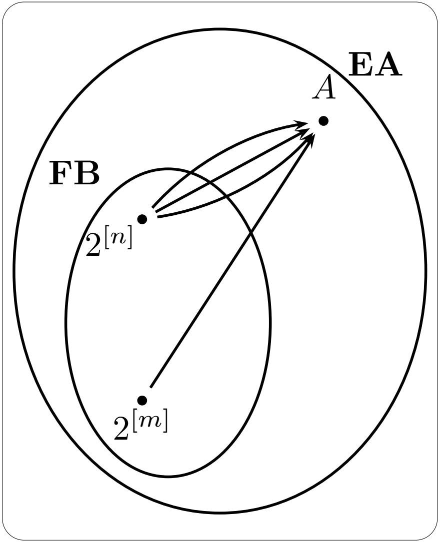


Look, Dedekind cuts!



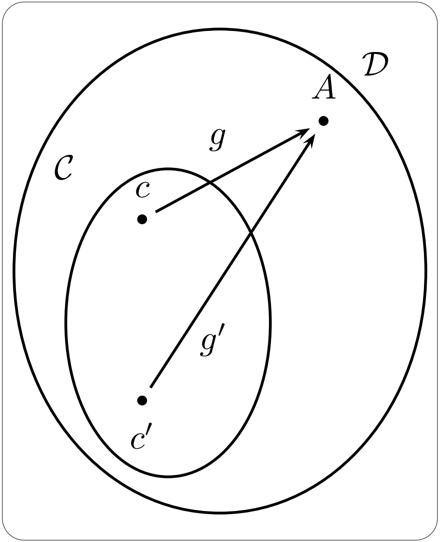
Finite observables

- An arrow $2^{[n]} \to A$ is called a finite A-valued observable.
- These are easy to describe by finite decompositions of the unit of A.



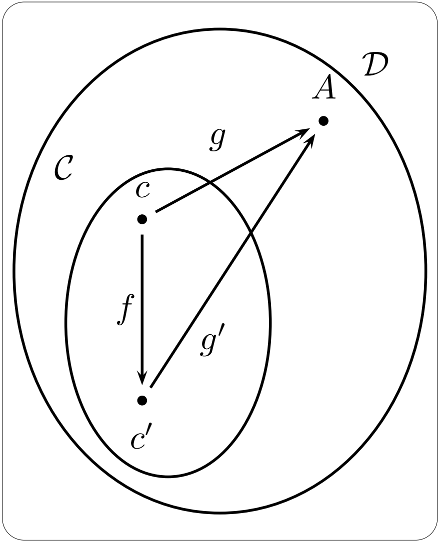
$\mathcal{C} \uparrow A$: arrows as objects

- Objects: all $\mathcal{D}$-arrows $c \to A$, where $c \in \mathcal{C}$.



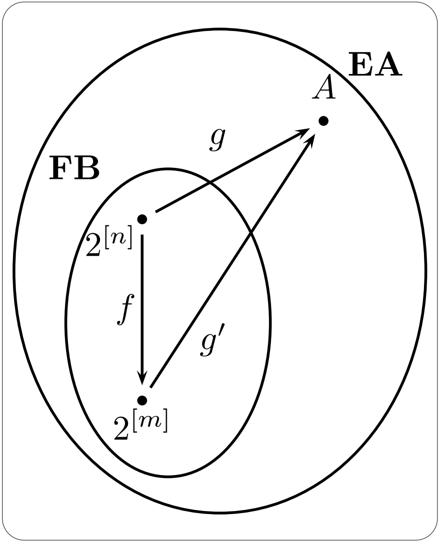
$\mathcal{C} \uparrow A$: morphisms of arrows are commutative triangles

- Morphisms: $f : g \to g'$ means that $g = g' \circ f$.
- Composition: pasting of triangles.



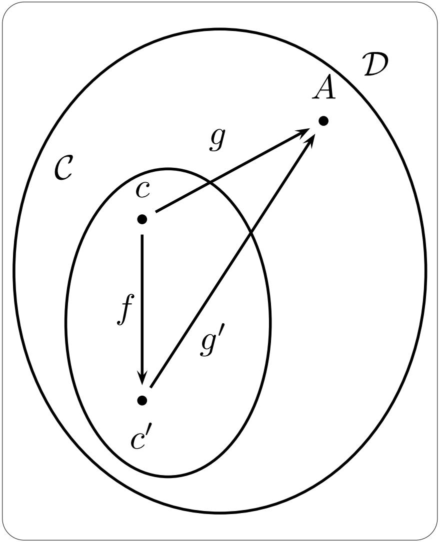
FB $\uparrow A$

- Objects: decompositions of unit in A.
- Morphisms: refinements.



$\mathcal{C} \uparrow A$: what does it say about A?

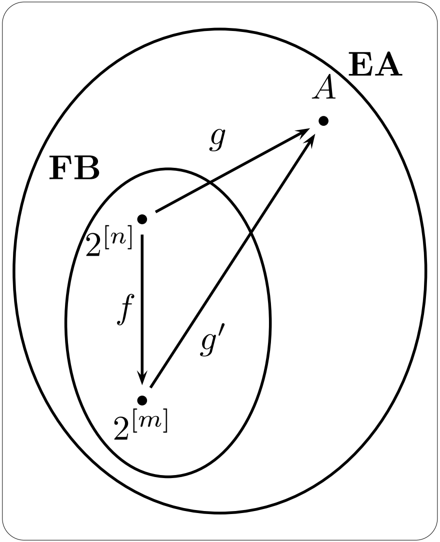
- We may look at various properties of A and determine whether they are reflected by categorical properties of $\mathcal{C} \uparrow A$.



FB $\uparrow A$: what does it say about A

Theorem

An effect algebra A is an orthoalgebra if and only if for every pair of morphisms $f_1, f_2 \colon g \to g'$ in $\mathcal{C} \uparrow A$ there is a coequalizing morphism $q \colon g' \to u$ such that $q \circ f_1 = q \circ f_2$.



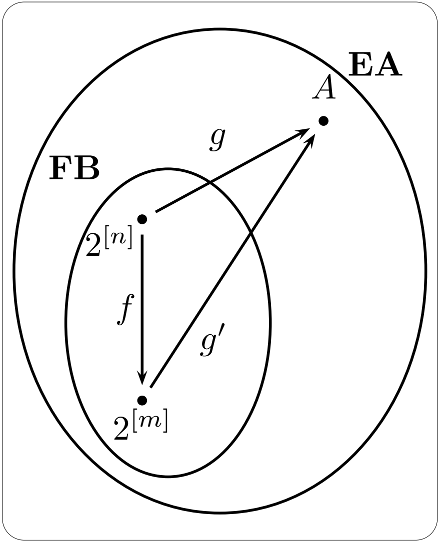
Can we reconstruct A from $\mathbf{FB} \uparrow A$?

- Yes!
- There is an obviously defined "projection" functor $P : \mathbf{FB} \uparrow A \to \mathbf{FB}$ that takes every observable $g : 2^{[n]} \to A$ to its domain $2^{[n]}$.

Theorem

A is the colimit of the diagram

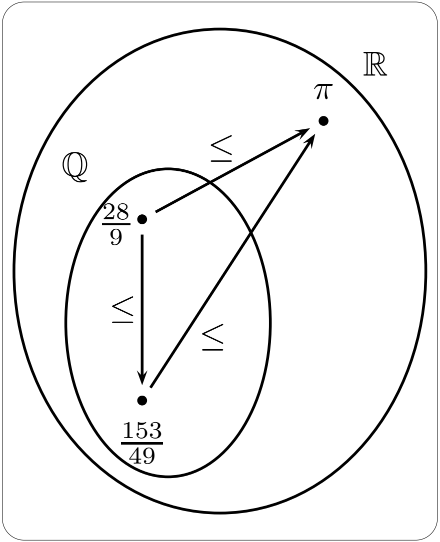
$$\mathbf{FB} \uparrow A \xrightarrow{p} \mathbf{FB} \hookrightarrow \mathbf{EA}$$



Can we reconstruct π from $\mathbb{Q} \uparrow \pi$?

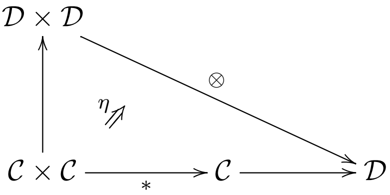
Yes, because

$$\pi = \sup\{x \in \mathbb{Q} \colon x \leq \pi\}$$

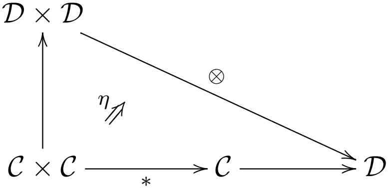


Transferring structure from $\mathcal{C}$ to $\mathcal{D}$

- Suppose that $\mathcal{C}$ is equipped with some sort of "tensor product" – a monoidal structure $(\mathcal{C}, *, I, \alpha, \lambda, \rho)$, satisfying the usual axioms.
- Can we extend $*$ from $\mathcal{C}$ to $\mathcal{D}$ in a universal way?



Transferring structure from $\mathcal{C}$ to $\mathcal{D}$



For $A, B \in \mathcal{D}$

$$A \otimes B = \operatorname{colim}(\, \mathcal{C} \uparrow A \times \mathcal{C} \uparrow B \xrightarrow{p \times p} \mathcal{C} \times \mathcal{C} \xrightarrow{*} \mathcal{C} \hookrightarrow \mathcal{D}\,)$$

This is the left Kan extension

$$\otimes = \mathrm{Lan}_{E \times E}(E \circ *)$$

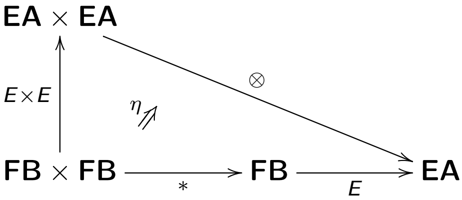
where E is the inclusion $\mathcal{C} \to \mathcal{D}$

Transferring structure from **FB** to **EA**

- There is a monoidal structure on **FB**

$$2^{[n]} * 2^{[m]} = 2^{[n]\times[m]}$$

- This is the <u>free product of Boolean algebras</u>.



Theorem

⊗ is the tensor product of effect algebras, introduced in 1995 by Dvurečenskij.

Transferring the structure from $\mathbb{Q}$ to $\mathbb{R}$

- There is a monoidal structure on $\mathbb{Q}$ – the product of rational numbers.
- For $a, b \in \mathbb{R}$, the left Kan extension colimit becomes

$$a \otimes b = \sup\{x.y\colon x, y \in \mathbb{Q},\ x \leq a,\ y \leq b\}$$

- This is just the product of real numbers.