Unitization and symmetrization of non-commutative partial abelian monoids

Gejza Jenča

Department of Mathematics and Descriptive Geometry
Faculty of Civil Engineering
Slovak Technical University

Summer School of General Algebra and Ordered Sets 2014

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Effect Algebras

Foulis and Bennett [1994], Kôpka and Chovanec [1994], Giuntini and Greuling [1989]

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.

(E4) If $a \oplus 1$ exists, then $a = 0$

Morphisms of effect algebras

- A mapping $f : A \to B$ of effect algebras is a *morphism* if it preserves all existing sums, 0 and 1.

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- This gives us the category of effect algebras **EA**.

Basic relationships

Let E be an effect algebra.

- *Neutral element:* $a \oplus 0 = a$.
- *Cancellativity:* $a \oplus b = a \oplus c \Rightarrow b = c$.
- *Partial difference:* If $a \oplus b = c$ then we write $a = c \ominus b$. $\ominus$ is well defined and $a' = 1 \ominus a$.
- *Positivity:* $a \oplus b = 0$ implies $a = b = 0$.
- *Poset:* Write $b \leq c$ iff $\exists a : a \oplus b = c$; $(E, \leq)$ is then a bounded poset.
- *Domain of* $\oplus$*:* $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$.

Important subclasses

The class of effect algebras is (essentially) a common superclass of several classes of algebras:

- orthomodular lattices
- orthoalgebras
- MV-algebras
- Boolean algebras

Generalized effect algebras

A *generalized effect algebra* is a partial algebra $(A; \oplus, 0, 1)$ satisfying the following conditions.

(GE1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(GE2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(GE3) *Neutral element:* $a \oplus 0 = a$.

(GE4) *Cancellativity:* $a \oplus b = a \oplus c \Rightarrow b = c$.

(GE5) *Positivity:* $a \oplus b = 0$ implies $a = b = 0$.

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- This gives us the category of generalized effect algebras **GEA**.
- There is a forgetful functor $U : \mathbf{EA} \to \mathbf{GEA}$.

Unitization of generalized effect algebras

Hedlíková and Pulmannová [1996]

Theorem

For every generalized effect algebra A there is an effect algebra $E(A)$ and an embedding of generalized effect algebras $\eta_A : A \to UE(A)$.

- The range of η_A is and ideal of $UE(A)$.

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- The range of η_A is and ideal of $UE(A)$.
- $UE(A)$ is a disjoint union of $\eta_A(A)$ and $\{(\eta_A(x))' : x \in A\}$.
- If A is upper-bounded, then $UE(A) \simeq A \times \mathbf{2}$, where $\mathbf{2}$ is the 2-element Boolean algebra.

Where does the unitization construction come from

Theorem

E is a left adjoint functor to U.

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Corollary

Let A, B be generalized effect algebras. let $f : A \to UE(B)$ be a morphism. There is a unique morphism of effect algebras $u : E(A) \to E(B)$ such that

$$\begin{array}{ccc} UE(A) & \overset{u}{\dashrightarrow} & UE(B) \\ \uparrow{\scriptstyle \eta_A} & \nearrow{\scriptstyle f} & \\ A & & \end{array}$$

commutes.

Corollary

The unitization construction preserves colimits.

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The unitization construction preserves colimits. In particular,

$$E(A \oplus B) \simeq E(A) \oplus E(B),$$

where on the left side $\oplus$ *is the* 0*-pasting and on the right side* $\oplus$ *is the* $0, 1$*-pasting.*

Pseudo effect algebras (PEAs)

Dvurečenskij and Vetterlein [2001a,b]

An *pseudo effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ satisfying the following conditions.

(PE1) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(PE2) For every $a \in E$ there are unique elements $a^-, a^\sim \in E$ such that $a \oplus a^\sim = a^- \oplus a = 1$.

(PE3) If $a \oplus b$ exists, then there are $c, d \in E$ such that $a \oplus b = c \oplus a = b \oplus d = 1$.

(PE4) If $a \oplus 1$ or $1 \oplus a$ exist, then $a = 0$.

Generalized pseudo effect algebras (GPEAs)

An *pseudo effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ satisfying the following conditions.

(GPE1) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(GPE2) If $a \oplus b$ exists, then there are $c, d \in E$ such that $a \oplus b = c \oplus a = b \oplus d = 1$.

(GPE3) If $a \oplus b = a \oplus c$ or $b \oplus a = c \oplus a$, then $b = c$.

(GPE4) $a \oplus 0 = 0 \oplus a = a$.

(GPE5) If $a \oplus b = 0$, then $a = b = 0$.

Is there a unitization?

Problem

Can every GPEA be embedded, as an ideal, into a PEA?

A unitization result

Foulis and Pulmannová [2014]

Theorem

Let P be a GPEA, suppose that there is an automorphism $\gamma : E \to E$ *(called* unitizing automorphism*) such that*

$$\gamma a \oplus b \text{ exists iff } b \oplus a.$$

Then P admits a nice unitization $E(P)$.

The adjunction generalizes to the non-commutative case

- There is a category "GPEAS equipped with an unitizing automorphism", call it $\mathbf{GPEA}\gamma$.
- There is a forgetful functor $U : \mathbf{GPEA} \to \mathbf{GPEA}\gamma$; the unitizing automorphism for $U(E)$ is just $x \mapsto x^{--}$.
- The unitization construction $F : \mathbf{GPEA}\gamma \to \mathbf{GPEA}$ is a functor left adjoint to U.

Symmetric pseudo effect algebras

Definition

An pseudo effect algebra is called *symmetric* if, for all elements a, $a^{\sim} = a^{-}$.

A symmetric pseudo effect algebra need not be commutative.

A canonical automorphism

Foulis and Pulmannová [2014]

Lemma

In a pseudo effect algebra, $x \mapsto x^{--}$ *is an automorphism.*

Symmetrization functor

Theorem

1. *There is an action of* $\mathbb{Z}$ *on every PEA E such that* $x \mapsto x^{--}$ *is the action of* $1 \in \mathbb{Z}$.

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2. *The orbits of this action determine a (strong) congruence* $\equiv$ *on* E *such that* $E/\equiv$ *is symmetric.*

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3. *The correspondence* $E \mapsto E/\equiv$ *determines a functor* S *from* **PEA** *to the category of symmetric pseudo effect algebras (***SPEA***) called* symmetrization.

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4. *This functor is left adjoint to the inclusion of symmetric pseudo effect into* **PEA***.*

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5. *It is an idempotent functor: $S^2 \simeq S$.*

Symmetrization functor

Theorem

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2. *The orbits of this action determine a (strong) congruence $\equiv$ on E such that $E/\equiv$ is symmetric.*
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4. *This functor is left adjoint to the inclusion of symmetric pseudo effect into* **PEA***.*
5. *It is an idempotent functor: $S^2 \simeq S$.*
6. **SPEA** *is a reflexive subcategory of* **PEA***, S being the reflection.*

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