A survey of homogeneous effect algebras

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Effect algebras

(Foulis and Bennett 1994, Chovanec and Kôpka 1994, Giuntini and Greuling 1989)

An effect algebra is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.

(E4) If $a \oplus 1$ exists, then $a = 0$

Basic Relationships

Let E be an effect algebra.

- Cancellativity: $a \oplus b = a \oplus c \Rightarrow b = c$.
- Partial difference: If $a \oplus b = c$ then we write $a = c \ominus b$. $\ominus$ is well defined and $a' = 1 \ominus a$.
- Poset: Write $b \leq c$ iff $\exists a : a \oplus b = c$; $(E, \leq)$ is then a bounded poset.
- Domain of $\oplus$: $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$.

Morphisms

Definition

Let E, F be effect algebras, let $\phi : E \to F$. We say that ϕ is a morphism of effect algebras iff

- $\phi(1) = 1$ and
- for all $a, b \in E$ such that $a \oplus b$ exists in E, $\phi(a) \oplus \phi(b)$ exists in F and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$

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- The notion of compatibility is very important in quantum logics, so it was natural to try to extend the theory of compatible sets from quantum logics.
- However, the general case appears to be very difficult.
- Next idea: try to find some conditions under which compatible sets behave sanely.

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Definition

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- An subset of A an effect algebra is compatible if and only if every finite subset of A is compatible.

Orthogonality, covers

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- An element a is covered by $\mathbf{b}$ if and only if a is a sum of a subsequence of $\mathbf{b}$.
- There is an obvious preorder relation, called refinement on the set of all decomposition of unit: "replace every b_i by an orthogonal word with sum equal to b_i."

Compatibility

characterization

Proposition

A finite subset A of an effect algebra E is compatible if and only of there is a decomposition of unit $\mathbf{b}$, such that $\mathbf{b}$ covers every element of A.

Lattice effect algebras are nice

- In [Rie00], Zdenka Riečanová proved a surprising theorem.

Theorem

Every maximal compatible subset (a block) of a lattice effect algebra E is an MV-algebra that is both a sublattice and a subeffect algebra of E.

- That means that lattice effect algebras look like orthomodular lattices, but their blocks are MV-algebras instead of Boolean algebras.

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- Is there a class of effect algebras that
  - includes lattice effect algebras,
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  - allows for a meaningful theory of compatibility and a notion of block?

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The solution

- In general, it may happen that the elements covered by a decomposition of unit are not closed with respect to $\oplus$, $\ominus$.
- It may happen even in a lattice effect algebra.
- But we do not need that.
- What we need is the following:
  - if we have a finite compatible set, covered by a decomposition of unit and
  - we have some x, y in the compatible set with $x \leq y$, then
  - we want to refine the decomposition of unit so that the finer decomposition of unit will cover $y \ominus x$.

The idea behind the definition

Definition

[Jen01]

- An effect algebra is homogeneous iff $u \leq v_1 \oplus \ldots \oplus v_n \leq u'$ implies that there exist $u_1, \ldots, u_n \in E$ such that $u_i \leq v_i$ and $u = u_1 \oplus \ldots \oplus u_n$.
- It is easy to prove that an effect algebra is homogeneous if and only if it satisfies the above condition with fixed $n = 2$.

An example of a "genuine" homogeneous effect algebra

Example

Let μ be the Lebesgue measure on $[0,1]$. Let $E \subseteq [0,1]^{[0,1]}$ be such that, for all $f \in E$,

(a) f is measurable

(b) $\mu(\mathrm{supp}(f)) \in \mathbb{Q}$

(c) $\mu(\{x \in [0,1] : f(x) \notin \{0,1\}\}) = 0$,

where $\mathrm{supp}(f)$ denotes the support of f. Then E is a homogeneous effect algebra which is not lattice ordered, not an orthoalgebra and does not satisfy the Riesz decomposition property.

Where does the name come from?

The characterization of finite homogeneous effect algebras

[Jen03]: a finite effect algebra E is homogeneous if and only if it satisfies the following condition.

- For every pair of decompositions of unit $\mathbf{b}, \mathbf{c}$,

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This characterization was recently extended to orthocomplete atomic case in [Ji14].

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$$u \leq v_1 \oplus v_2 \implies u = u_1 \oplus u_2, \text{ where } u_i \leq v_i$$

- If ϕ is a morphism from a Boolean algebra into a homogeneous effect algebra, then the range of ϕ is a subset of a block.

Sharp elements in HEAs

- An element a of an effect algebra is sharp if and only if $a \wedge a' = 0$.

Sharp elements in HEAs

- An element a of an effect algebra is sharp if and only if $a \wedge a' = 0$.
- The set of all sharp elements in a homogeneous effect algebra forms a subalgebra.

Questions

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- What if we have a lattice effect algebra?
- Does it look like some orthomodular lattice?

The finite case

Theorem

[Jen03] For every finite homogeneous effect algebra E there is an orthoalgebra $O(E)$ and a surjective morphism of effect algebras $\phi : O(E) \to E$ such that

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Moreover, if E is a lattice then $O(E)$ is a lattice.

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[Jen04a] For every MV-algebra M there is a Boolean algebra $B(M)$ and a surjective morphism of effect algebras $\phi_M : B(M) \to M$.

The infinite case

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Side note: the maps ϕ_M are a components of a natural transformation between two functors from the category of MV-algebras to the category of MV-effect algebras.

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- This condition can be expressed in terms of infinite sums.
- A lattice effect algebra is orthocomplete if and only if it is complete.

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- I do not know how to construct $O(E)$ for a general HEA E.
- In the finite case, the proof is based on an interplay between atoms and sharp elements.
- After taking appropriate generalizations, it turns out that the core problem is to describe the interaction between sharp elements, compatibility and blocks.

Sharp kernels and blocks

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- For every element x in an orthocomplete HEA, $x \ominus x^{\downarrow}$ is meager.

Triple representation for complete LEAs

Theorem

[Jen04b] Every complete lattice effect algebra is completely characterized by the following data

- *The orthomodular lattice $S(E)$ of sharp elements.*

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- *The orthomodular lattice $S(E)$ of sharp elements.*
- *The generalized effect algebra*[1] *$M(E)$ of meager elements.*
- *The mapping s from $S(E)$ to the ideal lattice of $M(E)$:*

$$s(x) = \{y \in M(E) : y \leq x\}$$

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Triple representation for orthocomplete HEAs

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- The proof was recently found by Paseka and Niederle in [NP12].

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[NP12, NP13a, NP13b]

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Open problem

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Is the following statement true?
For every orthocomplete homogeneous effect algebra E there is an orthoalgebra $O(E)$ and a surjective morphism of effect algebras $\phi : O(E) \to E$ such that

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Moreover, if E is a lattice then $O(E)$ is a lattice.

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This is open even in the finite case.

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