On a preorder relation induced by uninorms

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Abstract: *In this paper we study a pre-order $\preceq_U$ induced by uninorms U. We will be interested especially in cases when $\preceq_U$ is not an ordering. We will also study algebraic properties of equivalence classes $\sim_U$. We will show examples of uninorms when the uninorm restricted to an equivalence class A is an Abelian subgroup of the monoid $([0,1],U,e)$, examples when this is an Abelian group but not a subgroup of $([0,1],U,e)$, as well as examples when an equivalence class A cannot be organized into a group. We will further see that the Abelian groups occurring in the monoid $([0,1],U,e)$, may have non-trivial subgroups.*

Keywords: *uninorm, pre-order, partition*

1 Introduction and known facts

Uninorms since their introduction by Dombi [2] under the name *aggregative operator*, and later re-introduction by Yager and Rybalov [10], have found broad interest among researchers, and also broad applicability in many areas, such as decision making, fuzzy control, etc. Uninorms were proposed by Yager and Rybalov as a natural generalization of both, t-norms and t-conorms. Because of lack of space, for basic properties on t-norms and t-conorms we refer readers to [7, 9].

1.1 Uninorms

The definition of uninorm proposed by Yager and Rybalov [10] is the following.

Definition 1.1. *A uninorm U is a function $U\colon [0,1]^2 \to [0,1]$ that is increasing, commutative, associative and has a neutral element $e \in [0,1]$.*

An overview of basic properties of uninorms is in [1]. For an overview of known classes of uninorms see [8].

A uninorm U is said to be *conjunctive* if $U(x,0) = 0$, and U is said to be disjunctive if $U(1,x) = 1$, for all $x \in [0,1]$.

A uninorm U is called *representable* if it can be written in the form

$$U(x,y) = g^{-1}(g(x) + g(y)) \,,$$

where $g\colon [0,1] \to [-\infty,\infty]$ is a continuous strictly increasing function with $g(0) = -\infty$ and $g(1) = \infty$. Note yet that for each generator g there exist two different uninorms depending on convention we take $\infty - \infty = \infty$, or $\infty - \infty = -\infty$. In the former case we get a disjunctive uninorm, in the latter case a conjunctive uninorm.

Representable uninorms are “almost continuous”, i.e., they are continuous everywhere on $[0,1]^2$ except of points $(0,1)$ and $(1,0)$.

Conjunctive and disjunctive uninorms are dual in the following way

$$U_d(x,y) = 1 - U_c(1-x, 1-y) \,,$$

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where U_c is an arbitrary conjunctive uninorm and U_d its dual disjunctive uninorm. Assuming U_c has a neutral element e, the neutral element of U_d is $1-e$.

For an arbitrary uninorm U and arbitrary $(x,y) \in]0,e[\times]e,1] \cup]e,1] \times]0,e[$ we have

$$\min\{x,y\} \leq U(x,y) \leq \max\{x,y\}\,. \tag{1}$$

We say that a uninorm U contains a homomorphic image of a representable uninorm in $]a,b[^2$ for $0 \leq a < e < b \leq 1$ (where $a \neq 0$ and/or $b \neq 1$), if there exists a continuous strictly increasing function $\tilde{g}\colon [a,b] \to [-\infty,\infty]$ such that $\tilde{g}(a) = -\infty$, $\tilde{g}(b) = \infty$, $\tilde{g}(e) = 0$ and

$$U(x,y) = \tilde{g}^{-1}(\tilde{g}(x) + \tilde{g}(y)) \quad \text{for } x,y \in]a,b[\,. \tag{2}$$

1.2 Orders induced by t-norms

In [6] t-norms on bounded lattices were introduced.

Definition 1.2. *Let L be a bounded lattice. A function $T\colon L^2 \to L$ is said to be a t-norm if T is commutative, associative, monotone and $\mathbf{1}_L$ is its neutral element.*

Each uninorm U with a neutral element $0 < e < 1$, when restricted to the square $[0,e]^2$, is a t-norm (on the lattice $L = [0,e]$ equipped with meet and join) and when restricted to the square $[e,1]^2$, is a t-conorm (on the lattice $L = [e,1]$ equipped with meet and join). We will denote this t-norm by T_U and the t-conorm by S_U.

In [5], for a given t-norm T on a bounded lattice L a relation $\preceq_T$ generated by T was introduced. The definition is as follows

Definition 1.3. *Let $T\colon L^2 \to L$ be a given t-norm. For arbitrary $x,y \in L$ we denote $x \preceq_T y$ if there exists $\ell \in L$ such that $T(y,\ell) = x$.*

Proposition 1.4. ([5]) *Let T be an arbitrary t-norm. The relation $\preceq_T$ is a partial order. Moreover, if $x \preceq_T y$ holds for $x,y \in L$ then $x \leq y$, where $\leq$ is the order generated by lattice operations.*

Dually, we can introduce a partial order $\preceq_S$ for arbitrary t-conorm S by

$$x \preceq_S y \quad \text{if there exists } \ell \in [0,1] \text{ such that } S(y,\ell) = x\,.$$

However, in this case we have

$$x \preceq_S y \quad \Rightarrow \quad x \geq y\,.$$

1.3 Relation $\preceq_U$

As a generalization of the relation $\preceq_T$, Hliněná, Kalina and Král' in [3] introduced relation $\preceq_U$.

Definition 1.5 ([3])**.** *Let U be arbitrary uninorm. By $\preceq_U$ we denote the following relation*

$$x \preceq_U y \quad \textit{if there exists } \ell \in [0,1] \textit{ such that } U(y,\ell) = x\,.$$

Associativity of U implies transitivity of $\preceq_U$. Existence of a neutral element e implies reflexivity of $\preceq_U$. However, anti-symmetry of $\preceq_U$ is rather problematic.

Since for representable uninorm U and for arbitrary $x \in]0,1[$ and $y \in [0,1]$ there exists ℓ_y such that $U(x,\ell_y) = y$, the relation $\preceq_U$ is not necessarily anti-symmetric.

Lemma 1.6 ([3])**.** *Let U be arbitrary uninorm. The relation $\preceq_U$ is a pre-order.*

We introduce a relation $\sim_U$.

Definition 1.7 ([3])**.** *Let U be arbitrary uninorm. We say that $x,y \in [0,1]$ are U-indifferent if*

$$x \preceq_U y \quad \textit{and} \quad y \preceq_U x\,.$$

If x,y are U-indifferent, we write $x \sim_U y$.

Lemma 1.8 ([3])**.** *For arbitrary uninorm U the relation $\sim_U$ is an equivalence relation.*

2 Properties of uninorms induced by the relation $\preceq_U$

For arbitrary uninorm U with neutral element e the uninorm can be considered to be a binary operation on $[0,1]$. Thus $([0,1],U,e)$ becomes a commutative (i.e., Abelian) monoid which is moreover isotone with respect to the standard ordering of reals.

Lemma 2.1. *Let $U\colon [0,1]^2 \to [0,1]$ be an arbitrary uninorm and $x_1, x_2 \in [0,1]$ such that $x_1 \sim_U x_2$. Then*

$$U(x_1,x_1) \sim_U U(x_2,x_2) \sim_U U(x_1,x_2). \tag{3}$$

A direct corollary to Lemma 2.1 is the next assertion.

Proposition 2.2. *Let $U\colon [0,1]^2 \to [0,1]$ be an arbitrary uninorm and $e \in]0,1[$ be its neutral element. Assume there exists $x \in [0,1]$, $x \neq e$ and $y \in [0,1]$ such that $U(x,y) = e$. Then there exists a set $A_e \subset [0,1]$ such that for all $x \in A_e$, $x \sim_U e$. Moreover, if we denote by $\odot$ the binary operation on A_e defined by $x \odot y = U(x,y)$ for all $x, y \in A_e$, then $(A_e, \odot, e)$ is a non-trivial Abelian subgroup of $([0,1],U,e)$.*

Lemma 2.1 can be further generalized.

Proposition 2.3. *Let $U\colon [0,1]^2 \to [0,1]$ be an arbitrary uninorm and $x_1, x_2, y \in [0,1]$. Then the following holds*

$$(x_1 \sim_U x_2) \quad \Rightarrow \quad (U(x_1,y) \sim_U U(x_2,y)). \tag{4}$$

As Proposition 2.3 shows, the set A_e of all elements of $[0,1]$ which are indifferent from e generates classes of indifferent elements. In general, we have the following possibilities.

Proposition 2.4. *Let U be a fixed uninorm and e its neutral element. For arbitrary $x \in [0,1]$ denote the set $A_x = \{y \in [0,1]; x \sim_U y\}$. Then A_x is either a singleton or an infinite set.*
Further, assume that for a fixed x the set A_x is infinite and denote by $\odot_x$ the binary operation on A_x defined by $x \odot y = U(x,y)$ for all $x, y \in A_x$. Then there are the following possibilities.

- *$A_x = A_e$ and $(A_x, \odot_x, e)$ is a non-trivial Abelian subgroup of $([0,1],U,e)$.*
- *$A_x \neq A_e$ and $(A_x, \odot_x, \tilde{e})$ is a non-trivial Abelian group with a neutral element $\tilde{e} \neq e$. In this case $(A_x, \odot_x, \tilde{e})$ is not a subgroup of $([0,1],U,e)$.*
- *$x \not\sim_U U(x,x)$, i.e., $\odot_x$ is an operation on A_x but into $[0,1] \setminus A_x$. Moreover, there exists $y \in [0,1]$ such that $(A_y, \odot_y, \tilde{e})$ is a non-trivial Abelian group and for all $z \in A_y$ we have $U(x,z) \sim_U x$.*

3 Illustrative examples

In this section we provide illustrative examples. The first example shows uninorm U_1 which generates two infinite indifference classes with respect to $\sim_{U_1}$. One indifference class is A_e such that $(A_e, \odot_e, e)$ is a non-trivial subgroup of $([0,1],U_1,e)$. The other indifference class $A_{\frac{1}{8}}$ is such that for all $x \in A_{\frac{1}{8}}$ and all $y \in A_e$ we have $U(x,y) \in A_{\frac{1}{8}}$. The set $A_{\frac{1}{8}}$ cannot be organized into a group.

Example 3.1 ([3]). We recall the construction of a conjunctive uninorm which contains a homomorphic image of a representable uninorm U_r on $]\frac{1}{4},\frac{3}{4}[^2$. Further, on the rectangle $[0,\frac{1}{4}[\times[\frac{1}{4},\frac{3}{4}]$ the values of U_1

are given by the partial function $U_{\frac{1}{8}}(z) = \frac{z-\frac{1}{4}}{2}$. The explicit formula for the uninorm U_1 is the following

$$U_1(x,y) = \begin{cases} 0 & \text{if } \min\{x,y\} = 0 \\ & \text{ or if } \max\{x,y\} \le \frac{1}{4}, \\ 1 & \text{if } \min\{x,y\} \ge \frac{3}{4}, \\ \frac{1}{4} & \text{if } 0 < \min\{x,y\} \le \frac{1}{4} \\ & \text{ and if } \max\{x,y\} \ge \frac{3}{4}, \\ & \text{or if } \min\{x,y\} = \frac{1}{4} \\ & \text{ and } \max\{x,y\} > \frac{1}{4}, \\ U_r(x,y) & \text{if } (x,y) \in]\frac{1}{4},\frac{3}{4}[^2, \\ \max\{x,y\} & \text{if } \frac{1}{4} < \min\{x,y\} < \frac{3}{4} \\ & \text{ and } \max\{x,y\} \ge \frac{3}{4}, \end{cases}$$

and values on $]0,\frac{1}{4}[\times]\frac{1}{4},\frac{3}{4}[$ and $]\frac{1}{4},\frac{3}{4}[\times]0,\frac{1}{4}[$ are given by the partial function $U_{\frac{1}{8}}$ by formula (5) showing the computation of the value at a point $(x_2,y_2) \in \;]0,\frac{1}{4}[\;\times\;]\frac{1}{4},\frac{3}{4}[$. Assume that $x_2 = U(\frac{1}{8},y_1)$. Then

$$U(x_2,y_2) = U\left(\frac{1}{8}, U(y_1,y_2)\right). \tag{5}$$

The uninorm U_1 and its level-set functions of levels $\frac{1}{16},\frac{1}{8},\frac{3}{16}$ are sketched on Fig. 1.

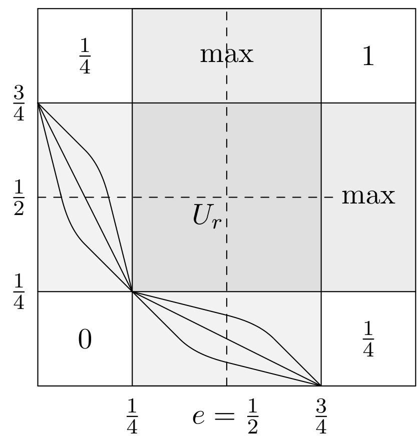

Figure 1: Uninorm U_1

The next example shows a uninorm U_2 which generates two infinite indifference classes with respect to $\sim_{U_2}$. One indifference class is A_e such that $(A_e,\odot_e,e)$ is a non-trivial subgroup of the monoid $([0,1],U_2,e)$. The other indifference class A equipped with operation $\odot_A = U_2 \restriction A$ is also an Abelian group, but not a subgroup of the monoid $([0,1],U_2,e)$.

Example 3.2. The construction of the uninorm U_2 we are going to present in this example, is based on the idea of paving that was introduced in [4]. The idea is the following. We split the unit interval into countably many disjoint subintervals $\{I_i\}_{i\in\mathbb{Z}}$ (in such a way we split the unit square into countably many disjoint sub-rectangles $I_i \times I_j$). Then we choose an operation $\otimes\colon [0,1]^2 \to [0,1]$ we want to use for paving, choose increasing bijective transformations $\varphi_i\colon I_i \to [0,1]$ and we "pave" the whole unit square (see Fig. 2 for a graphical schema of paving) by

$$\varphi_{i+j}^{-1}(\varphi_i(x) \otimes \varphi_j(y)). \tag{6}$$

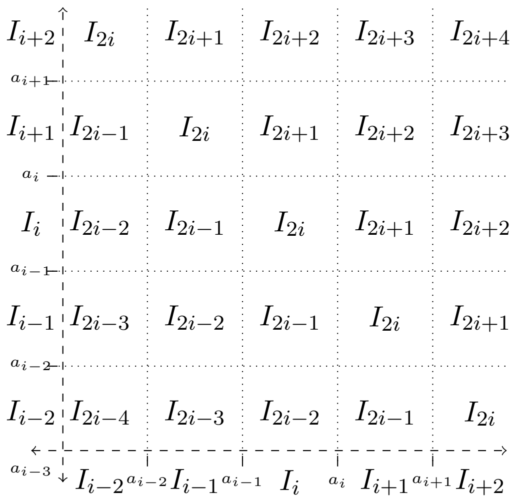

Figure 2: Graphical schema of paving

To make each point $(x, y) \in [0,1]^2$ uniquely identifiable with a rectangle $I_i \times I_j$, we will use left-open intervals. Then the operation $\otimes$ used for paving must be without zero-divisors. In this case we choose a representable uninorm U_r as the operation $\otimes$, and the following partition of $]0,1[$:

$$I_i = \begin{cases}]\frac{1}{4}, \frac{3}{4}] & \text{if } i = 0, \\ ]\frac{2^{i+1}-1}{2^{i+1}}, \frac{2^{i+2}-1}{2^{i+2}}] & \text{if } i > 0, \\ ]\frac{1}{2^{2-i}}, \frac{1}{2^{1-i}}] & \text{if } i < 0. \end{cases} \tag{7}$$

The uninorm U_2 is defined by

$$U_2(x,y) = \begin{cases} \varphi_{i+j}^{-1}\left(U_r\left(\varphi_i(x), \varphi_j(y)\right)\right) & \text{if } x \in I_i,\ y \in I_j, \\ 0 & \text{if } \min\{x,y\} = 0, \\ 1 & \text{if } \max\{x,y\} = 1 \text{ and } \min\{x,y\} \neq 0. \end{cases}$$

Let us remark that the increasing bijective transformations $\varphi_i \colon I_i \to [0,1]$ are chosen arbitrarily (and for every choice we get a different uninorm). The relation $\preceq_{U_2}$ generates two indifference classes – $A_{\frac{3}{4}} = \{\frac{2^{i+1}-1}{2^{i+1}}; i \in \mathbb{N}\} \cup \{\frac{1}{2^{1+i}}; i \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of positive integers, and $A_e =]0,1[\setminus A_{\frac{3}{4}}$. Then $(A_e, \odot_e, e)$ is a non-trivial subgroup of $([0,1], U_2, e)$ and $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ is an Abelian group that is not a subgroup of $([0,1], U_2, e)$.

Remark 3.3. If we look at the two Abelian groups induced by the uninorm U_2 (Example 3.2), they are in some sense different. While $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ has no non-trivial subgroups, $(A_e, \odot_e, e)$ has a non-trivial subgroup, namely $\left(\,]\frac{1}{4}, \frac{3}{4}[\,, (\odot_e \restriction]\frac{1}{4}, \frac{3}{4}[), e\right)$.

In the last example we modify the uninorm U_2 from Example 3.2 in two ways.

Example 3.4. We take the product t-norm T_Π for the operation $\otimes$ used for paving. We split the interval $]0,1[$ in the same way as in Example 3.2, i.e., the partition is given by formula (7). As the result of paving we get uninorm U_3 defined by the following

$$U_3(x,y) = \begin{cases} \varphi_{i+j}^{-1}\left(T_\Pi\left(\varphi_i(x), \varphi_j(y)\right)\right) & \text{if } x \in I_i,\ y \in I_j, \\ 0 & \text{if } \min\{x,y\} = 0, \\ 1 & \text{if } \max\{x,y\} = 1 \text{ and } \min\{x,y\} \neq 0. \end{cases} \tag{8}$$

Also in this case we can choose arbitrarily the increasing bijective transformations $\varphi_i \colon I_i \to [0,1]$. I.e., correctly speaking, we have got a system of uninorms. But they all induce the same pre-order $\preceq_{U_3}$.

Denote a_i and b_i the left- and right-end-points of the interval I_i, respectively. Then for $x_i \in I_i$ and $x_j \in I_j$ we have $x_i \sim_{U_3} x_j$ if and only if $\frac{x_i - a_i}{b_i - a_i} = \frac{x_j - a_j}{b_j - a_j}$. The set A_{x_0} for $x_0 \in I_0$, $x_0 \neq \frac{3}{4}$, cannot be organized into a group, and $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ is a subgroup of $\left([0,1], U_3, \frac{3}{4}\right)$.

If we choose the minimum t-norm, T_M, instead of the product in the formula (8), and use the same partition given by formula (7), we get again the same system of equivalence classes. But in this case for all $x_0 \in I_0$ the algebraic system $(A_{x_0}, \odot_{x_0}, x_0)$ is an Abelian group, and for $x_0 = \frac{3}{4}$ it is a subgroup of $\left([0,1], U_3, \frac{3}{4}\right)$.

Remark 3.5. We have seen in Example 3.4 that uninorms U_3, U_4 induce the same pre-order, i.e., $\preceq_{U_3} = \preceq_{U_4}$. If we look at algebraic properties of equivalence classes got by the pre-orders $\preceq_{U_3}$ and $\preceq_{U_4}$, they are different. This means, in some cases, when different uninorms induce the same pre-order the underlying algebraic properties of equivalence classes may help to distinguish types of uninorms in question.

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