Generalization of the discrete Choquet integral

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Abstract: *In this paper, we present an approach to generalization of the discrete Choquet integral. We replace the product operator joining capacity m of criteria sets and values of score vector by a fusion function F satisfying some constraints, similarly as was already done for another form of the discrete Choquet integral in [2]. The properties of obtained functional C_F^m are studied and some examples for particular capacities m are given.*

Keywords: *Choquet integral, fusion function*

1 Introduction

The evaluation of a score vector achieved in some set of criteria is a long-term point of interest in multicriteria decision making theory. One of the useful tools used for that purpose is the Choquet integral [1], which was generalized in several ways, see, for instance, [3], [4].

Our generalization was inspired by that of Mesiar et al. in [2], where the authors generalized one of the two usually used discrete forms of the Choquet integral (see below, the formula (1)) replacing the product operator by fusion function satisfying certain conditions. Using the same idea, we generalize the other formula (see the formula (2)) for the discrete Choquet integral. Note that, in general, the resulting functional differs from that obtained in [2].

We recall the definition of the Choquet integral on a general monotone measure space $(X, \mathfrak{S}, m)$, where X is a non-empty set X, $\mathfrak{S}$ is a σ-algebra of its subsets and $m : \mathfrak{S} \to [0, \infty]$ a monotone measure, i.e., a set function satisfying the properties $m(\emptyset) = 0$ and $m(A) \leq m(B)$ for all $A, B \in \mathfrak{S},\ A \subseteq B$.

Definition 1.1. *Let $(X, \mathfrak{S}, m)$ be a monotone measure space. For any $\mathfrak{S}$-measurable function $f : X \to [0, 1]$ the Choquet integral $Ch_m(f)$ is given by*

$$Ch_m(f) = \int_0^1 m(\{x \in X | f(x) \geq t\}) \,\mathrm{d}t,$$

where the integral on the right-hand side is the Riemann integral.

In this paper we will only deal with finite spaces $X = \{1, \ldots, n\}$ for some $n \in N$, $n \geq 2$, $\mathfrak{S} = 2^X$ and normalized monotone measures $m :\ 2^X \to [0, 1]$, i.e., monotone measures with $m(X) = 1$, calling them capacities [5]. The set of all capacities $m :\ 2^X \to [0, 1]$ will be denoted by $\mathcal{M}_n$. Any 2^X-measurable function $f : X \to [0, 1]$ will be identified with a vector $\mathbf{x} = (x_1, \ldots, x_n) \in [0, 1]^n$, where $x_i = f(i),\ i = 1, \ldots, n$.

A discrete form of the Choquet integral is of a great importance in decision making theory, regarding a finite set $X = \{1, \ldots, n\}$ as some criteria set, a vector $\mathbf{x} \in [0, 1]^n$ as a score vector and a capacity $m :\ 2^X \to [0, 1]$ as the weights of particular sets of criteria.

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Proposition 1.2. *Let $X = \{1, \ldots, n\}$ and let $m : 2^X \to [0,1]$ be a capacity. Then for any $\mathbf{x} \in [0,1]^n$ the discrete Choquet integral is given by*

$$\mathcal{C}h_m(\mathbf{x}) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)}) \cdot m(E_{(i)}), \tag{1}$$

where $(\cdot) : X \to X$ is a permutation such that $x_{(1)} \leq \cdots \leq x_{(n)}$, $E_{(i)} = \{(i), \ldots, (n)\}$ for $i = 1, \ldots, n$, and $x_{(0)} = 0$,

or, equivalently, by

$$\mathcal{C}h_m(\mathbf{x}) = \sum_{i=1}^{n} x_{(i)} \cdot (m(E_{(i)}) - m(E_{(i+1)})), \tag{2}$$

with $x_{(i)}$ and $E_{(i)}$, $i = 1, \ldots, n$, as above, and $E_{(n+1)} = \emptyset$.

Observe that information contained in a score vector and that in a capacity are joined by the standard product operator. Replacing the product in formulae (1) and (2) by a function $F : [0,1]^2 \to [0,1]$ (a binary fusion function), we obtain the formulae:

$$C_m^F(\mathbf{x}) = \sum_{i=1}^{n} F(x_{(i)} - x_{(i-1)}, m(E_{(i)})) \tag{3}$$

and

$$C_F^m(\mathbf{x}) = \sum_{i=1}^{n} F(x_{(i)}, m(E_{(i)}) - m(E_{(i+1)})), \tag{4}$$

respectively.

The functionals C_F^m defined by (3) were deeply studied in [2] including a complete characterization of functionals C_m^F as aggregation functions.

In this paper, we will analyse the functionals defined by (4). The paper is organized as follows. In the next section, we provide the conditions under which a functional C_F^m is correctly defined for any capacity $m \in \mathcal{M}_n$ and any $\mathbf{x} \in [0,1]^n$ and, for suitable fusion functions, we exemplify C_F^m for several particular capacities. In Section 3, we provide several properties of functionals C_F^m and show their connection with the discrete Choquet integral. Finally, some concluding remarks are added.

2 Operators C_F^m

Let us first analyse conditions under which the functionals C_F^m introduced in (4) are well defined.

Evidently, for a score vector $\mathbf{x} \in [0,1]^n$ with $\text{card}\,\{x_1, \ldots, x_n\} = n$ there is a unique permutation $(\cdot) : X \to X$ such that $x_{(1)} \leq \cdots \leq x_{(n)}$ (in fact, all inequalities are strict). Thus C_F^m is correctly defined by formula (4). If some ties occure, i.e., if $\text{card}\,\{x_1, \ldots, x_n\} < n$, we have to analyse the following two cases.
Case 1: Let $n = 2$. Consider $\mathbf{x} = (x_1, x_2) = (x, x)$, and a capacity $m_{a,b} \in \mathcal{M}_2$ defined by $m_{a,b}(\{1\}) = a$ and $m_{a,b}(\{2\}) = b$, where $a, b \in [0,1]$. Then $C_F^{m_{a,b}}(x, x)$ is well defined only if formula (4) gives back the same value for both possible permutations (1,2) and (2,1) ordering the vector $\mathbf{x}$ increasingly, i.e., if it holds

$$F(x, 1-a) + F(x, a) = F(x, 1-b) + F(x, b)$$

for all $a, b \in [0,1]$.

Consequently, we obtain the following proposition.

Proposition 2.1. *Let $n = 2$. Then $C_F^m : [0,1]^2 \to [0,2]$ introduced in (4) is well defined if and only if*

$$F(x, y) + F(x, 1-y) = 2F(x, 1/2), \qquad \text{for any } x, y \in [0,1]\,.$$

We can immediately characterize all well defined functionals $C_F^{m_{a,b}}$:

$$C_F^{m_{a,b}}(x,y) = \begin{cases} F(x,1-b)+F(y,b) & \text{if } x<y, \\ 2F(x,1/2) & \text{if } x=y, \\ F(x,a)+F(y,1-a) & \text{if } x>y. \end{cases}$$

Example 2.2. *Consider $F:[0,1]^2 \to [0,1]$ defined by $F(x,y)=\frac{x}{2}((2y-1)^3+1)$, see Fig. 1. Then F satisfies the constraints of Proposition 2.1. and thus C_F^m is correctly defined for any $m_{a,b} \in \mathcal{M}_2$. Note that then*

$$C_F^{m_{a,b}}(x,y) = \begin{cases} \frac{x+y}{2} + \frac{(y-x)}{2}(2b-1)^3 & \textit{if } x \le y, \\ \frac{x+y}{2} + \frac{(x-y)}{2}(2a-1)^3 & \textit{otherwise,} \end{cases}$$

see Fig. 2.

If $a=b$, i.e., $m_{a,a}$ is a symetric capacity, then

$$C_F^{m_{a,a}}(x,y) = \frac{x+y}{2} + \frac{|x-y|}{2}(2a-1)^3 \qquad \text{for all } x,y \in [0,1]\,.$$

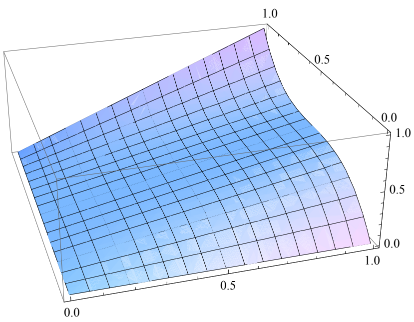

Figure 1: $F(x,y)=\frac{x}{2}((2y-1)^3+1)$

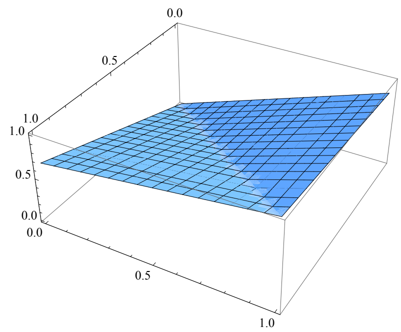

Figure 2: $C_F^{m_{a,b}},\ a=0.85, b=0.95$

Case 2: Now, consider $n>2$ and a vector $\mathbf{x}=(x_1,\cdots,x_n) \in [0,1]^n$ such that $\operatorname{card}\{x_1,\cdots,x_n\} < n$. Without loss of generality we can suppose that $\operatorname{card}\{x_1,\cdots,x_n\} = n-1$ and $x_1 = x_2 = \min\{x_1,\cdots,x_n\} = x$. Then, similarly as before, we find out that $C_F^m(\mathbf{x})$ is well defined only if

$$F(x,1-m(\{2,3,\cdots,n\})) + F(x,m(\{2,3,\cdots,n\})-m(\{3,\cdots,n\})) = \\ F(x,1-m(\{1,3,\cdots,n\})) + F(x,m(\{1,3,\cdots,n\})-m(\{3,\cdots,n\}))\,.$$

The last equality has to be satisfied for any capacity $m \in \mathcal{M}_n$, i.e., for any $\alpha,\beta,\gamma,\delta \in [0,1]$ such that $\alpha+\beta=\gamma+\delta \in [0,1]$ it should hold that

$$F(x,\alpha)+F(x,\beta)=F(x,\gamma)+F(x,\delta).$$

The only solution of this Cauchy's equation is of the form

$$F(x,y)=f(x)\cdot y, \tag{5}$$

where $f:[0,1]\to[0,1]$ is an arbitrary function. On the other hand, any function F of the form (5) yields a well defined functional $C_F^m:[0,1]^n \to [0,n]$.

Proposition 2.3. *Let $n>2$. The functional $C_F^m:[0,1]^n \to [0,n]$ is well defined for any $m \in \mathcal{M}_n$ if and only if $F(x,y)=f(x)\cdot y$ for all $x,y \in [0,1]$ and some function $f:[0,1]\to[0,1]$. In that case*

$$C_F^m(\mathbf{x}) = \sum_{i=1}^n f(x_{(i)})\cdot(m(E_{(i)})-m(E_{(i+1)})). \tag{6}$$

Example 2.4. *Consider $F : [0,1]^2 \to [0,1]$ given by $F(x,y) = (1-x)y$, which satisfies Proposition 2.3. Then for each $m \in \mathcal{M}_n$ and $\mathbf{x} \in [0,1]^n$ it holds:*

$$C_F^m(\mathbf{x}) = 1 - \mathcal{C}h_m(\mathbf{x}) = \mathcal{C}h_{m^d}(\mathbf{1} - \mathbf{x}),$$

where m^d is a dual capacity to m, given by $m^d(E) = 1 - m(E^c)$. Note that C_F^m is a decreasing operator, $C_F^m(0,\dots,0) = 1$ and $C_F^m(1,\dots,1) = 0$.

Using (6), for a fixed suitable fusion function F given by (5), we can derive C_F^m for some particular capacities $m \in \mathcal{M}_n$, see the following table.

$m \in \mathcal{M}_n$	$C_F^m; \quad F(x,y) = f(x) \cdot y$
$m^*(E) = \begin{cases} 1 & \text{if } E \neq \emptyset, \\ 0 & \text{if } E = \emptyset \end{cases}$	$C_F^{m^*}(\mathbf{x}) = f(x_{(n)}) = f(\max\limits_{1 \le i \le n} x_i)$
$m_*(E) = \begin{cases} 1 & \text{if } E = \{1,\cdots,n\}, \\ 0 & \text{otherwise} \end{cases}$	$C_F^{m_*}(\mathbf{x}) = f(x_{(1)}) = f(\min\limits_{1 \le i \le n} x_i)$
$m_H(E) = \begin{cases} 1 & \text{if } H \subseteq E, \\ 0 & \text{otherwise} \end{cases}$ $\emptyset \neq H \subseteq X$	$C_F^{m_H}(\mathbf{x}) = f(x_i)$, where $\{j \in \{1,\cdots,n\} \mid x_j \ge x_i\} \supseteq H$ but $\{j \in \{1,\cdots,n\} \mid x_j > x_i\} \supseteq H$ does not hold
$\overline{m}(E) = \frac{card\ E}{n}$	$C_F^{\overline{m}}(\mathbf{x}) = \frac{1}{n} \sum\limits_{i=1}^{n} f(x_i)$

Note that m^* and m_* are the greatest and the smallest elements of $\mathcal{M}_n$, respectively, and that $m_* = m_H$ for $H = X$.

3 C_F^m with some particular properties

In this section, we formulate several properties of functionals C_F^m and also show the connection C_F^m with the discrete Choquet integral.

At first, we recall that the functional C_F^m is:

- an *aggregation function*, if C_F^m is monotone increasing and $C_F^m(\mathbf{0}) = 0$, $C_F^m(\mathbf{1}) = 1$;
- a *mean*, if for each $\mathbf{x} \in [0,1]^n$ it holds $\text{Min}(\mathbf{x}) \le C_F^m(\mathbf{x}) \le \text{Max}(\mathbf{x})$, where $\text{Min}(\mathbf{x}) = \min\{x_1,\dots,x_n\}$, $\text{Max}(\mathbf{x}) = \max\{x_1,\dots,x_n\}$;
- *translation invariant*, if $C_F^m(x_1 + c,\dots,x_n + c) = c + C_F^m(x_1,\dots,x_n)$ for all $c \in]0,1]$ and $(x_1,\dots,x_n) \in [0,1]^n$ such that $(x_1 + c,\dots,x_n + c) \in [0,1]^n$;
- *idempotent*, if $C_F^m(x,\dots,x) = x$ for each $x \in [0,1]$.

For the functionals C_F^m of the form (6) with F satisfying (5), the following properties can be directly derived.

Proposition 3.1. *Let $F : [0,1]^2 \to [0,1]$, $F(x,y) = f(x) \cdot y$, where $f : [0,1] \to [0,1]$. Then for any fixed $n \geq 2$ it holds that*

(i) C_F^m is an aggregation function for each $m \in \mathcal{M}_n$ if and only if f is an increasing function satisfying $f(0) = 0$ and $f(1) = 1$.

(ii) $C_F^m \geq \mathrm{Min}$ for each $m \in \mathcal{M}_n$ if and only if f is an increasing function satisfying $f(x) \geq x$ for all $x \in [0,1]$.

(iii) $C_F^m \leq \mathrm{Max}$ for each $m \in \mathcal{M}_n$ if and only if f is an increasing function satisfying $f(x) \leq x$ for all $x \in [0,1]$.

(iv) C_F^m is a mean for each $m \in \mathcal{M}_n$ if and only if F is the product operator.

(v) C_F^m is translantion invariant for each $m \in \mathcal{M}_n$ if and only if F is the product operator.

(vi) C_F^m is idempotent for each $m \in \mathcal{M}_n$, if and only if F is the product operator.

Note that for the standard product $F(x,y) = x.y$ the functional C_F^m coincides with $\mathcal{C}h_m$, therefore the properties (iv), (v), (vi) hold only for the Choquet integral itself.

Since an increasing function preserves ordering of an input vector and a decreasing one inverts it, we obtain the following propositions that show the connection between C_F^m and the discrete Choquet integral.

Proposition 3.2. *Let $F : [0,1]^2 \to [0,1]$, $F(x,y) = f(x) \cdot y$, where $f : [0,1] \to [0,1]$ is an increasing function. Then, for each $m \in \mathcal{M}_n$ and $\mathbf{x} \in [0,1]^n$,*

$$C_F^m(\mathbf{x}) = \mathcal{C}h_m(f(\mathbf{x})),$$

where $f(\mathbf{x}) = (f(x_1), \cdots, f(x_n))$.

Proposition 3.3. *Let $F : [0,1]^2 \to [0,1]$, $F(x,y) = f(x) \cdot y$, where $f : [0,1] \to [0,1]$ is a decreasing function. Then, for each $m \in \mathcal{M}_n$ and $\mathbf{x} \in [0,1]^n$,*

$$C_F^m(\mathbf{x}) = 1 - \mathcal{C}h_m(1 - f(\mathbf{x})) = \mathcal{C}h_{m^d}(f(\mathbf{x}))$$

where $m^d \in \mathcal{M}_n$ is a capacity dual to m.

Note that the last property was already ilustrated for a special function F in Example 2.4.

4 Concluding remarks

We have generalized the formula (2) for the discrete Choquet integral, replacing the standard product operator by a function $F : [0,1]^2 \to [0,1]$. Several particular operators C_F^m were discussed, based either on a fixed capacity $m \in \mathcal{M}_n$ or on a fixed function F. We expect applications of our results in all domains where the generalizations of the discrete Choquet integral are considered, for example in medicine.

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