Fuzzy interval orders and aggregation process

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Abstract: *In this contribution, conditions for n-argument functions to preserve fuzzy interval orders during aggregation process are presented. The considered properties of a fuzzy interval order (Ferrers property and connectedness) depend on binary operations including t-norms and t-conorms and, more generally, fuzzy conjunctions and disjunctions. Moreover, some existing results on Ferrers property are generalized and applied for fuzzy interval orders.*

Keywords: *fuzzy relation, fuzzy connective, aggregation function, Ferrers property, fuzzy interval order*

1 Introduction

Fuzzy order structures, such as linear orders, semi-orders and interval orders are often used to model preferences in decision making problems. In this contribution we pay attention to fuzzy interval orders which definitions are based on the notions of Ferrers property and total connectedness. Ferrers property is less demanding than transitivity property used in the most of orders, so it is worthy do examine Ferrers property and fuzzy interval orders from the application point of view.

We will consider fuzzy interval orders in the context of their preservation in aggregation process (cf. [6, 8, 11, 12, 15, 17]) which is due to the possible applications, e.g. in fuzzy preference modelling, multicriteria decision making problems and solving other issues related to imprecise and uncertain information. In decision making problems a set $X = \{x_1, ..., x_m\}$ represents a set of objects, where $m \in \mathbb{N}$. There is also considered a set $K = \{k_1, ..., k_n\}$ of criteria under which the objects are supposed to be evaluated. Fuzzy relations $R_1, ..., R_n$ reflect judgements of decision makers. The considered aggregation process involves also an n-argument function F. With the use of given fuzzy relations $R_1, \dots, R_n$ and the function F, we consider a new fuzzy relation $R_F = F(R_1, \dots, R_n)$ representing a final decision on evaluated objects (after considering the involved criteria). Although we focus on aggregation functions, the aim of this paper is to give the results under the weakest assumptions on F used for the aggregation process. Therefore, we start our considerations with an arbitrary n-ary function.

The notions of fuzzy relation properties, in their simplest forms, may involve functions $\min$ and $\max$. These ones were generalized by the use of a t-norm and t-conorm, respectively [11, Chapter 2.5]. In particular, the following properties were examined: T-asymmetry, T-antisymmetry, S-connectedness, T-transitivity, negative S-transitivity, T-S-semitransitivity, and T-S-Ferrers property of fuzzy relations, where T is a t-norm and S a t-conorm, also with regard to their preservation in aggregation process [9]. However, the assumptions put on widely used t-norms are not always necessary or desired. This is why a lot of definitions of binary operations which can play a role of weaker fuzzy connectives were introduced and studied, for example fuzzy conjunctions: weak t-norms, overlap functions, t-seminorms (or semicopulas, or conjunctors), and pseudo-t-norms, sometimes along with their dual disjunctions.

In this article, we consider the properties of fuzzy relations which definitions are based on fuzzy conjunctions and disjunctions including t-norms and t-conorms. In order to obtain the most general results we start with binary operations in the unit interval without any additional assumptions. As a result we examine fuzzy interval orders which are totally B-connected and fulfil B_1-B_2-Ferrers property, where $B, B_1, B_2 : [0, 1]^2 \to [0, 1]$ are binary operations.

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In Section 2, we provide basic definitions and results concerning n-ary functions in $[0,1]$ including fuzzy connectives and dominance between functions. Next, in Section 3, we present basic information about fuzzy relations and some useful results related to preservation of fuzzy relation properties in aggregation process. Finally, in Section 4 we put the main results of this contribution connected with fuzzy interval orders in aggregation process.

2 Preliminaries

In this section we present the notions useful in further considerations, i.e. properties of n-ary functions in $[0,1]$, fuzzy connectives, dominance between operations.

Definition 2.1 ([5]). *Let $n \in \mathbb{N}$. A function $A : [0,1]^n \to [0,1]$ which is increasing, i.e.*

$$A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n) \text{ for } x_i, y_i \in [0,1], x_i \leqslant y_i, i = 1, \ldots, n$$

is called an aggregation function if $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

Example 2.2. *Aggregation functions are:*

- *median*

$$med(t_1, \ldots, t_n) = \begin{cases} \frac{s_k + s_{k+1}}{2}, & \text{for } n = 2k \\ s_{k+1}, & \text{for } n = 2k+1 \end{cases},$$

  where $(s_1, \ldots, s_n)$ is the increasingly ordered sequence of the values $t_1, \ldots, t_n$, which means that $s_1 \leqslant \ldots \leqslant s_n$.

- *a weighted arithmetic mean*

$$A_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k, \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1,$$

- *a quasi-linear mean*

$$F(x_1, ..., x_n) = \varphi^{-1}(\sum_{k=1}^{n} w_k \varphi(x_k)), \quad \text{for} \quad w_k > 0, \sum_{k=1}^{n} w_k = 1,$$

where $x_1, \ldots, x_n \in [0,1]$, $\varphi : [0,1] \to \mathbb{R}$ is a continuous, strictly increasing function.

Definition 2.3. *Let $n \in \mathbb{N}$. We say that a function $F : [0,1]^n \to [0,1]$:*
• *has a zero element $z \in [0,1]$ if for each $k \in \{1, \ldots, n\}$ and each $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in [0,1]$ one has*

$$F(x_1, ..., x_{k-1}, z, x_{k+1}, ..., x_n) = z,$$

• *is without zero divisors if it has a zero element z and*

$$\underset{x_1,...,x_n \in [0,1]}{\forall} \left(F(x_1, ..., x_n) = z \Rightarrow \left(\underset{1 \leqslant k \leqslant n}{\exists} x_k = z\right)\right).$$

Definition 2.4 ([10]). *An operation $C : [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and $C(1,1) = 1$, $C(0,0) = C(0,1) = C(1,0) = 0$. An operation $D : [0,1]^2 \to [0,1]$ is called a fuzzy disjunction if it is increasing with respect to each variable and $D(0,0) = 0$, $D(1,1) = D(0,1) = D(1,0) = 1$.*

Corollary 2.5. *A fuzzy conjunction has a zero element* 0. *A fuzzy disjunction has a zero element* 1.

Definition 2.6. *An operation $C\colon [0,1]^2 \to [0,1]$ is called:*

- *an overlap function [3] if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition* $C(x, y) = 1$ *if and only if* $xy = 1$,
- *a t-norm [18] if it is a commutative, associative, increasing operation with neutral element* 1.

Definition 2.7. *An operation* $D\colon [0,1]^2 \to [0,1]$ *is called:*

- *a grouping function [4] if it is a commutative, continuous fuzzy disjunction without zero divisors, fulfilling condition* $D(x, y) = 0$ *if and only if* $x = y = 0$,
- *a t-conorm [14] if it is a commutative, associative, increasing operation with neutral element* 0,
- *a strict t-conorm* $S : [0,1]^2 \to [0,1]$ *if it is a t-conorm which is continuous and strictly increasing in* $[0,1)^2$.

Example 2.8 ([14])**.** *The Łukasiewicz t-norm and t-conorm are described in the following way* $T_L(s,t) = \max(s+t-1, 0)$ *and* $S_L(s,t) = \min(s+t, 1)$*, respectively.*

Definition 2.9. *A t-norm* T *is called nilpotent if it is continuous and each* $x \in (0,1)$ *is a nilpotent element of* T*, i.e. for each* $x \in (0,1)$ *there exists* $n \in \mathbb{N}$ *such that* $x_T^{(n)} = 0$.

Theorem 2.10. *Any nilpotent t-norm is isomorphic to the Łukasiewicz t-norm* T_L*, i.e.*

$$T(x,y) = \varphi^{-1}(T_L(\varphi(x), \varphi(y))), \ \ x, y \in [0,1],$$

where $\varphi : [0,1] \to [0,1]$ *is an increasing bijection.*

Definition 2.11 ([13])**.** *A rotation invariant t-norm is a t-norm* T *that verifies for all* $x, y, z \in [0,1]$

$$T(x,y) \leqslant z \Leftrightarrow T(x, 1-z) \leqslant 1-y.$$

Definition 2.12 ([5])**.** *Let* $F : [0,1]^n \to [0,1]$*. A function* F^d *is called a dual function to* F*, if for all* $x_1, \ldots, x_n \in [0,1]$

$$F^d(x_1, \ldots, x_n) = 1 - F(1-x_1, \ldots, 1-x_n).$$

F *is called a self-dual function, if it holds* $F = F^d$.

Fuzzy disjunctions are dual to fuzzy conjunctions, grouping functions are dual to overlap functions, t-conorms are dual functions to t-norms, in particular S_L is dual to T_L, $\max$ is dual to $\min$. Now, we recall the notion of dominance.

Definition 2.13 ([19])**.** *Let* $m, n \in \mathbb{N}$*. A function* $F\colon [0,1]^m \to [0,1]$ *dominates function* $G\colon [0,1]^n \to [0,1]$ *(* $F \gg G$*) if for an arbitrary matrix* $[a_{ik}] = A \in [0,1]^{m\times n}$ *the following inequality holds*

$$F(G(a_{11},...,a_{1n}),...,G(a_{m1},...,a_{mn})) \geqslant G(F(a_{11},...,a_{m1}),...,F(a_{1n},...,a_{mn})).$$

Example 2.14 ([1])**.** *Any weighted arithmetic mean dominates t-norm* T_L *and any weighted arithmetic mean is dominated by* S_L*. Minimum dominates any fuzzy conjunction. Fuzzy disjunctions dominate maximum.*

3 Fuzzy relations

Here we recall the notion of a fuzzy relation, some properties of fuzzy relations and their preservation in aggregation process.

Definition 3.1 ([20])**.** *A fuzzy relation in a set* $X \neq \emptyset$ *is an arbitrary function* $R : X \times X \to [0,1]$*. The family of all fuzzy relations in* X *is denoted by* $FR(X)$*.*

Definition 3.2 (cf. [11, 16]). *Let $B, B_1, B_2 : [0,1]^2 \to [0,1]$ be binary operations. Relation $R \in FR(X)$ is:*
• *reflexive, if* $\underset{x \in X}{\forall} R(x,x) = 1$,
• *totally B-connected, if*
$\underset{x,y \in X}{\forall} B(R(x,y), R(y,x)) = 1$,
• *B_1-B_2-Ferrers, if*
$\underset{x,y,z,w \in X}{\forall} B_1(R(x,y), R(z,w)) \leqslant B_2(R(x,w), R(z,y))$,
• *a B-B_1-B_2-fuzzy interval order, if it is totally B-connected and B_1-B_2-Ferrers,*
• *a B_1-B_2-fuzzy interval order, if it is totally B_2-connected and B_1-B_2-Ferrers.*

We present the notions of the given properties in the most general version, i.e. with operations $B, B_1, B_2 : [0,1]^2 \to [0,1]$. However, the natural approach is to consider a fuzzy disjunction B in definition of B-connectedness, a fuzzy conjunction B_1 and a fuzzy disjunction B_2 in the Ferrers property.

Let $F : [0,1]^n \to [0,1]$, $R_1, \dots, R_n \in FR(X)$. An aggregated fuzzy relation $R_F \in FR(X)$ is described by the formula $R_F(x,y) = F(R_1(x,y), \dots, R_n(x,y))$, $x, y \in X$. A function F preserves a property of fuzzy relations if for every $R_1, \dots, R_n \in FR(X)$ having this property, R_F also has this property. Preservation of the properties listed above and also other properties of this kind was considered in [1]. We will recall here only the results for the properties that will be useful in the sequel.

Theorem 3.3. *Let $R_1, \dots, R_n \in FR(X)$ be reflexive. The relation R_F is reflexive, if and only if the function F satisfies the condition $F(1, \dots, 1) = 1$.*

Theorem 3.4. *Let card $X \geqslant 2$, B have a zero element 1 and be without zero divisors. A function F preserves total B-connectedness (B-connectedness) if and only if it satisfies the following condition for all $s, t \in [0,1]^n$*

$$\underset{1 \leqslant k \leqslant n}{\forall} \max(s_k, t_k) = 1 \Rightarrow \max(F(s), F(t)) = 1. \tag{1}$$

Example 3.5. *Let B be a fuzzy disjunction without zero divisors (e.g. a strict t-conorm or a grouping function). Examples of functions fulfilling* (1) *for all $s, t \in [0,1]^n$ are $F = \max$, $F =$med or functions F with the zero element $z = 1$ with respect to a certain coordinate, i.e.*

$$\underset{1 \leqslant k \leqslant n}{\exists} \; \underset{i \neq k}{\forall} \; \underset{t_i \in [0,1]}{\forall} F(t_1, \dots, t_{k-1}, 1, t_{k+1}, \dots, t_n) = 1.$$

Theorem 3.6. *If a function $F \colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F \gg B_1$ and $B_2 \gg F$, then it preserves B_1-B_2-Ferrers property.*

Lemma 3.7. *Let $B : [0,1]^2 \to [0,1]$ and B^d be a corresponding dual operation. If $F : [0,1]^n \to [0,1]$ is a self-dual function, then $F \gg B$ implies $B^d \gg F$.*

The condition given in Lemma 3.7 is only the sufficient one. Let us consider projections $F = P_k$, $B = T$ being a t-norm, $S = T^d$. Then $S \gg P_k$ and $P_k \gg T$, but $F \neq F^d$.

Example 3.8. *Any weighted arithmetic mean preserves B_1-B_2-Ferrers property for t-norm $T_L = B_1$ and t-conorm $S_L = B_2$.*

Corollary 3.9. *Any quasi-linear mean preserves T-S-Ferrers property for a nilpotent t-norm T and $S = T^d$.*

Conditions given in Theorem 3.6 are only the sufficient ones. Let us consider function $F(s,t) = st$ (so $F = T_P$) and fuzzy relations presented by the matrices

$$R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Relations R_1, R_2 are min-max-Ferrers ([11]). Moreover $R = F(R_1, R_2)$ is min-max-Ferrers, where $R \equiv 0$. However, it is not true that $F \gg \min$ (the only t-norm that dominates minimum is minimum itself).

4 Aggregation of fuzzy interval orders

Now, we will consider fuzzy interval orders and their properties in the process of aggregation. In this section B, B_1, B_2 denote binary operations on unit interval, i.e. $B, B_1, B_2 : [0,1]^2 \to [0,1]$.

Theorem 4.1. *Let B_1 have an idempotent element 1. A reflexive B_1-B_2-Ferrers relation is totally B_2-connected.*

Proof. If R is a reflexive, B_1-B_2-Ferrers fuzzy relation, then we get

$$1 = B_1(1,1) = B_1(R(x,x), R(y,y)) \leqslant B_2(R(x,y), R(y,x)),$$

which means that $B_2(R(x,y), R(y,x)) = 1$ and R is totally B_2-connected. □

Corollary 4.2. *Let T be a t-norm and S a t-conorm. A reflexive T-S-Ferrers relation is totally S-connected.*

Theorem 4.3. *Let B_1 have a zero element 0, an idempotent element 1 and for each $x, y \in [0,1]$ such that $x + y > 1$ fulfil $B_1(x,y) = B_1(y,x)$ and let B_2 be dual to B_1 such that $B_1 \leqslant B_2$. The following assertions are equivalent:*
(1) A reflexive B_1-B_2-Ferrers relation is totally S_L-connected.
(2) Operation $B_1 : [0,1]^2 \to [0,1]$ fulfils $B_1(x,y) > 0$ for any pair $(x,y) \in [0,1]^2$ such that $x + y > 1$.

Proof. (1) ⇒ (2) Let us consider an operation B_1 such that there exists a pair $(x,y) \in [0,1]^2$ fulfilling $x+y > 1$ and $B_1(x,y) = 0$. Then a reflexive relation that is B_1-B_2-Ferrers but not totally S_L-connected may be build. For example, let $X = \{x_1, x_2\}$ and $R(x_1,x_2) = 1 - x$, $R(x_2,x_1) = 1 - y$.
(2) ⇒ (1) Let $R \in FR(X)$, $x, y \in X$ and R be reflexive and B_1-B_2-Ferrers. Applying these assumptions we obtain

$$1 = B_1(R(x,x), R(y,y)) \leqslant B_2(R(x,y), R(y,x)) = 1 - B_1(1 - R(x,y), 1 - R(y,x)),$$

which implies that $B_1(1 - R(x,y), 1 - R(y,x)) = 0$. As a result from (2) it follows that $1 - R(x,y) + 1 - R(y,x) \leqslant 1$, which means that R is totally S_L-connected. □

Corollary 4.4 ([7]). *Let us consider a t-norm T and its dual t-conorm S. The following assertions are equivalent:*
(1) A reflexive T-S-Ferrers relation is totally S_L-connected.
(2) The t-norm T fulfils $T(x,y) > 0$ for any pair $(x,y) \in [0,1]^2$ such that $x + y > 1$.

In particular, the above corollary applies to all rotation invariant t-norms ([7]). The next results concern total max-connectedness. Let us observe that this notion is also named as strong completeness (cf. [11]).

Theorem 4.5. *Let B have a zero element 1 and have no zero divisors. Then total B-connectedness is equivalent to total* max*-connectedness.*

Proof. Let $R \in FR(X)$, B have a zero element 1 and have no zero divisors. Total B-connectedness is equivalent to

$$B(R(x,y), R(y,x)) = 1 \Leftrightarrow R(x,y) = 1 \vee R(y,x) = 1 \Leftrightarrow \max(R(x,y), R(y,x)) = 1,$$

which is equivalent to the fact that R is totally max-connected. □

Corollary 4.6 ([2]). *Let S be a t-conorm without zero divisors. Then total S-connectedness is equivalent to total* max*-connectedness.*

Theorem 4.7. *Let a commutative operation B_1 have a zero element 0, an idempotent element 1 and let B_2 be dual to B_1 such that $B_1 \leqslant B_2$. The following assertions are equivalent:*
(1) A reflexive B_1-B_2-Ferrers relation is totally max*-connected.*
(2) B_1 has no zero divisors.

Proof. (1) $\Rightarrow$ (2) Let us suppose, to the contrary, that B_1 is not without zero divisors. Then there exist $x, y \in (0, 1]$ such that $B_1(x, y) = 0$. Let us now consider the relation $R \in FR(X)$, where $X = \{x_1, x_2\}$ and $R(x_1, x_2) = 1 - x$, $R(x_2, x_1) = 1 - y$. R is B_1-B_2-Ferrers relation but it is not totally $\max$-connected which contradicts to (1).
(2) $\Rightarrow$ (1) Let R be a reflexive, B_1-B_2-Ferrers relation. By Theorem 4.1 it is also totally B_2-connected. From (2) and the assumption that B_2 is dual to B_1 it follows that B_2 has a zero element 1 and has no zero divisors. By Theorem 4.5 we obtain that R is totally $\max$-connected. □

Corollary 4.8 ([7])**.** *Let T be a t-norm and S its dual t-conorm. Then the following conditions are equivalent:*
(1) A reflexive T-S- Ferrers relation is totally $\max$*-connected.*
(2) The t-norm T has no zero divisors.

The above results from Section 4 simplify the considerations on aggregation of fuzzy interval orders (condition on F for preservation of reflexivity is much easier than the one for total connectedness). Applying these results and results of Section 3 we get for example the following statements.

Theorem 4.9. *Let T be a rotation invariant t-norm, $R_1, ..., R_n \in FR(X)$ be reflexive and T-S_L-Ferrers. If a function $F : [0, 1]^n \to [0, 1]$, which is increasing in each of its arguments, fulfils $F(1, ..., 1) = 1$, $F \gg T$ and $S_L \gg F$, then $R_F = F(R_1, ..., R_n)$ is a T-S_L fuzzy interval order.*

Since T_L is an example of a rotation invariant t-norm, in particular we get the following results.

Theorem 4.10. *Let $R_1, ..., R_n \in FR(X)$ be reflexive and T_L-S_L-Ferrers. If a function $F : [0, 1]^n \to [0, 1]$, which is increasing in each of its arguments fulfils $F(1, ..., 1) = 1$, $F \gg T_L$ and $S_L \gg F$, then $R_F = F(R_1, ..., R_n)$ is a T_L-S_L fuzzy interval order.*

Corollary 4.11. *Let $R_1, ..., R_n \in FR(X)$ be reflexive and T_L-S_L-Ferrers. Then fuzzy relation $R_F = F(R_1, ..., R_n)$ is a T_L-S_L fuzzy interval order, where F is a weighted arithmetic mean. Moreover, fuzzy relation R_F is a T-S fuzzy interval order, where F is a quasi-linear mean and T is a nilpotent t-norm, $S = T^d$.*

Theorem 4.12. *Let $R_1, ..., R_n \in FR(X)$ be reflexive and* $\min$-$\max$*-Ferrers. If a function $F : [0, 1]^n \to [0, 1]$, which is increasing in each of its arguments fulfils $F(1, ..., 1) = 1$, $F \gg \min$ and $\max \gg F$, then $R_F = F(R_1, ..., R_n)$ is a* $\min$-$\max$ *fuzzy interval order.*

Examples of increasing functions which dominate minimum and are dominated by maximum are projections ([1]), so they fulfil assumptions on F in the above theorem.

5 Conclusion

In this paper fuzzy interval orders were considered in the context of aggregation process. In future work it would be interesting to consider other orders and their preservation in aggregation process, in particular total preorder, total order, strict total order, partial preorder, partial order, strict partial order, or semiorder and their preservation in aggregation process.

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