Transitivity of interval-valued fuzzy relations

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Abstract: *In this contribution a new relation for the set of interval-valued fuzzy relations is introduced. This relation is more suitable for the epistemic setting of these relations. This is an interval order for the family of intervals and consequences of considering such order are studied in the context of operations on interval-valued fuzzy relations. Moreover, the new transitivity property, namely pos-T-transitivity is studied. This transitivity property is connected with the new relation proposed here.*

Keywords: *partial order, interval order, T-transitivity, pos-T-transitivity, interval-valued fuzzy relation*

1 Introduction

Interval-valued fuzzy relations were introduced by L. A. Zadeh [17] as a generalization of the concept of a fuzzy relation [16]. Interval valued fuzzy sets and relations have applications in diverse types of areas, for example in classification, image processing and multicriteria decision making.

In [13], a comparative study of the existing definitions of order relations between intervals, analyzing the level of acceptability and shortcomings from different points of view were presented. Orders used for interval-valued fuzzy relations may be connected with ontic and epistemic setting ([5, 6]). Epistemic uncertainty represents the idea of partial or incomplete information. Simply, it may be described by means of a set of possible values of some quantity of interest, one of which is the right one. A fuzzy set represents in such approach incomplete information, so it may be called disjunctive [5]. On the other hand, fuzzy sets may be conjunctive and can be called ontic fuzzy sets [5]. In this situation the fuzzy set is used as representing some precise gradual entity consisting of a collection of items.

The aim of this work is to examine dependencies between the natural (partial) order and the here introduced relation in the set of interval-valued fuzzy relations. Moreover, consequences of considering such relation are studied in the context of operations on interval-valued fuzzy relations, among others the new transitivity property called pos-T-transitivity is discussed.

The paper is structured as follows. Firstly, some concepts and results useful in further considerations are recalled (Section 2). Next, results connected with the interval order are presented (Section 3). Moreover, some properties of operations on interval-valued fuzzy relations are studied (Section 4). To finish, in Section 5 pos-T-transitivity, based on the definition of the given new relation, is presented and its preservation by basic operations is considered.

2 Interval-valued fuzzy relations

First, we recall definition of the lattice operations and the order for interval-valued fuzzy relations. Let X, Y, Z be non-empty sets.

Definition 2.1 (cf. [15, 17]). *An interval-valued fuzzy relation R between universes X, Y is a mapping $R : X \times Y \to L^I$ such that*

$$R(x,y) = [\underline{R}(x,y), \overline{R}(x,y)] \in L^I,$$

for all couples $(x,y) \in (X \times Y)$, where $L^I = \{[x_1, x_2] :\ x_1, x_2 \in [0,1],\ x_1 \leq x_2\}$.

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The class of all interval-valued fuzzy relations between universes X, Y will be denoted by $\mathcal{IVFR}(X \times Y)$ or $\mathcal{IVFR}(X)$ for $X = Y$.
We use the following partial order for intervals:

$$[x_1, y_1] \leq [x_2, y_2] \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2. \tag{1}$$

For every $(x, y) \in (X \times Y)$, $P = [\underline{P}, \overline{P}]$, $R = [\underline{R}, \overline{R}] \in IVFR(X)$ we have

$$P(x, y) \leq R(x, y) \Leftrightarrow \underline{P}(x, y) \leq \underline{R}(x, y), \overline{P}(x, y) \leq \overline{R}(x, y).$$

The boundary elements in $\mathcal{IVFR}(X \times Y)$ are $\mathbf{1} = [1, 1]$ and $\mathbf{0} = [0, 0]$.
Let $P, R \in \mathcal{IVFR}(X \times Y)$, then

$$(P \vee R)(x, y) = [\max(\underline{P}(x, y), \underline{R}(x, y)), \max(\overline{P}(x, y), \overline{R}(x, y))],$$

$$(P \wedge R)(x, y) = [\min(\underline{P}(x, y), \underline{R}(x, y)), \min(\overline{P}(x, y), \overline{R}(x, y))].$$

The structure $(\mathcal{IVFR}(X \times Y), \leq)$ is a partially ordered set, i.e. the relation $\leq$ is:
reflexive, $R(x, y) \leq R(x, y)$,
antisymmetric, $R(x, y) \leq P(x, y)$ and $P(x, y) \leq R(x, y) \Rightarrow R(x, y) = P(x, y)$,
transitive, $R(x, y) \leq P(x, y)$ and $P(x, y) \leq Q(x, y) \Rightarrow R(x, y) \leq Q(x, y)$
for every $(x, y) \in (X \times Y)$ and $P, Q, R \in \mathcal{IVFR}(X \times Y)$.

For an arbitrary index set $D \neq \emptyset$ it holds that

$$(\bigvee_{d \in D} R_d)(x, y) = [\sup_{d \in D} \underline{R_d}(x, y), \sup_{d \in D} \overline{R_d}(x, y)],$$

$$(\bigwedge_{d \in D} R_d)(x, y) = [\inf_{d \in D} \underline{R_d}(x, y), \inf_{d \in D} \overline{R_d}(x, y)].$$

More general classes of operations are triangular norms.

Definition 2.2 ([2])**.** *A triangular norm* $\mathbf{T}$ *on a bounded poset* $\boldsymbol{P} = (\boldsymbol{P}, \leq, 0, 1)$ *is an increasing, commutative, associative operation* $\mathbf{T} : \boldsymbol{P^2} \to \boldsymbol{P}$ *with a neutral element* 1.

One construction method for triangular norms is presented below.
An operation $\mathcal{T} : (L^I)^2 \to L^I$ is called a representable triangular norm if there exist triangular norms $T_1, T_2 : [0, 1]^2 \to [0, 1]$ such that for all $x = [x_1, x_2], y = [y_1, y_2] \in L^I$ and $T_1 \leq T_2$:

$$\mathcal{T}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)].$$

Many authors, for example in [12, 14], used the following definition of transitivity.

Definition 2.3. *Let* $\mathcal{T} = [T_1, T_2]$ *and* $T_1 \leq T_2$*,* $R \in \mathcal{IVFR}(X)$*. Relation* R *is called* $\mathcal{T}$*-transitive if*

$$T_1(\underline{R}(x, y), \underline{R}(y, z)) \leq \underline{R}(x, z)$$

and

$$T_2(\overline{R}(x, y), \overline{R}(y, z)) \leq \overline{R}(x, z).$$

In the next part of the paper we will introduce another type of transitivity.

3 Interval order in L^I

To begin with, we recall the definition of an interval order for crisp relations. The name 'interval order' first appeared in print in Fishburn [7, 8, 9].

Definition 3.1 ([10], p. 42)**.** *A relation $R \subset X \times X$ is an interval order if it is complete and has the Ferrers property, i.e.:*
$R(x, y)$ or $R(y, x)$, for $x, y \in X$,
$R(x, y)$ and $R(z, w) \Rightarrow R(x, w)$ or $R(z, y)$, for $x, y, z, w \in X$,
respectively.

Now we consider the following relation between intervals:

$$[x_1, y_1] \preceq [x_2, y_2] \Leftrightarrow x_1 \leq y_2. \tag{2}$$

This relation is more adequate in the epistemic setting of the interval-valued fuzzy relations. If $[x_1, y_1]$ is an unprecise description of a variable x and $[x_2, y_2]$ is an unprecise description of a variable y, then $[x_1, y_1] \preceq [x_2, y_2]$ denotes that it is possible that the true value of x is smaller than or equal to the true value of y. The relation $\preceq$ thus has a possibilistic interpretation [4].

Theorem 3.2. *In the structure $(L^I, \preceq)$, the relation $\preceq$ is an interval order.*

Proof. Let $[a_1, b_1] \preceq [a_2, b_2]$ and $[a_3, b_3] \preceq [a_4, b_4]$ for $[a_i, b_i] \in L^I, i \in \{1, 2, 3, 4\}$, so $a_1 \leq b_2$, $a_3 \leq b_4$.
If $a_1 > b_4$, then $a_3 \leq a_1$, i.e. $a_3 \leq b_2$.
If $a_3 > b_2$, then $a_1 \leq a_3$, i.e. $a_1 \leq b_4$.
So $\preceq$ has the Ferrers property, i.e.

$$[a_1, b_1] \preceq [a_2, b_2] \ \ and \ \ [a_3, b_3] \preceq [a_4, b_4] \Rightarrow [a_1, b_1] \preceq [a_4, b_4] \ \ or \ \ [a_3, b_3] \preceq [a_2, b_2],$$

If $a_1 \leq b_2$, then $[a_1, b_1] \preceq [a_2, b_2]$.
If $a_1 \geq b_2$, then $[a_2, b_2] \preceq [a_1, b_1]$.
So $\preceq$ is complete, i.e.

$$[a_1, b_1] \preceq [a_2, b_2] \ \ or \ \ [a_2, b_2] \preceq [a_1, b_1].$$

□

Directly from (1) and (2), we note the following connection between the natural (partial) order $\leq$ and the interval order $\preceq$.

Corollary 3.3. *If the natural order* (1) *holds, then also the interval order holds* (2)*.*

The converse implication does not hold, as can be seen from the following example.

Example 3.4. *For intervals $A = [0.2, 0.8]$ and $B = [0.1, 1]$ we observe that $A \preceq B$ but it is not true that $A \leq B$.*

We would like to use the new relation on the class $\mathcal{IVFR}(X \times Y)$ and examine the consequences of this choice. Thus, for every $(x, y) \in (X \times Y)$, $P = [\underline{P}, \overline{P}]$, $R = [\underline{R}, \overline{R}] \in \mathcal{IVFR}(X \times Y)$ we have

$$P(x, y) \preceq R(x, y) \Leftrightarrow \underline{P}(x, y) \leq \overline{R}(x, y).$$

Let us notice that the relation $\preceq$ in the family $\mathcal{IVFR}(X)$ has the reflexivity property only. Thus it is not an order relation in this family.

4 Dependencies between operations on $\mathcal{IVFR}(X \times Y)$

Firstly, we consider connections between basic operations on $\mathcal{IVFR}(X \times Y)$ and the considered relation $\preceq$. For $(x, y) \in (X \times Y)$, $P = [\underline{P}, \overline{P}]$, $R = [\underline{R}, \overline{R}] \in \mathcal{IVFR}(X \times Y)$ we have

$$P \wedge R \preceq P \preceq P \vee R,$$

$$P \wedge R \preceq R \preceq P \vee R.$$

Moreover, if $\overline{P} \geq \overline{R}$, then $R \preceq P \wedge R$ and if $\overline{P} \leq \overline{R}$, then $P \preceq P \wedge R$.
Interesting differences between the considered relation $\preceq$ and the natural (partial) order present the following conditions

$$P \preceq R \Leftarrow (P \wedge R = P, \ \ P \vee R = R)$$

and

$$\text{if } \ \overline{P} \leq \underline{R}, \ \text{ then } \ P \preceq R \Rightarrow (P \wedge R = P, \ \ P \vee R = R).$$

Moreover, we have the implication

$$(R \preceq P, P \preceq R) \Leftarrow (\underline{R} = \underline{P}, \ \overline{R} = \overline{P}),$$

but the converse implication we obtain if the relation $\preceq$ is replaced with the natural order $\leq$. If we consider the converse operation $R^t(x, y) = R(y, x)$, then it holds

$$P \preceq R \Leftrightarrow P^t \preceq R^t.$$

Another interesting properties for here considered relation $\preceq$, we present in the following result.

Theorem 4.1. *Let $(x, y) \in (X \times Y)$, $P = [\underline{P}, \overline{P}]$, $Q = [\underline{Q}, \overline{Q}]$, $R = [\underline{R}, \overline{R}] \in \mathcal{IVFR}(X \times Y)$. Then we have*

$$\bullet P(x, y) \preceq R(x, y), \ P(x, y) \preceq Q(x, y) \Leftrightarrow P(x, y) \preceq R(x, y) \wedge Q(x, y),$$

$$\bullet R(x, y) \preceq P(x, y), \ Q(x, y) \preceq P(x, y) \Leftrightarrow R(x, y) \vee Q(x, y) \preceq P(x, y),$$

$$\bullet P(x, y) \preceq R(x, y) \text{ and } W(x, y) \preceq Q(x, y) \Rightarrow$$

$$P(x, y) \vee W(x, y) \preceq R(x, y) \vee Q(x, y) \text{ and } P(x, y) \wedge W(x, y) \preceq R(x, y) \wedge Q(x, y).$$

Proof. Let $P(x, y) \preceq R(x, y)$ and $P(x, y) \preceq Q(x, y)$, so $\underline{P} \leq \overline{R}$ and $\underline{P} \leq \overline{Q}$, then $\underline{P} \leq \overline{R} \wedge \overline{Q}$ because $\wedge$ is the infimum in the lattice $([0, 1], \wedge, \vee)$. Similarly, we obtain the second condition by the property of supremum $\vee$. Moreover, by isotonicity of these operations we obtain the third condition. □

Lets us now recall the notion of the composition for interval-valued fuzzy relations.

Definition 4.2 (cf. [1, 11])**.** *Let $P \in \mathcal{IVFR}(X \times Y)$, $R \in \mathcal{IVFR}(Y \times Z)$. The* $\sup -\mathbf{T}$ *composition of the relations P and R is called the relation $P \circ R \in \mathcal{IVFR}(X \times Z)$,*

$$(P \circ R)(x, z) = \bigvee_{y \in Y} \mathbf{T}(P(x, y), R(y, z)).$$

Especially, if $\mathbf{T}$ is a representable triangular norm $\mathcal{T}$ we have $\sup -T_1 T_2$ *composition,*

$$(P \circ R)(x, z) = [(\underline{P} \circ_{T_1} \underline{R})(x, z), (\overline{P} \circ_{T_2} \overline{R})(x, z)],$$

where $T_1 \leq T_2$ and

$$(\underline{P} \circ_{T_1} \underline{R})(x, z) = \sup_{y \in Y} T_1(\underline{P}(x, y), \underline{R}(y, z)), \ (\overline{P} \circ_{T_2} \overline{R})(x, z) = \sup_{y \in Y} T_2(\overline{P}(x, y), \overline{R}(y, z)).$$

In our further considerations in the whole paper we will use the composition with a representable triangular norm and the symbol $\circ$ will mean $\sup -T_1T_2$ composition. For simplicity of notations we present the results for composition in the class $\mathcal{IVFR}(X)$.

Theorem 4.3. *If $T_1,\ T_2, T_1 \le T_2$ are triangular norms, then*

$$P \preceq R \Rightarrow P \circ Q \preceq R \circ Q, \quad Q \circ P \preceq Q \circ R,$$

$$P \circ (Q \vee R) = P \circ Q \vee P \circ R.$$

Moreover, if T_1, T_2, $T_1 \le T_2$ are supremum preserving then

$$P \circ (Q \circ R) = (P \circ Q) \circ R.$$

Proof. Let $P \preceq R$, i.e. $\underline{P} \le \overline{R}$ and $T_1 \le T_2$, then by Theorem 4.1 we have for $x, y \in X$ $\bigvee_{z\in X} T_1(\underline{P}(x,z), \underline{Q}(z,y)) \le \bigvee_{z\in X} T_2(\overline{P}(x,z), \overline{Q}(z,y))$, so $P \circ Q \preceq R \circ Q$. The second inequality in the first condition can be proven similarly. By distributivity of a triangular norm with respect to maximum we obtain the second condition. Moreover, since triangular norms $T_1,\ T_2$ are supremum preserving, we have associativity. □

In a semigroup $(\mathcal{IVFR}(X), \circ)$ we can consider the powers of its elements, i.e. relations R^n for $R \in \mathcal{IVFR}(X)$, $n \in \mathbb{N}$.

Definition 4.4. *Let $R \in \mathcal{IVFR}(X)$. The powers of R are defined in the following way*

$$R^1 = R, \quad R^{n+1} = R^n \circ R, \qquad n \in \mathbb{N}.$$

The upper operation $R^\vee$ and the lower operation $R^\wedge$ of R are defined in the following way

$$R^\vee = \bigvee_{k=1}^{\infty} R^k, \quad R^\wedge = \bigwedge_{k=1}^{\infty} R^k,$$

where $R^k = [\underline{R}^k, \overline{R}^k]$. Now we will examine connections between powers and upper (lower) operations and operations $\vee$ and $\wedge$.

Theorem 4.5. *Let T_1, T_2, $T_1 \le T_2$ be supremum preserving and $P, R \in \mathcal{IVFR}(X)$.*

$$\textit{If} \quad R \preceq P, \ \textit{then} \quad R^n \preceq P^n, \quad R^\vee \preceq P^\vee, \quad R^\wedge \preceq P^\wedge, \quad n \in \mathbb{N}.$$

Moreover,

$$(P \vee R)^\vee \succeq P^\vee \vee R^\vee,$$

$$(P \wedge R)^\vee \preceq P^\vee \wedge R^\vee,$$

$$(P \vee R)^\wedge \succeq P^\wedge \vee R^\wedge,$$

$$(P \wedge R)^\wedge \preceq P^\wedge \wedge R^\wedge.$$

Proof. By isotonicity of composition we obtain isotonicity of powers, moreover by isotonicity of supremum and infimum we have dependencies for lower and upper operations. By Theorem 4.1 and the above conditions we obtain the rest of results. □

5 Possible T-transitivity

Now we will consider the transitivity property connected with the introduced relation $\preceq$ in the epistemic setting. This definition of transitivity naturally follows from the introduced relation $\preceq$, namely replacing the natural order $\leq$ with the relation $\preceq$ we get by Definition 2.3 for a representable triangular norm $\mathcal{T}$ the formula $\mathcal{T}(R(x,y), R(y,z)) \preceq R(x,z)$. As a result, applying definition of the relation $\preceq$ we get the following notion.

Definition 5.1 ([3]). *Let $T : [0,1]^2 \to [0,1]$ be a triangular norm. A relation $R \in \mathcal{IVFR}(X)$ is possibly T-transitive (pos-T-transitive), if*

$$T(\underline{R}(x,y), \underline{R}(y,z)) \leq \overline{R}(x,z). \tag{3}$$

This transitivity property is called possible T-transitivity which follows from the interpretation of the relation $\preceq$. Again, if $R(x,y)$ is an imprecise description of the relation between x and y, and similarly for $R(y,z)$ and $R(x,z)$, then formula (3) expresses that it is possible to choose values in these intervals such that usual T-transitivity holds.

Theorem 5.2. *Let $D \neq \emptyset$ and $R_d \in \mathcal{IVFR}(X)$, $d \in D$. If (R_d) is a family of pos-T-transitive relations, then the fuzzy relation $\bigwedge_{d\in D} R_d$ is pos-T-transitive.*

Proof. If R_d are pos-T-transitive relations, i.e., $T(\underline{R_d}(x,y), \underline{R_d}(y,z)) \leq \overline{R_d}(x,z)$, then by isotonicity of triangular norms, we know that $\min$ dominates any triangular norm T, i.e. $T(\bigwedge_{d\in D} \underline{R_d}(x,y), \bigwedge_{d\in D} \underline{R_d}(y,z)) \leq \bigwedge_{d\in D} T(\underline{R_d}(x,y), \underline{R_d}(y,z)) \leq \bigwedge_{d\in D} \overline{R_d}(x,z)$. □

Theorem 5.3. *Let $R \in \mathcal{IVFR}(X)$. If R is pos-T-transitive, then R^t is also pos-T-transitive.*

Proof. For an arbitrary $R \in \mathcal{IVFR}(X)$ which is pos-T-transitive and by commutativity of a triangular norm we have
$T(\underline{R^t}(x,y), \underline{R^t}(y,z)) = T(\underline{R}(y,x), \underline{R}(z,y)) = T(\underline{R}(z,y), \underline{R}(y,x)) \leq \overline{R}(z,x) = \overline{R^t}(x,z)$. □

In the following theorems we use the fact (which follows from definition of pos-T-transitivity and definition of composition) that

Lemma 5.4. *Let $R \in \mathcal{IVFR}(X)$. R is pos-T-transitive if and only if $\underline{R}^2 \leq \overline{R}$, where $\underline{R}^2 = \underline{R} \circ_T \underline{R}$.*

Theorem 5.5. *Let $P, R \in \mathcal{IVFR}(X)$. If P, R are pos-T-transitive relations and $\underline{R} \circ_T \underline{P} \vee \underline{P} \circ_T \underline{R} \leq \overline{R} \vee \overline{P}$, then $R \vee S$ is pos-T-transitive.*

Proof. Let P, R be interval-valued fuzzy pos-T-transitive relations. By Lemma 5.4, and the assumption $\underline{R} \circ_T \underline{P} \vee \underline{P} \circ_T \underline{R} \leq \overline{R} \vee \overline{P}$ and by Theorem 4.3 we have

$$(\underline{R \vee P})^2 = (\underline{R \vee P}) \circ_T (\underline{R \vee P}) = \underline{R}^2 \vee \underline{R} \circ \underline{P} \vee \underline{P} \circ \underline{R} \vee \underline{P}^2 \leq \overline{R} \vee \overline{R} \vee \overline{P} \vee \overline{P} = \overline{R} \vee \overline{P},$$

so $R \vee S$ is pos-T-transitive. □

Theorem 5.6. *Let T_1,T_2, $T_1 \leq T_2$ be supremum preserving $P, R \in \mathcal{IVFR}(X)$. If P, R are pos-T_1-transitive and $\underline{R} \circ_{T_1} \underline{P} = \underline{P} \circ_{T_1} \underline{R}$, then $R \circ P$ is pos-T_1-transitive.*

Proof. Let P, R be interval-valued fuzzy pos-T-transitive relations. By associativity of composition and the assumption $\underline{R} \circ_{T_1} \underline{P} = \underline{P} \circ_{T_1} \underline{R}$, by Lemma 5.4 we have
$(\underline{R \circ_{T_1} P})^2 = (\underline{R} \circ_{T_1} \underline{P})^2 = \underline{R} \circ_{T_1} (\underline{P} \circ_{T_1} \underline{R}) \circ_{T_1} \underline{P} = \underline{R}^2 \circ_{T_1} \underline{P}^2 \leq \overline{R} \circ_{T_1} \overline{P} \leq \overline{R} \circ_{T_2} \overline{P} = \overline{P \circ_{T_2} R}$.
Thus, $R \circ P$ is a pos-T_1-transitive relation. □

Corollary 5.7. *Let $T_1, T_2, T_1 \leq T_2$ be supremum preserving and $R \in \mathcal{IVFR}(X)$. If R is pos-T_1-transitive, then R^n is also pos-T_1-transitive.*

Proof. By isotonicity of composition and powers, we obtain $\underline{R}^n \leq \overline{R}^{n-1}$, so R^n preserves pos-T_1-transitivity. □

Theorem 5.8. *Let $R \in \mathcal{IVFR}(X)$. If $\underline{R}$ is T-transitive, then R is pos-T-transitive.*

Proof. Let $\underline{R}$ be T-transitive. Then $\underline{R}^2 \leq \underline{R} \leq \overline{R}$. Thus by Lemma 5.4, we obtain pos-T-transitivity of R. □

We also notice the connection between $\mathcal{T}$-transitivity and pos-T-transitivity.

Proposition 5.9. *Let $R \in \mathcal{IVFR}(X)$. If R is $\mathcal{T}$-transitive, then R is pos-T_1-transitive.*

Moreover, we know directly by definitions of $\mathcal{T}$-transitivity and composition, that

Proposition 5.10. *Let $R \in \mathcal{IVFR}(X)$, $T_1, T_2, T_1 \leq T_2$ be triangular norms. R is $\mathcal{T}$-transitive if and only if $\underline{R}$ is T_1-transitive and $\overline{R}$ is T_2-transitive.*

Moreover, we have the following property.

Theorem 5.11. *Let T_1, T_2 be triangular norms and $T_1 \leq T_2$. If $R \in \mathcal{IVFR}(X)$ is pos-T_2-transitive, then R is pos-T_1-transitive.*

Proof. Let R be pos-T_2-transitive. Then $T_2(\underline{R}(x,y), \underline{R}(y,z)) \leq \overline{R}(x,z)$ and by the fact that $T_1 \leq T_2$ we have $T_1(\underline{R}(x,y), \underline{R}(y,z)) \leq T_2(\underline{R}(x,y), \underline{R}(y,z)) \leq \overline{R}(x,z)$ for $x, y, z \in X$. As a result R is pos-T_1-transitive.

□

6 Conclusion

In future work other operations and some properties for interval-valued fuzzy relations for the relation $\preceq$ may be considered. Next, generalization of the here considered composition, i.e. $\sup - A$ composition (where A is an aggregation function), may be discussed. Moreover, other types of transitivity and other relations between interval-valued fuzzy relations may be studied.

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