A note on the construction of large graphs and digraphs of given degree and diameter

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Abstract: *The undirected (directed) version of the degree - diameter problem is determination of the largest order of a graph (digraph) of a given maximum (out -) degree and diameter. In this note we present some of known constructions of the large graphs and digraphs and discuss using of the voltage assignment approach as well as the property to be a Cayley graph (digraph) in some known constructions of large graphs and digraphs.*

1 Introduction

The undirected (directed) degree-diameter problem is to determine the largest order n of a graph (digraph) of a given maximum (out -) degree Δ and diameter D. It also involves the problem of determination of the corresponding extremal graphs and digraphs. For more details we refer to the survey [6] for a summary of the history and the current state-of-the-art.

A spanning tree argument shows that for the number of vertices n

$$n(\Delta, D) \leq M(\Delta, D),$$

where

$$M(\Delta, D) = 1 + \Delta + \Delta(\Delta - 1) + ... + \Delta(\Delta - 1)^{D-1} \tag{1}$$

is the Moore bound. The graphs of maximum degree Δ and diameter at most D will be referred to as (Δ, D) **- graphs**. A Moore graph is a (Δ, D)-graph of order equal to the Moore bound (Δ, D). Another way to study large graphs close to the Moore bound is constructing large graphs of a given degree and diameter in order to improve the lower bound on the maximum possible order of graphs for given Δ and D.

In the case of directed graphs, there is also an upper bound $\overrightarrow{n}(\Delta, D)$ on the order of directed graphs, for given maximum out-degree Δ and diameter D. Let $\overrightarrow{n}_i$, for $0 \leq i \leq D$, be the number of vertices at distance i from v. Then $\overrightarrow{n_i} \leq \Delta^i$, for $0 \leq i \leq D$, and therefore

$$\begin{aligned} \overrightarrow{n}(\Delta, D) = \sum_{i=1}^{D} n_i &\leq 1 + \Delta + \Delta^2 + \ldots + \Delta^D \\ &= \begin{cases} \frac{\Delta^{D+1}-1}{\Delta-1} & \text{if } \Delta > 1 \\ D+1 & \text{if } \Delta = 1 \end{cases} \end{aligned} \tag{2}$$

The right-hand side of (2), denoted $\overrightarrow{M}_{\Delta,D}$ is called *Moore bound* for digraphs. If the order of the digraph is equal to the Moore bound, such digraph is called a *Moore digraph*.

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Various restricted versions of the problem have been studied and we shall be interested in an analysis of certain large *vertex-transitive* graphs and digraphs of a given degree and diameter. Interest in vertex-transitive graphs and digraphs within the undirected as well as in the directed degree-diameter problem is also motivated by computer generation of graphs (digraphs) with an extremely large number of vertices, where fast diameter checking is essential. Chcecking diameter in the process of computer generation of graphs of order "more than a millon", would not be possible in real time if the graphs were not vertex - transitive. This gave rise to investigation of the vertex - transitive and Cayley version of the (Δ, D) - problem.

The aim of this note is to show some known constructions of (Δ, D) - graphs and digraphs and to show how it is posible to describe them in the language of voltage assignment approach and discuss which of the selected construction have the property to be a Cayley graph (digraph).

The paper is organized as follows. In Section 2 are given Preliminaries, in Section 3 will be described the voltage assignment technique [4]. In Section 4 we discusse the construction of Kautz digraph [3, 5].

2 Preliminaries

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. The order of the graph is the number of vertices. The degree of a vertex is the number of darts adjacent to the vertex. A graph is Δ - regular if the degree of all vertices is equal to Δ. The distance $d(u, v)$ from a vertex u to a vertex v is the number of darts of a shortest directed path from u to v, and its maximum value over all pairs of vertices, $D = max_{u,v \in V} d(u, v)$ is the diameter of the digraph. A graph is vertex symmetric if its automorphism group acts transitively on its set of vertices.

Let Γ be a finite group and let X be unit-free generating set for X such that $X = X^{-1}$, that is, we assume that X is closed under taking inverse elements. The Cayley graph $Cay(\Gamma, X)$ has vertex set Γ, and two vertices $g, h \in \Gamma$ are joined by an edge if $g^{-1}h \in X$. Since this condition is equivalent to $h^{-1}g \in X$ because of $X = X^{-1}$, the Cayley graph $Cay(\Gamma, X)$ is undirected. The degree of $Cay(\Gamma, X)$ is $|X|$.

Let $G = (V, D)$ be a digraph with the vertex set V and the dart set D. The order of the digraph is the number of vertices. The out - degree (in - degree) of a vertex is the number of darts leaving (entering) the vertex. A digraph is Δ - regular if the in - degree and out - degree of all vertices is equal to Δ. All digraphs considered in this article are strongly connected. The distance $d(u, v)$ from a vertex u to a vertex v is the number of darts of a shortest directed path from u to v, and its maximum value over all pairs of vertices, $D = max_{u,v \in V} d(u, v)$ is the diameter of the digraph. A digraph is vertex symmetric if its automorphism group acts transitively on its set of vertices.

Let Γ be a finite group and let X be a subset of Γ which generates Γ and does not contain the identity, the Cayley digraph of Γ with respect to X is the directed graph with vertex set Γ and dart set $\{(u, v); v = ux$ for some $x \in X\}$.

3 Covering techniques

Graph coverings are a special case of coverings of topological spaces, for more details see [4]. A graph can be regarded as one-dimensional simplicial complex. Therefore, known results from algebraic topology can be transferred to graphs. This method enables to "blow up" a given "base graph" to a larger graph (called lift) which is a regular covering space of the base graph.

The lift is best described in terms of the base graph and a mapping, called a voltage assignment. As shown in [1], many of the currently known largest examples of graphs of given degree and diameter can indeed be obtained by the covering graph construction. The covering graph construction has a very good potential for producing examples of large graph of given degree and diameter. For more details see the monograph of Gross and Tucker [4].

3.1 Ordinary voltage assignments-undirected case

Let G be an undirected graph, possibly with loops and / or parallel edges and semiedges, that is, dangling edges with just one end incident to a vertex. Although the graph G itself is undirected, it will be advantage to assign (for auxiliary purposes) directions to its edges. Each edge (inclusive loops) which is not a semiedge can be assigned one of the two possible directions; an edge with a direction is called a dart. A semiedge has only one possible direction (toward the incident vertex). In this way, every edge which is not a semiedge underlies a pair of mutually reverse darts. The reverse of a dart e is denoted by e^{-1}. For convenience, if e is a semiedge we may still used the symbol e^{-1} but we understand that $e = e^{-1}$ in such a case.

We say that e is a dart at v if the orientation of e points towards the (incident) vertex v. If e is a dart at v than e is also said to terminate at v; at the same time e^{-1} is said to emanate from v. If a dart e emanates from a vertex u and terminates at the vertex v we often say that e is a dart from u to v. The set of all darts of a graph G will be denoted by $D(G)$. Note that $|D(G)| = 2|E(G)| + |S(G)|$, where $S(G)$ stands for the set of all semiedges of G.

By a $u - v$ walk of length k we understand a sequence $W = e_1e_2...e_k$ where e_i are darts of G, such that e_1 emanates from u, e_k terminates at v, and terminal vertex of e_{i-1} coincides with the initial vertex of $e_i, 2 \leq i \leq k$. (If, say, e_i is a dart arising from a semiedge, then its initial and terminal vertices are identical.) We also admit a trivial walk based as u; it consists just of the vertex u and has no darts. By W^{-1} we denote the reverse of W; formally $W^{-1} = e_k^{-1}...e_2^{-1}e_1^{-1}$.

Let Γ be an arbitrary finite group. A mapping $\alpha : D(G) \to \Gamma$ is a **voltage assignment** if, for each dart $e \in D(G)$,

$$\alpha(e^{-1}) = (\alpha(e))^{-1}. \tag{3}$$

The values of α are called voltages an the group Γ is the voltage group. Observe that for each semiedge e we have $(\alpha(e))^2 = 1_\Gamma$.

The pair $\langle G, \Gamma \rangle$ enables us to define a new graph G^α, called an ordinary lift of G. The vertex set of the lift is $V(G^\alpha) = V(G) \times \Gamma$ and the dart set of the lift is $D(G) \times \Gamma$. Incidence in G^α is defined as follows: A dart $(e, i) \in D(G^\alpha)$ emanates from the vertex (u, i) and terminates at the vertex (v, j) if and only if e is a dart from u to v and $j = i\alpha(e)$. We will sometimes use u_i and e_i in place of (u, i) and (e, i).

The ordinary lift G^α is an undirected graph, since the darts (e, i) and $(e^{-1}, i\alpha(e))$ are mutually reverse and form an undirected edge on G^α. Note that a semiedge e in G lifts in G^α either to $|\Gamma|$ loops (if $\alpha(e) = 1_\Gamma$) or to $|\Gamma|/2$ edges which are not loops (if $\alpha(e)$ is a nontrivial element of order 2 in Γ).

The ordinary voltage-graph construction was first suggested by Gross (1974) and immediately improved by Gross and Tucker (1974). Its advantage over various formalistic ”covering graph” constructions, all essentially equivalent, is largely its visual suggestiveness. Voltage graphs are usually given by pictures, rather than combinatorial descriptions.

3.1.1 Fibers and Natural Projection

Let us have two graphs H and G. A homomorphism from H to G is a mapping $f : D(H) \to D(G)$, from H into G if f maps any two darts at a common vertex of H onto a pair of darts at a common vertex of G, and if $f(e^{-1}) = (f(e))^{-1}$ for any dart $e \in D(H)$.

A bijective homomorphism $G \to G$ is called an automorphism of G. The collection of all automorphism forms the automorphism group $Aut(G)$ of the graph G.

A covering is a homomorphism $f : H \to G$ of two graphs if for each vertex $v \in V(H)$ the set of darts at v is mapped bijectively by f onto the sets of darts at $f(v)$.

The sets $f^{-1}(u)$ and $f^{-1}(e)$ are called fibers above a vertex u and a dart e, respectivelly. If e is a dart running from a vertex u to a vertex v in the graph G and if e is assigned voltage b, the each edge e_a in the fiber over e runs from the vertex u_a in the fiber over the initial point u to the vertex v_{ab} in the fiber over the terminal point v. We can say that the edge fiber over e matches the vertices in the vertex fiber over u one-to-one onto the vertices of the vertex fiber over v. Thus, the fiber over a proper edge is isomorphic to the disjoint union of $|\Gamma|$ copies of K_2 and the fiber over a loop forms a set of cycles (if the voltage group is fined). If the voltage b on a v-based loop e has order n in the group Γ then each cycle in the edge fiber over e must have length n, and there must be $|\Gamma|/n$ such cycles.

3.1.2 Walk Lifting

Many properties of the lift can be identified by examing walks in the base graph. Let $W = e_1e_2...e_k$ be a walk in the graph G. Then the voltage $\alpha(W)$ of the walk W is defined by $\alpha(W) = \alpha(e_1)\alpha(e_2)...\alpha(e_k)$. Observe that the voltage of W and the voltage of its reverse W^{-1} are related by $\alpha(W^{-1}) = (\alpha(W))^{-1}$. By default the voltage of a trivial walk is defined to be 1_Γ, the identity element of the group Γ.

A 'lift' of a walk W in the base graph G is a walk $\widetilde{W} = \widetilde{e_1}\widetilde{e_2}...\widetilde{e_k}$ in the derived graph G^α such that for $i = 1, ..., n$ the edge $\widetilde{e_i}$ is in the fiber over the edge e_i.

Let W be a walk in the ordinary voltage graph such that the initial vertex W is u. For each vertex u_a in the fiber over u, there is unique lift of W that starts at u_a. It makes sense to designate the lift of a walk W starting at the vertex u_a by W_a. Observe that if W is a walk from u to v in G and b is a net voltage on W, then the lift W_a starting at u_a terminates at the vertex v_{ab}. We can say that for each $u \to v$ walk W in G and each $g \in \Gamma$ there exists a unique walk W_g^α in the lift G^α emanating from u_g and such that $f(W_g^\alpha) = W$; it has the same length as W and terminates at the vertex $v_{g\alpha(W)}$.

3.1.3 Ordinary voltage assignments-directed case

The covering and lifting method described in previously subsections can be use for generating large constructing large digraphs. Most of facts can be applied to digraphs without minor changes. Let G be a *base digraph*, let $A(G)$ be its dart set and let Γ be a finite group. We define a *voltage assignment* on G in Γ as *any* mapping $\alpha : D(G) \to \Gamma$. We need no extra voltages, because edge directions are a part of the description of the digraphs G. The description of the lift is the same as it was by undirected case. The lift is automatically a digraph. For more details see [6].

3.2 Permutation voltage assignments

We now introduce permutation voltage assignment that allow for an alternative description of graph coverings. The permutation on voltage-graph construction is also due to Gross and Tucker(1977).

Let $\sum_n$ be the symmetric group on the set $[n] = \{1, 2, ..., n\}$. A mapping $\sigma : D(G) \to \sum_n$ is said to be a permutation voltage assignment on the graph G if $\sigma(e^{-1}) = (\sigma(e))^{-1}$, for any dart $e \in D(G)$. The permutation lift G^σ is defined by setting $V(G^\sigma) = V(G) \times [n]$ and $D(G^\sigma) = D(G) \times [n]$. As for ordinary derived graphs, one uses the pair (v, i) and (e, i), or the subscripted notations, v_i and e_i. If the dart e of the base graph G runs from the vertex u to vertex v, and if the voltage on e is the permutation π, then for $i = 1, ..., n$, the edge e_i of the derived graph G^σ runs from the vertex u_i to the vertex $v_{\pi(i)}$.

The natural projection $p : G^\alpha \to G$ for permutation voltage graph $\langle G, \alpha\rangle_n$ is the graph map that takes any vertex v_i or edge e_i of the derived graph to the vertex v or the edge e of the base graph. The set of vertices $\{v_i | i = 1, .., n\}$ is called the "fiber" over v and the set of edges $\{e_i | i = 1, .., n\}$ is called the "fiber" over e. The natural projection $p : G^\alpha \to G$ associated with any permutation voltage graph $\langle G, \alpha\rangle_n$ is a covering projection on each component of its domain.

We can define the net permutation voltage on a walk as the product of the voltages encountered in traversal of that walk, exactly as for ordinary voltages. A lift of a walk W in the base graph G is a walk $\widetilde{W}$ in the permutation derived graph such that the natural projection Π maps the edges $\widetilde{W}$ onto the edges of W precisely in the order of traversal. If W is a walk in the base space of a permutation voltage graph $\langle G, \alpha \rangle_n$ such that the initial vertex of W is u, then for each vertex u_i in the fiber over u, there is a unique lift of W that starts at u_i.

4 Kautz digraphs

Kautz digraphs [5] $K(\Delta, D)$ gives the general lower bound on the largest order for the degree diameter problem. These digraphs can be obtained by $(D-1)$ - fold iteration of the line digraph construction applied to the complete digraph of order $\Delta + 1$. As regards symmetry properties of Kautz digraphs, $K(\Delta, D)$ are vertex-transitive if and only if $D \leq 2$. Because of growing interest in vertex-transitive and Cayley digraphs in the degree-diameter problem our aim was in [7] to determine all the values of n for which the line digraph of the complete digraph of order n is a Cayley digraph.

Line digraphs of complete digraphs are a special case of the so - called Kautz digraphs $K(\Delta, D)$ obtained by applying $(D-1)$ - times the line digraphs construction to the complete digraph on $\Delta+1$ vertices [5]. (We recall that in a complete digraph, for any ordered pair of distinct vertices u, v there is an arc from u to v). The digraph $K(\Delta, D)$ is Δ-regular, has $\Delta^D + \Delta^{D-1}$ vertices and diameter D.

Kautz digraphs of diameter two are line digraphs of complete digraphs (in which for any ordered pair of vertices u, v there is an arc from u to v). We have determined [7] all the values of n for which the line digraph of a complete digraph of order n is a Cayley digraph.

Let K_n be the complete digraph of order n (that is with n vertices) and let $L(K_n)$ be its line digraph. As we know, $L(K_n)$ is the Kautz digraph $K(n-1, 2)$. We are now ready to present our characterization.

Theorem 1 *[7] The Kautz digraph $L(K_n)$ is a Cayley digraph if and only if n is a prime power.*

The Kautz digraphs $K(\Delta, D), \Delta \geq 2$ can alternatively be described as follows [2]:

Vertices are labeled with words $x_1 x_2 \ldots x_{n-1}$, where x_i belongs to an alphabet of $\Delta+1$ letters and $x_i \neq x_{i+1}$ for $1 \leq i \leq D-1$. A vertex $x_1 x_2 \ldots x_{n-1}$ is adjacent to the Δ-vertices $x_2 x_3 \ldots x_D x_{D+1}$, where x_{D+1} can be any letter different from x_D. Hence, the digraph $K(\Delta, D)$ is Δ-regular, has $\Delta^D + \Delta^{D-1}$ vertices and diameter D. For $D = 2$ the Kautz digraphs are vertex symmetric.

We remark that assertion of Thm. 1 can be interpreted as follows.

Theorem 2 *If $D = 2$ and $\Delta = q$, where q is a prime power, Kautz digraphs are Cayley digraphs.*

Proof. We recall that for $D = 2$ Kautz digraphs are in fact the digraphs of Faber-Moore-Chen construction [3] and hence the proof of this result in Thm.1. □

Observation 1

$K(\Delta, 2)$ can be described as ordinary lift of bouquet of circles in the following way:
From 2 follows, that for $\Delta = q-1$, where q is a prime power, and $D = 2$, Kautz digraphs are Cayley digraphs.

The base digraph is a bouquet of circles, that means single vertex digraph with $\Delta = q-1$ loops. Then we define an ordinary voltage assignment α in a group $A(1, q) = \{x \mapsto ax + b,\ a \neq 0, a, b \in GF(q)\}$ and on every loop we give the voltage $f_{a,1} = ax + 1$, where $a \neq 0, a \in GF(q)$. Then we will obtain a lift, which is isomorphic to the Kautz digraph of diameter 2 and $\Delta = q-1$.

Observation 2

$K(\Delta, 2)$ can be described as lifts.

As the base graph H we consider complete graphs K_Δ, with one loop at each vertex and $V(H) = Z_{d+1} \setminus \{O\}$. The voltage assignment α is defined: $\alpha(e) = i$, in the voltage group $Z_{\Delta+1}$. Then we will obtained the lift H^α, where $V(H^\alpha) = \Delta.(\Delta + 1)$ which is isomorphic to the Kautz digraph of degree Δ and diameter 2. For more details see [1].

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