A note on a copula construction method

Vadoud Najjari *†‡ Hasan Bal † Salih Çelebioğlu †

Abstract: *The main endeavor in this work is to comment on the study by Kim et al. which generalizes Rodríguez-Lallena and Úbeda-Flores' result to any given copula family. In this study we concentrate on the proposed interval of parameter in their work and then we comment on the proposed interval.*

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1 Introduction

A copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies:

(a) for every $u,\ v$ in $[0,1]$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$ and $C(1,v) = v$;

(b) for every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \le u_2$ and $v_1 \le v_2$, $V_C(R) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$ (in other word, for all rectangles $R = [u_1,u_2] \times [v_1,v_2]$ whose vertices lie in $[0,1]^2$, C-volume is non-negative).

Copulas are multivariate distributions in modeling the dependence structure between variables, irrespective of their marginal distributions. Obviously with a wide range of copulas we are able to capture more miscellaneous dependence structures. Hence there is a wide effort on constructions of copulas in the literature (see for instance, [1, 2, 3, 4, 5, 6]). Also Nelsen [7] summarizes different methods of constructing copulas.

Rodríguez-Lallena and Úbeda-Flores [6] introduced a class of bivariate copulas of the form:

$$C_\lambda(u,v) = uv + \lambda\, f(u)g(v), \qquad (u,v)\ \in\ [0,1]^2 \tag{1.1}$$

where f and g are two non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$ and the admissible range of the parameter λ is

$$\frac{-1}{\max(\alpha\gamma, \beta\delta)} \le \lambda \le \frac{-1}{\min(\alpha\delta, \beta\gamma)} \tag{1.2}$$

where

$$\begin{aligned} &\alpha = \inf\{f^{'}(u):\ u \in A\} < 0 \quad , \quad \beta = \sup\{f^{'}(u):\ u \in A\} > 0 \\ &\gamma = \inf\{g^{'}(v):\ v \in B\} < 0 \quad , \quad \delta = \sup\{g^{'}(u):\ u \in B\} > 0 \\ &A = \{u \in [0,1] : f^{'}(u)\ exists\} \quad , \quad B = \{v \in [0,1] : g^{'}(u)\ exists\}. \end{aligned} \tag{1.3}$$

*Corresponding author, *vnajjari@gazi.edu.tr*, Tel: +90 312 2021465, Fax: +90 312 212 2279

†Department of Mathematics, Islamic Azad University, Maragheh Branch, Maragheh, Iran

‡Gazi University, Faculty of Science, Statistics Department, Ankara, Turkey

This class of copulas provides a method for constructing bivariate distributions with a variety of dependence structures and generalizes several known families such as the Farlie-Gumble-Morgenstern (FGM) distributions. Dolati and Úbeda-Flores [2] provided procedures to construct parametric families of multivariate distributions which generalize (1.1).

Kim et al. [8] generalized Rodríguez-Lallena and Úbeda-Flores' study to any given copula family. They presented an extension for any given copula family C as below

$$C^*_\lambda(u,v) = C(u,v) + \lambda\, f(u)g(v), \qquad (u,v)\ \in\ [0,1]^2 \tag{1.4}$$

where for any non-trivial rectangle R, the parameter λ satisfies on the following inequalities

$$\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma,\beta\delta)} \le \lambda \le \frac{-V_C(R)}{\triangle \times \min(\alpha\gamma,\beta\delta)} \tag{1.5}$$

in these inequalities $\alpha,\ \beta,\ \gamma,\ \delta$ are same as (1.3), $\triangle = (u_2 - u_1)(v_2 - v_1)$, u_1, u_2, v_1, v_2 are in $[0,1]$ such that $u_1 \le u_2$ and $v_1 \le v_2$ and f, g are two non-zero absolutely continuous functions defined on $[0,1]$ such that $f(0) = f(1) = g(0) = g(1) = 0$. Kim et al.'s results had been discussed by Mesiar et al. in [9] and also by Bekrizadeh et al. in [10, 11].

In this study we concentrate on interval of the λ parameter given by (1.5) and we investigate inaccuracy of this interval.

2 Comments on inaccuracy of the interval

In this section we concentrate on the bounds in (1.5) to specify some comments about this relation. We impose on C^* in (1.4) the property of being 2-increasing. For any u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 < u_2$ and $v_1 < v_2$ and $R = [u_1, u_2] \times [v_1, v_2]$, with simple calculation we get

$$V_{C^*}(R) = V_C(R) + \lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \ge 0 \tag{2.1}$$

then

$$\lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \ge -V_C(R). \tag{2.2}$$

If f and g are absolutely continuous functions, as in Rodríguez-Lallena and Úbeda-Flores' study we get

$$\frac{-V_C(R)}{\max(\alpha\gamma,\beta\delta)(u_2-u_1)(v_2-v_1)} \le \lambda \le \frac{-V_C(R)}{\min(\alpha\delta,\beta\gamma)(u_2-u_1)(v_2-v_1)} \tag{2.3}$$

and to get the optimal interval we have

$$\sup\left(\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma,\beta\delta)}\right) \le \lambda \le \inf\left(\frac{-V_C(R)}{\triangle \times \min(\alpha\delta,\beta\gamma)}\right) \tag{2.4}$$

where $\triangle,\ \alpha,\ \beta,\ \gamma,\ \delta$ are same as (1.3). From the comparison of (2.4) and (1.5) the inaccuracy of the interval by (1.5) is clear. Moreover we show inaccuracy of the mentioned interval in (1.5) by several counter-examples:

Counterexample 2.1. Let f, g are of the form $f(u) = u(1-u),\ g(v) = v(1-v)$ then $\alpha = -1,\ \beta = 1,\ \gamma = -1\ ,\ \delta = 1$ and $\max(\alpha\gamma, \beta\delta) = \min(\alpha\gamma, \beta\delta) = 1$. For any rectangle R

$$\lambda = \frac{-V_C(R)}{(u_2 - u_1)(v_2 - v_1)} \tag{2.5}$$

and then there does not exist any λ satisfying the above equalities whenever C is different from the product copula. Moreover, for the product copula, there is the unique solution $\lambda = -1$, i.e., this fact also contradicts the result of Rodríguez-Lallena and Úbeda-Flores' study if we let $C(u, v) = uv$.

Counterexample 2.2. Let $C(u, v) = min(u, v)$ and f, g for all u, v in $[0, 1]$ are non-zero absolutely continuous functions as below

$$f(u) = u^2(1-u) \quad , \quad g(v) = v(1-v) \tag{2.6}$$

then $\alpha = -0.3333,\ \beta = 1,\ \gamma = -1\ ,\ \delta = 1$ and hence

$$\max(\alpha\gamma, \beta\delta) = 1 \quad , \quad \min(\alpha\gamma, \beta\delta) = 0.3333. \tag{2.7}$$

In the Kim's approach we obtain empty set for lambda (see R=$[0.2, 0.3] \times [0.1, 0.2]$ and R=$[0, 1]^2$). Note that by (2.4) we get $\lambda = 0$.

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