Ordering Based on Implications

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Abstract: *Implication function I on a bounded lattice L is defined by means of some boundary conditions and monotonicity constraints. On the other hand, each implication function I on L defines a special relation which, in some cases, can be a (partial) order on L. We study the properties of I resulting into such (partial) orders. A special attention is given do situations yielding new bounded lattices.*

Keywords: *Implication, bounded lattice, partial order.*

1 Introduction

Fuzzy implications are one of the most important operations in fuzzy logic having a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, etc. (see [1, 5, 12, 13, 16]). They generalize the classical implication, which takes values in $\{0, 1\}$, to fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. In general situation, since $[0, 1]$ is a bounded lattice, like in the case of other logical operators, the problem of introducing implications on a bounded lattice laid bare and Ma and Wu [11] have introduced them at first. Several authors have investigated the implications on a bounded lattice and their relations to the other logical operators [9, 14, 15, 18, 19, 20].

In this paper, we introduce an order by means of an implication possessing some special properties on a lattice and discuss some of its properties. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we determine the relationship between the order induced by an implication and the order on the lattice. Giving example, we show that a bounded lattice needs not be a lattice with respect to the order induced by an implication. Also, we give an example for an implication making the unit interval $[0, 1]$ a lattice with respect to the order induced by it. Moreover, we obtain that such a generating method of an order is independent from the order induced by an adjoint t-norm (T-partial order) [8]. We prove that under the conditions required to define implication based order, the considered implication must be an S-implication, and so we obtain that the order induced by an implication coincides with the order which is generated in a similar way from a t-conorm. Consequently, we obtain that an implication on the unit interval $[0, 1]$ is continuous if and only if the implication based order and the dual of the natural order on $[0, 1]$ coincide.

2 Notations, definitions and a review of previous results

Definition 1. [2] Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation T (S) on L is called a t-norm (t-conorm) if it satisfies the following conditions:

(1) $T(T(a, b), c) = T(a, T(b, c))$ (associative law),

(2) $T(a, b) = T(b, a)$ (commutative law),

(3) $b \leq c \Rightarrow T(a, b) \leq T(a, c)$ (monotonicity),

(4) $T(a, 1) = a$ ($S(a, 0) = a$) (boundary condition),

where a, b and c are any elements of L.

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Definition 2. [11] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \rightarrow L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. An implication N on L is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

On each bounded lattice L we have two extremal negations $N^+, N^- : L \rightarrow L$ given by $N^-(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & \text{otherwise} \end{cases}$ and $N^+(x) = \begin{cases} 0 & if \quad x = 1, \\ 1 & \text{otherwise.} \end{cases}$
Obviously, for any negation $N : L \rightarrow L$, it holds $N^- \leq N \leq N^+$.

Definition 3. [1, 11] Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operator $I : L^2 \rightarrow L$ is said to be an implication function, shortly an implication, if it satisfies

(I1) For every elements a, b with $a \leq b$, $I(a, y) \geq I(b, y)$ for all $y \in L$.
(I2) For every elements a, b with $a \leq b$, $I(x, a) \leq I(x, b)$ for all $x \in L$.
(I3) $I(1, 1) = 1$, $I(0, 0) = 1$ and $I(1, 0) = 0$.

Note that from the definition, it follows that
$I(0, x) = 1$ and $I(x, 1) = 1$, for all $x \in L$. Special interesting properties for implications are:

- The exchange principle (EP)
  $I(x, I(y, z)) = I(y, I(x, z))$ for all $x, y, z \in L$
- The left neutrality principle (NP)
  $I(1, y) = y$, for every $y \in L$
- The contrapositive symmetry to a negation N (CP-N)
  $I(x, y) = I(N(y), N(x))$, for every $x, y \in L$
- The left contrapositive symmetry to a negation N (L-CP(N))
  $I(N(x), y) = I(N(y), x)$, for every $x, y \in L$

Obviously, for a strong negation N, the left contrapositive symmetry and the contrapositive symmetry coincide.

Definition 4. [1] Let $(L, \leq, 0, 1)$ be a lattice and I be an implication on L. The function $N_I : L \rightarrow L$ given by
$N_I = I(x, 0)$ for all $x \in L$
is a negation and it is called the natural negation of I.

Definition 5. [9] Let $(L, \leq, 0, 1)$ be a lattice. An implication $I : L^2 \rightarrow L$ is called an S- implication if there exists a t-conorm S and a strong negation N such that for every $x, y \in L$

$$I(x, y) = S(N(x), y).$$

Definition 6. [8] Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a $T-$ *partial order* (triangular order) for t-norm T:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

From the definition, it follows that $a \preceq_T b$ implies that $a \leq b$ for any elements $a, b \in L$.

Definition 7. [10] Let $T : [0, 1]^2 \rightarrow [0, 1]$ be a left-continuous t-norm. The function $I_T : [0, 1]^2 \rightarrow [0, 1]$ given by

$$I_T(x, y) = sup\{z \in [0, 1] | T(x, z) \leq y\}. \tag{1}$$

is an implication and it is called as a residual implication.

Observe that the definition (1) can be applied to any t-norm $T : L^2 \rightarrow L$ acting on a bounded lattice L, and the resulting function $I_T : L^2 \rightarrow L$ is an implication on L.

3 An *I*-based ordering

Definition 8. Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ be an implication. Define the relation $\preceq_I$ on L as follows:
For every $x, y \in L$

$$y \preceq_I x :\Leftrightarrow \exists \ell \in L \quad \text{such that} \quad I(\ell, x) = y. \tag{2}$$

Proposition 1. *The relation $\preceq_I$ is a partial order on L, whenever $I : L^2 \to L$ is an implication satisfying the exchange property (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I.*

We will call such an order defined in (2) as an *I- based ordering*.

Proposition 2. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ be an implication satisfying (EP) and (CP) with respect to the strong natural negation N_I. If $(x, y) \in \preceq_I$, then $(y, x) \in \leq$.*

Remark 1. Let $(L, \leq, 0, 1)$ be a bounded lattice and I be an implication satisfying (EP) and (CP-N_I).
(i) It is clear that 0 and 1 are the greatest and the least element with respect to $\preceq_I$, respectively.
(ii) The converse of Proposition 2 may not be satisfied. For example:
Consider the lattice $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 1:

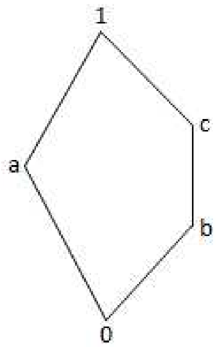

Figure 1: $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$

Define the function $I : L^2 \to L$ as follows:

I	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	c	1	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Obviously, I is an implication on L satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I defined as

$$N_I(x) = \begin{cases} a & if \quad x = a, \\ c & if \quad x = b, \\ b & if \quad x = c, \\ 1 & if \quad x = 0, \\ 0 & if \quad x = 1. \end{cases}$$

Although $b \leq c$, $c \not\preceq_I b$ since there does not exist an element $k \in L$ such that $I(k, b) = c$. The order $\preceq_I$ on L has its Hasse diagram as follows:

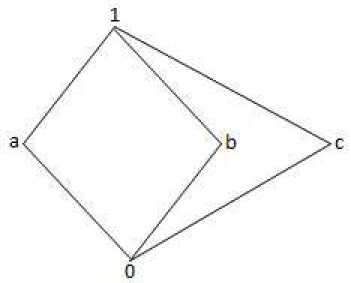

Figure 2: $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$

(iii) Even if $(L, \leq, 0, 1)$ is a chain, the partially ordered set $(L, \preceq_I)$ may not be a chain. For example: consider $L = [0, 1]$ and take the Fodor implication $I = I_{FD}$ defined as

$$I_{FD}(x, y) = \begin{cases} 1 & if \quad x \leq y, \\ max(1 - x, y) & if \quad x > y. \end{cases} \tag{3}$$

It is clear that I_{FD} satisfies the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation $N_{I_{FD}} = N_C$, $N_C(x) = 1 - x$. Since $1/2$ and $3/4$ are not comparable with respect to $\preceq_{I_{FD}}$, $([0, 1], \preceq_{I_{FD}})$ is not a chain.

Remark 2. Let T be a left continuous t-norm on $[0, 1]$ and I_T be the corresponding residual implication. Then, the implication based ordering and the T-partial order are independent.

L needs not be a lattice w.r.t. $\preceq_I$. The following example illustrates this case.

Example 1. Let $L = [0, 1]$ and take the implication I_{FD} given by (3). $([0, 1], \preceq_{I_{FD}})$ is not a lattice.

Proposition 3. *For every implication I satisfying (EP) and (CP-N_I), there exists a t-conorm S such that*

$$I(x, y) = S(N_I(x), y),$$

that is, I is an S- implication.

Corollary 4. *Let the implication $I : L^2 \to L$ satisfy (EP) and CP-N_I. Then, for any $a, b \in L$*
$a \preceq_I b$ if and only if $N_I(a) \preceq_T N_I(b)$,
where $T : L^2 \to L$ is a t-norm given by $T(x, y) = N_I(I(x, N_I(y)))$.

Theorem 5. *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication satisfying (EP) and the contrapositive symmetry (CP) with respect to the natural strong negation N_I and $\preceq_I$ be the order linked to the implication I. Then, I is continuous if and only if $\preceq_I = \geq$.*

One can wonder whether L is a bounded lattice w.r.t. an order obtained from an implication (under which conditions). In the next Proposition, we give some sufficient conditions.

Proposition 6. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ an implication on L defined as $I(x, y) = 1$ when $x \neq 1$ and $y \neq 0$, satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I. Then, $(L, \preceq_I)$ is a lattice.*

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