Properties of OML operations and the role of computers in proofs

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Motivation

The lack of distributivity of operations in OMLs causes problems with evaluation of complex expressions which cannot be transformed to a unique normal form, e.g., the word problem. As a substitute for distributivity, several tools were designed (see [4, 6, 7]); however all these approaches fail whenever a variable does not commute with at least two other variables. Thus their use is limited to formulas with up to 2 variables or satisfying compatibility restrictions. This does not allow, e.g., to test associativity. Particular results have been obtained in [3, 12] in the case when one variable commutes with the other two. Here we clarify the question of which operations on OMLs are associative.

Basic tools

The approach of [13] is based on the use of the free orthomodular lattice with 2 free generators a, b, denoted by $\mathrm{F}(a, b)$. It has $2^4 \times 6 = 96$ elements, their complete list is presented in [1] together with their unique codes (called *Beran's numbers* in [12] and subsequent papers).

Preceding results

In Boolean algebras, there are $2^4 = 16$ Boolean operations, among them the following 8 operations are associative:

the disjunction and conjunction,
the equivalence and its negation (XOR),
the left and right projection, and
the constants 0 and 1.

Each Boolean operation has 6 corresponding OML operations. E.g., the Boolean implication gives rise to 6 (quantum) implications $\to_i$, $i = 0, \ldots, 5$, which can be characterized as those binary OML operations which satisfy the Birkhoff-von Neumann requirement [2] $a \to_i b = 1$ iff $a \le b$ (see [10]). In Boolean algebras, all these 6 operations coincide with the classical implication, and the same happens in OMLs if a, b commute.

The 6 implications $\to_i$, $i = 0, \ldots, 5$, give rise to the corresponding (quantum) conjunctions $\wedge_i$ and disjunctions $\vee_i$ defined as $a \vee_i b = a' \to_i b$ and $a \wedge_i b = (a \to_i b')'$ [12]. In OMLs, a triple satisfies the associativity equation with respect to $\wedge_i$ and $\vee_1$, $i = 1, \ldots, 5$, under the condition that one of the elements commutes with the other two [3, 12]. For $i = 0$ these operations reduce to the classical conjunction and disjunction and are associative.

Kröger [11] showed that in a Boolean skew lattice, the condition of commuting elements is a sufficient and necessary condition.

New results

There are 8 associative operations in a Boolean algebra, each of them induces 6 OML operations, thus we have to test 48 OML operations.

We considered the special forms of the associativity equation where only two variables are present, $x, y \in \{a, a', b, b'\}$ where x and y do not commute. Then there are 6 possible equations which have to be tested for associativity: $(x * x) * y = x * (x * y)$, $(x * x') * y = x * (x' * y)$, $(x * y) * y = x * (y * y)$, $(x * y') * y = x * (y' * y)$, $(x * y) * x = x * (y * x)$, and $(x * y) * x' = x * (y * x')$. This was done by the computational tool[1] [8].

It turns out that 40 of these 48 OML operations violate at least one of the above equations. However, this approach is not applicable to the lower and upper commutator. These satisfy all the 6 equations above and we have to use another approach to prove their non-associativity.

In [5], we have proved that there are only 6 OML operations which are associative; they are

the disjunction and conjunction,
the left and right projection, and
the constants 0 and 1.

Other OML operations satisfy the associativity equation only under additional conditions on their arguments, e.g., that one of the variables commutes with the remaining two.

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