Finite Volume Scheme for Tensor Anisotropic Diffusion in Image Processing

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1. Introduction

Finite volume method is one of modern discretization method. Since this favourable numerical technique is well suited for the numerical simulation of conservation laws, it has been applied in a large number of scientific fields. One of them is just image processing.

This paper consists of three basic parts. First we present tensor anisotropic diffusion and derive a diffusion tensor for this nonlinear model. We also provide some information on this type diffusion. In the second part we derive a semi-implicit finite volume scheme for nonlinear tensor diffusion with the help of co-volume mesh. Finally in the last part we prove existence and uniqueness of a discrete solution for this numerical scheme.

2. Derivation of the diffusion tensor

Effort to gain precessed image more quickly and not so computionally expensive leads to inventions of new diffusion models and also to their improvements. One of them was introduced by Weickert (see [9], [10], [11] and [12]) in the following form

$$\begin{aligned}
\frac{\partial u}{\partial t} - \nabla \cdot (D\nabla u) &= 0, \qquad \text{in } Q_T \equiv I \times \Omega, && (1)\\
u(x,0) &= u_0(x), \quad \text{in } \Omega, && (2)\\
< D\nabla u, n > &= 0, \qquad \text{on } I \times \partial\Omega, && (3)
\end{aligned}$$

where D is a matrix depending on the eigenvalues and on the eigenvectors of the so-called (regularized) structure tensor $J = \nabla u (\nabla u)^T$ (for details see next subsections). This modification is useful in any situation, where is diserable strong smoothing in one direction and low smoothing in the perpendicular direction. Owing to this property, tensor anisotropic diffusion has applied mainly for images with interrupted coherence of structures.

2.1. Analysing coherent structures

In order to enhance a coherence of structures, we need a reliable tool for analysing coherent structures.

A very simple structure descriptor is given e.g., by the properties of $\nabla u_{\tilde{t}}$, where

$$u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot,t))(x), \qquad (\tilde{t} > 0). \tag{4}$$

We can use e.g., absolute value of $\nabla u_{\tilde{t}}$ for detecting edges in some images (see [1]) but for images with line structures this descriptor is unuseful. We know that for small $\tilde{t}$ high fluctuations remain, while larger $\tilde{t}$ leads to entirely useless results. This is due to the fact that for larger $\tilde{t}$ neighbouring gradients with same orientation, but opposite sign cancel each other. We need the structure descriptor invariant under sign changes, so we replace $\nabla u_{\tilde{t}}$ by its tensor product

$$J_0(\nabla u_{\tilde{t}}) = \nabla u_{\tilde{t}} \otimes \nabla u_{\tilde{t}} = \nabla u_{\tilde{t}} \nabla u_{\tilde{t}}^T. \tag{5}$$

The matrix J_0 is symmetric and positive semidefinite and its eigenvectors are parallel and orthogonal to $\nabla u_{\tilde{t}}$, respectively. We can average J by applying other convolution with Gaussian G_ρ

$$J_\rho(\nabla u_{\tilde{t}}) = G_\rho * (\nabla u_{\tilde{t}} \otimes \nabla u_{\tilde{t}}), \qquad (\rho \geq 0). \tag{6}$$

In computer vision community the matrix

$$J_\rho = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is well-known as structure tensor or interest operator or second moment matrix. Its exploitation is possible to find in many tasks, for example in analysis of flow-like textures (see [8]), corners and T-junctions (see [4] and [7]), shape cues (see [6]) and also spatio-temporal image sequences (see [5]).

This matrix J_ρ is symmetric and positive semidefinite and its eigenvalues are given as follows

$$\mu_{1,2} = \frac{1}{2}\left((a + c \pm \sqrt{(a-c)^2 + 4b^2})\right), \qquad \mu_1 \geq \mu_2. \tag{7}$$

Since the eigenvalues integrate the variation of the grey values within a neighbourhood of size $O(\rho)$, they describe the average contrast in the eigendirections v and w. The integration scale ρ reflects the characteristic size of the texture and in the most cases, it is large in comparison to the noise scale $\tilde{t}$.

With the help of the eigenvalues of J_ρ we can obtain useful information on the coherence of a structure. The expression $(\mu_1 - \mu_2)^2$ is large for anisotropic structures and tends to zero for isotropic structures. We can also identify kind of the image structures. Constant areas are characterized by $\mu_1 = \mu_2 = 0$, straight edges by $\mu_1 \gg \mu_2 = 0$ and corners by $\mu_1 \geq \mu_2 \gg 0$.

The corresponding orthonormal set of eigenvectors (v, w) to eigenvalues (μ_1, μ_2) is given by

$$v = (v_1, v_2), \qquad w = (w_1, w_2), \tag{8}$$

$$v_1 = 2b, \qquad v_2 = c - a + \sqrt{(a-c)^2 + 4b^2},$$

$$w \perp v, \qquad w_1 = -v_2, \qquad w_2 = v_1.$$

The orientation of the eigenvector w, which corresponds to the smaller eigenvalue μ_2 is called coherence orientation. This orientation has the lowest fluctuations.

2.2. Coherence-enhancing anisotropic diffusion

Since we have a tool for analysing coherence, we draw our goals to enhance of image coherence. One of possibilities, how to do it, can be done by embedding the structure tensor analysis into a nonlinear diffusion filter.

The idea of nonlinear diffusion filtering is as follows. We get a processed version $u(x,t)$ of an original image $u_0(x)$ with a scale parameter $t \geq 0$ as the solution of mathematical model (1)-(3), where u denotes an unknown function $u(x,t)$, n is the outer normal unit vector and $< \cdot, \cdot >$ the usual Euclidean scalar product. In our application the matrix D depends on solution u and satisfies the following properties: smoothness, symmetry and uniform positive definiteness. It is called diffusion tensor because it steers the diffusion process and its eigenvalues determine the diffusivities in the directions of the eigenvectors. For enhancing coherence, D must steers a filtering process such that diffusion is strong mainly along the coherence direction w and it increases with the coherence $(\mu_1 - \mu_2)^2$. To obtain it, we require that D must possess the same eigenvectors v and w as the structure tensor $J_\rho(\nabla u_{\tilde{t}})$ and we choose the eigenvalues of D as

$$
\begin{aligned}
\kappa_1 &= \alpha, \quad \alpha \in (0,1), \alpha \ll 1, \\
\kappa_2 &= \begin{cases} \alpha, & \text{if} \mu_1 = \mu_2, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{(\mu_1-\mu_2)^2}\right), \ C > 0 & \text{else.} \end{cases}
\end{aligned}
$$

The matrix D has a form

$$
D = ABA^{-1}, \tag{9}
$$

where

$$
A = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix}
$$

and

$$
B = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.
$$

We use the exponential function in choice of κ_2 because it ensures that the smoothness of the structure tensor carries over to the diffusion tensor and that κ_2 does not exceed 1. The positive parameter α guarantees that the process never stops. Even if $(\mu_1 - \mu_2)^2$ tends to zero so the structure becomes isotropic, there still remains some small linear diffusion with diffusivity $\alpha > 0$. Such α is a regularization parameter, which keeps the diffusion tensor uniformly positive definite. C has a role of a treshold parameter. Since if $(\mu_1 - \mu_2)^2 \gg C$ then $\kappa_2 \approx 1$ and if $(\mu_1 - \mu_2)^2 \ll C$ then $\kappa_2 \approx \alpha$. Due to the convolutions in (4) and (6), the elements of matrix D are C^1 functions.

3. Finite Volume Scheme for Tensor Anisotropic Diffusion in Image Processing

The aim of this section is to prove existence of unique discrete solution for the model (1)-(3) which satisfies to semi-implicit finite volume scheme obtained with the help of co-volume mesh. Let us consider a rectangular image domain $\Omega = (0, n_1) \times (0, n_2)$ and let an image $u(x)$ be represented by a bounded mapping $u : \Omega \to R$. Our image is represented by $n_1 \times n_2$ pixels (finite volumes) such that it looks as mesh with n_1 rows and n_2 columns (see Fig.1).

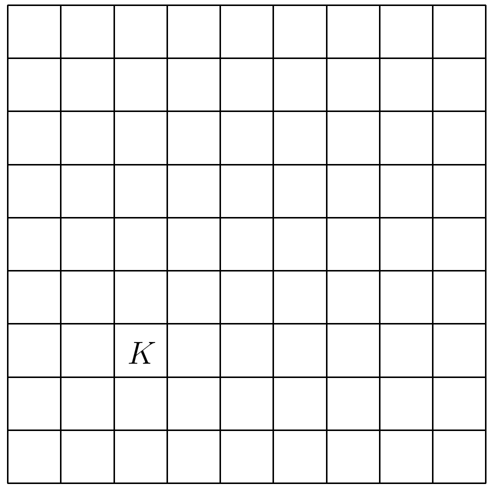

Fig.1 A mesh consists of pixels (finite volumes) K.

Let τ_h be a mesh of Ω (see Fig.2).

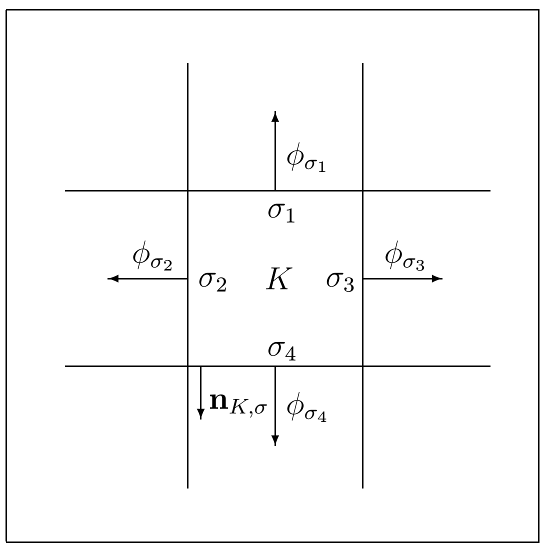

Fig.2 A detail of an image mesh - a finite volume K, its boundary $\sigma = \cup\sigma_i, i = 1, 2, 3, 4$ and the fluxes outward to a finite volume K.

We consider it in a scaling(time) interval $I = [0, T]$. Let $0 = t_0 \le t_1 \le \cdots \le t_{N_{max}} = T$ denote the time discretization with $t_n = t_{n-1} + k$, where k is the time(scale) step. For $n = 0, \dots, N_{max}$ we will look for u^n an approximation of solution at time t_n.

The value of grey level intensity of each pixel at $n-th$ discrete time level is given by $u^n[i][j]$, where i is a number of the row and j is a number of the column, in which this pixel is situated in the image. In our numerical scheme we need to compute the diffusion tensor D for each pixel at each discrete time step for uniform mesh with spatial step h. We calculate elements of $\nabla u_{\tilde{t}}[i][j] = \begin{pmatrix} u_x[i][j] \\ u_y[i][j] \end{pmatrix}$ as

$$u_x[i][j] = \frac{u^{n-1}[i][j+1] - u^{n-1}[i][j-1]}{2h},$$

$$u_y[i][j] = \frac{u^{n-1}[i+1][j] - u^{n-1}[i-1][j]}{2h}.$$

Then we continue in a computation of $D^{n-1}[i][j]$ by (5)-(9). Because of the convolutions in (4) and (6), the elements of matrix D^{n-1} are C^1 functions. By integrating equation (1) over finite volume K, we obtain

$$\int_K \frac{\partial u}{\partial t} dx - \int_K \nabla \cdot (D \nabla u) dx \quad = \quad 0. \tag{10}$$

We provide a semi-implicit in time discretization and use a divergence theorem to get

$$\frac{u_K^n - u_K^{n-1}}{k} m(K) - \sum_{\sigma \in \mathcal{E}_K} \int_\sigma D_K^{n-1} \nabla u^n \cdot \mathbf{n}_{K,\sigma} ds \quad = \quad 0, \tag{11}$$

where u_K^n, $K \in \tau_h$ represents the mean value of u^n on K, $m(K)$ is the measure of the finite volume K with boundary ∂K, σ is an edge of the control volume K, $\mathcal{E}_K$ is a subset of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \sigma$, $\mathcal{E} = \bigcup_{K \in \tau_h} \mathcal{E}_K$, where τ_h is admissible finite volume mesh (see [3]), D_K^{n-1} is a mean value of $D^{n-1} \equiv D(u^{n-1})$ on K, that is $D_K^{n-1} = \frac{1}{m(K)} \int_K D^{n-1} dx$ and $\mathbf{n}_{K,\sigma}$ is the normal unit vector to σ outward to K. Let us define the discrete solution by

$$u_{h,k}(x,t) = \sum_{n=0}^{N_{max}} \sum_{K \in \tau_h} u_K^n \chi\{x \in K\} \chi\{t_{n-1} < t \leq t_n\}, \tag{12}$$

where the function $\chi(A)$ is defined as

$$\chi_{\{A\}} = \begin{cases} 1, & \text{if A is true,} \\ 0, & \text{elsewhere.} \end{cases} \tag{13}$$

Equation (11) can also be written as

$$\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K} \phi_\sigma^n(u_{h,k}^n) m(\sigma) = 0, \tag{14}$$

with

$$\phi_\sigma^n(u_{h,k}^n) \approx \frac{1}{m(\sigma)} \int_\sigma D_K^{n-1} \nabla u^n \cdot \mathbf{n}_{K,\sigma} ds \tag{15}$$

and

$$u_{h,k}^n(x) = \sum_{K \in \tau_h} u_K^n \chi\{x \in K\}, \tag{16}$$

where $m(\sigma)$ is the measure of edge σ.

One possibility how to get an approximation of the flux (i.e. $\phi_\sigma^n(u_{h,k}^n)$) is obtained with the help of co-volume mesh. The specific name (diamond-cell) of this method (see [2]) is due to the choice of co-volume as a diamond-shaped polygon. The co-volume χ_σ associated to σ is constructed around each edge by joining all four co-volume vertices (i.e. endpoints of this edge and midpoints of finite volumes which are common to this edge) (see Fig.3).

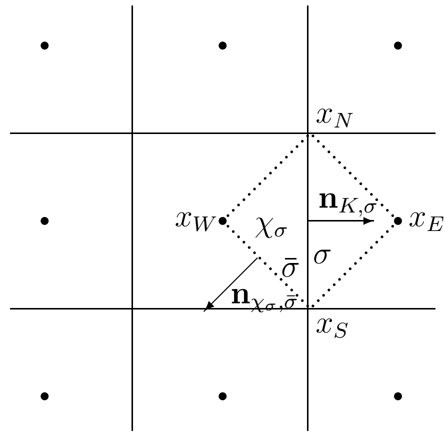

Fig.3 A detail of mesh. A co-volume associated to edge σ.

We denote the endpoints of an edge $\bar\sigma \subset \partial\chi_\sigma$ by $N_1(\bar\sigma)$ and $N_2(\bar\sigma)$ and $\mathbf{n}_{\chi_\sigma,\bar\sigma}$ is the normal unit vector to $\bar\sigma$ outward to χ_σ. In order to have an approximation of the diffusion flux, we first derive, using divergence theorem, an approximation of the averaged gradient on σ

$$\frac{1}{m(\chi_\sigma)}\int_{\chi_\sigma}\nabla u^n dx = \frac{1}{m(\chi_\sigma)}\int_{\partial\chi_\sigma} u^n \mathbf{n}_{\chi_\sigma,\bar\sigma} ds, \tag{17}$$

and then we denote it

$$p_\sigma^n = \frac{1}{m(\chi_\sigma)}\sum_{\bar\sigma\in\partial\chi_\sigma}\frac{1}{2}\left(u^n_{N_1(\bar\sigma)} + u^n_{N_2(\bar\sigma)}\right) m(\bar\sigma)\mathbf{n}_{\chi_\sigma,\bar\sigma}. \tag{18}$$

The value at the centres x_E and x_W are u_E and u_W while the values at the vertices x_N and x_S are computed as the arithmetic mean of values on finite volumes which are common to this vertex (for general nonuniform meshes see [2]).

Since our mesh is uniform squared, for simplification, we can use the following relations: $m(\chi_\sigma) = \frac{h^2}{2}$, $m(\bar\sigma) = \frac{\sqrt{2}}{2}h$ and after a short calculation we are ready to write

$$p_\sigma^n = \begin{pmatrix} \frac{u_E^n - u_W^n}{h} \\ \frac{u_N^n - u_S^n}{h} \end{pmatrix}. \tag{19}$$

The relation (19) can be written as

$$p_\sigma^n = \frac{u_E^n - u_W^n}{h}\mathbf{n}_{K,\sigma} + \frac{u_N^n - u_S^n}{h}\mathbf{t}_{K,\sigma}, \tag{20}$$

where $\mathbf{t}_{K,\sigma}$ is a unit vector parallel to σ such that $(x_N - x_S)\cdot\mathbf{t}_{K,\sigma} > 0$. Although such u_N^n, u_W^n, u_E^n and u_S^n correspond to particular edge σ, we should denote them by $u^n_{N_\sigma}$, $u^n_{W_\sigma}$, $u^n_{E_\sigma}$ and $u^n_{S_\sigma}$, we use those simpler notations. Replacing the exact gradient ∇u^n by the numerical gradient p_σ^n in approximation $u_{h,k}$ we get the numerical flux in the form

$$\phi_\sigma^n(u_{h,k}) = \frac{1}{m(\sigma)}\int_\sigma D_K^{n-1} p_\sigma^n \cdot \mathbf{n}_{K,\sigma} ds = D_\sigma p_\sigma^n \cdot \mathbf{n}_{K,\sigma}, \tag{21}$$

where $D_\sigma = \frac{1}{m(\sigma)} \int_\sigma D_K^{n-1} ds = \begin{pmatrix} \bar{\lambda}_\sigma & \bar{\beta}_\sigma \\ \bar{\beta}_\sigma & \bar{\nu}_\sigma \end{pmatrix}$ in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t}_{K,\sigma})$.

If $D = \begin{pmatrix} \lambda & \beta \\ \beta & \nu \end{pmatrix}$ then $D_{\sigma_2} = D_{\sigma_3} = \begin{pmatrix} \lambda_\sigma & \beta_\sigma \\ \beta_\sigma & \nu_\sigma \end{pmatrix}$, i.e. $\bar{\lambda}_\sigma = \lambda_\sigma,\ \bar{\beta}_\sigma = \beta_\sigma,\ \bar{\nu}_\sigma = \nu_\sigma$ and

$D_{\sigma_1} = D_{\sigma_4} = \begin{pmatrix} \nu_\sigma & -\beta_\sigma \\ -\beta_\sigma & \lambda_\sigma \end{pmatrix}$, i.e. $\bar{\lambda}_\sigma = \nu_\sigma,\ \bar{\beta}_\sigma = -\beta_\sigma,\ \bar{\nu}_\sigma = \lambda_\sigma$,

where $\lambda_\sigma = \frac{1}{m(\sigma)} \int_\sigma \lambda^{n-1} ds$ and β_σ and ν_σ correspondingly. And it in turn implies

$$\phi_\sigma^n(u_{h,k}) = \begin{pmatrix} \bar{\lambda}_\sigma & \bar{\beta}_\sigma \\ \bar{\beta}_\sigma & \bar{\nu}_\sigma \end{pmatrix} \begin{pmatrix} \frac{u_E^n - u_W^n}{h} \\ \frac{u_N^n - u_S^n}{h} \end{pmatrix} \bar{\mathbf{n}}_{K,\sigma} = \bar{\lambda}_\sigma \frac{u_E^n - u_W^n}{h} + \bar{\beta}_\sigma \frac{u_N^n - u_S^n}{h}, \tag{22}$$

where $\bar{\mathbf{n}}_{K,\sigma}$ is a normal unit vector to σ outward to K in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t}_{K,\sigma})$, i.e. $\bar{\mathbf{n}}_{K,\sigma} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all σ.

In order to prove of existence and uniqueness of u_K^n, $K \in \tau_h$, we estimate the expressions $u_N^n - u_S^n$ by means of $u_E^n - u_W^n$ for all edges σ. Let us note that due to simpler notation, we will use in the sequel u_N, u_S, u_W and u_E instead of u_N^n, u_S^n, u_W^n and u_E^n.

Let P_σ be the set of all edges δ perpendicular to σ. Since we use uniform squared mesh, we compute values u_N and u_S as arithmetic mean of the values of solution in the neighbouring finite volumes and therefore we can write $u_N - u_S$ in the following way (see Fig.4)

$$\begin{aligned} u_N - u_S &= \frac{1}{2}\left(\frac{1}{2}\left[(u_E^1 - u_W^1) + (u_W^3 - u_E^3)\right] + \frac{1}{2}\left[(u_E^2 - u_W^2) + (u_W^4 - u_E^4)\right]\right) \\ &= \frac{1}{4}\left[u_E^1 - u_W^1) + (u_W^3 - u_E^3) + (u_E^2 - u_W^2) + (u_W^4 - u_E^4)\right], \end{aligned} \tag{23}$$

where u_E^1 and u_W^1 correspond to edge δ_1 and similarly u_E^2, u_W^2, u_E^3, u_W^3, u_E^4 and u_W^4 correspond to edges δ_2, δ_3 and δ_4.

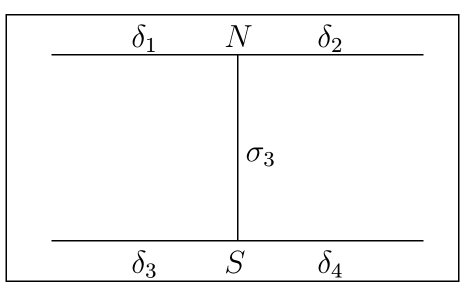

Fig.4 An edge σ and edges δ_1, δ_2, δ_3 and δ_4 perpendicular to σ.

Applying the inequality $(a-b)^2 \le 2a^2 + 2b^2$ in (23) we get

$$\begin{aligned} (u_N - u_S)^2 &\le \frac{1}{8}\left[(u_E^1 - u_W^1) + (u_W^3 - u_E^3)\right]^2 + \frac{1}{8}\left[(u_E^2 - u_W^2) + (u_W^4 - u_E^4)\right]^2 \\ &\le \frac{1}{4}(u_E^1 - u_W^1)^2 + \frac{1}{4}(u_W^3 - u_E^3)^2 + \frac{1}{4}(u_E^2 - u_W^2)^2 + \frac{1}{4}(u_W^4 - u_E^4)^2. \end{aligned} \tag{24}$$

It can be written as

$$(u_{N_\sigma} - u_{S_\sigma})^2 \leq \sum_{\delta \in P_\sigma} \frac{1}{4}(u_{E_\delta} - u_{W_\delta})^2, \tag{25}$$

where for clarity we put subindexes σ and δ.
Multiplying (25) by $\left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \frac{\bar{\lambda}_\sigma}{h^2}$ and summing over for all $\sigma \in \mathcal{E}_{int}$ we obtain

$$\sum_{\sigma \in \mathcal{E}_{int}} \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \left(\frac{u_{N_\sigma} - u_{S_\sigma}}{h}\right)^2 \bar{\lambda}_\sigma \leq \sum_{\sigma \in \mathcal{E}_{int}} \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \sum_{\delta \in P_\sigma} \frac{1}{4} \left(\frac{(u_{E_\delta} - u_{W_\delta})}{h}\right)^2 \bar{\lambda}_\sigma. \tag{26}$$

The next step is achieved by swapping the two sums

$$\sum_{\sigma \in \mathcal{E}_{int}} \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \left(\frac{u_{N_\sigma} - u_{S_\sigma}}{h}\right)^2 \bar{\lambda}_\sigma \leq \sum_{\delta \in \mathcal{E}} \gamma_\delta \left(\frac{(u_{E_\delta} - u_{W_\delta})}{h}\right)^2 \bar{\lambda}_\delta \tag{27}$$

with

$$\gamma_\delta = \sum_{\sigma \in P_\delta \cap \mathcal{E}_{int}} \frac{1}{4} \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \frac{\bar{\lambda}_\sigma}{\bar{\lambda}_\delta}. \tag{28}$$

Let us consider the matrix $\begin{pmatrix} \lambda_{\sigma^\perp} & \beta_{\sigma^\perp} \\ \beta_{\sigma^\perp} & \nu_{\sigma^\perp} \end{pmatrix}$, which is the matrix D in the basis $(\mathbf{t}_{K,\delta}, -\mathbf{n}_{K,\delta})$ on edge σ. Due to smoothness of D we get

$$\bar{\lambda}_\sigma = \bar{\nu}_{\sigma^\perp} = \bar{\nu}_\delta(1 + O(h)) = \bar{\lambda}_{\delta^\perp}(1 + O(h)), \quad \delta \in P_\sigma, \tag{29}$$
$$\bar{\beta}_\sigma = -\bar{\beta}_{\sigma^\perp} = -\bar{\beta}_\delta(1 + O(h)) = \bar{\beta}_{\delta^\perp}(1 + O(h)), \quad \delta \in P_\sigma, \tag{30}$$
$$\bar{\nu}_\sigma = \bar{\lambda}_{\sigma^\perp} = \bar{\lambda}_\delta(1 + O(h)) = \bar{\nu}_{\delta^\perp}(1 + O(h)), \quad \delta \in P_\sigma. \tag{31}$$

Applying (29)-(31) in (28) we have

$$\gamma_\delta = \sum_{\sigma \in P_\delta \cap \mathcal{E}_{int}} \frac{1}{4} \left(\frac{\bar{\beta}_{\delta^\perp}}{\bar{\lambda}_{\delta^\perp}}\right)^2 \frac{\bar{\lambda}_{\delta^\perp}}{\bar{\lambda}_\delta} (1 + O(h)) = \left(\frac{\bar{\beta}_{\delta^\perp}}{\bar{\lambda}_{\delta^\perp}}\right)^2 \frac{\bar{\lambda}_{\delta^\perp}}{\bar{\lambda}_\delta} (1 + O(h)). \tag{32}$$

Using the positive definiteness of the diffusion tensor $\begin{pmatrix} \lambda_\delta & \beta_\delta \\ \beta_\delta & \nu_\delta \end{pmatrix}$ we obtain

$$|D| = \lambda_\delta \nu_\delta - \beta_\delta^2 > 0. \tag{33}$$

We have two possibilities for γ_δ. Let δ is arbitrary edge parallel to σ_3. Then

$$\gamma_\delta = \left(\frac{-\beta_\delta}{\nu_\delta}\right)^2 \frac{\nu_\delta}{\lambda_\delta}(1 + O(h)) = \frac{(\beta_\delta)^2}{\lambda_\delta \nu_\delta}(1 + O(h)) < 1 \tag{34}$$

for h sufficiently small. If $\delta \perp \sigma_3$ then we have

$$\gamma_\delta = \left(\frac{\beta_\delta}{\lambda_\delta}\right)^2 \frac{\lambda_\delta}{\nu_\delta}(1 + O(h)) = \frac{(\beta_\delta)^2}{\lambda_\delta \nu_\delta}(1 + O(h)) < 1 \tag{35}$$

for h sufficiently small. Thus, due to fact that $\lambda_\sigma \geq C > 0$ and $\nu_\sigma \geq C > 0$, we obtain $0 \leq \gamma_\delta$ for h sufficiently small. Since this condition is fulfiled for each edge δ we can rewrite (27) as

$$\sum_{\sigma \in \mathcal{E}_{int}} \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \left(\frac{u_N - u_S}{h}\right)^2 \bar{\lambda}_\sigma \leq \gamma \sum_{\sigma \in \mathcal{E}} \left(\frac{u_E - u_W}{h}\right)^2 \bar{\lambda}_\sigma, \tag{36}$$

where

$$0 \leq \gamma < 1, \quad \gamma = \max_{\sigma \in \mathcal{E}} \gamma_\sigma. \tag{37}$$

Let us now introduce the space of piecewise constant functions associated to our mesh and discrete H_0^1 norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.
Definition 3.1. Let Ω be an open bounded polygonal subset of R^2. We define $\mathcal{P}_0(\tau_h)$ as the set of functions from Ω to R which are constant over each finite volume of the mesh.
Definition 3.2. Let Ω be an open bounded polygonal subset of R^2. For $u \in \mathcal{P}_0(\tau_h)$ we define

$$||u_{h,k}^n||_{1,\mathcal{T}} = \left(\sum_{\sigma \in \mathcal{E}} \left(\frac{u_E - u_W}{h}\right)^2 m(\chi_\sigma)\right)^{\frac{1}{2}}. \tag{38}$$

We can define discrete operator for (1)-(3) by

$$\mathcal{L}_h(u_{h,k}^n) = u_K^n m(K) - k \sum_{\sigma \in \mathcal{E}_K} \phi_\sigma^n(u_{h,k}^n) m(\sigma), \tag{39}$$

such that $u_{h,k}^n$ is the solution in $\mathcal{P}_0(\tau_h)$ of

$$\mathcal{L}_h(u_{h,k}^n) = f_{h,k}(u_{h,k}^{n-1}), \tag{40}$$

where $f_{h,k}(u_{h,k}^{n-1}) = u_K^{n-1} m(K)$ and u_K^{n-1} is a value of the piecewise constant function $u_{h,k}^{n-1}$ in K. This equality is a linear system of N equations with N unknowns, namely u_K, $K \in \tau_h$, where $N = card(K)$.

Multiplying $\mathcal{L}_h(u_{h,k})$ by u_K^n, summing over K and splitting into a part A and B leads to

$$\sum_{K \in \tau_h} \mathcal{L}_h(u_{h,k}^n) u_K^n = A + B, \tag{41}$$

with

$$A = \sum_{K \in \tau_h} (u_K^n)^2 m(K) = ||u_{h,k}^n||_{L^2(\Omega)}^2 \tag{42}$$

and

$$\begin{aligned} B &= k \sum_{K \in \tau_h} u_K^n \sum_{\sigma \in \mathcal{E}_K} -\phi_\sigma^n(u_{h,k}^n) m(\sigma) \\ &= k \sum_{\sigma \in \mathcal{E}} \phi_\sigma^n(u_{h,k}^n) \frac{u_E - u_W}{h} 2m(\chi_\sigma) = Q(u_{h,k}^n) \end{aligned} \tag{43}$$

and we know due to Neumann boundary condition that $\phi_\sigma(u^n_{h,k}) = 0$ if $\sigma \in \partial\Omega$. B can be written as

$$Q(u^n_{h,k}) = k \sum_{\sigma\in\mathcal{E}_{int}} (D_\sigma p^\star_\sigma) \cdot p_\sigma 2m(\chi_\sigma) = 2k(D_h p^\star_h, p_h)_{L^2(\Omega)}, \tag{44}$$

where

$$p^\star_\sigma = \frac{u_E - u_W}{h} \mathbf{n}_{K,\sigma} \tag{45}$$

and D_h is the piecewise constant function of value D_σ on χ_σ for each $\sigma \in \mathcal{E}_{int}$. Further, we use the following inequality

$$(D_h p^\star_h, p_h)_{L^2(\Omega)} \geq (D_h p^\star_h, p^\star_h)_{L^2(\Omega)} - |(D_h p^\star_h, p_h - p^\star_h)_{L^2(\Omega)}|. \tag{46}$$

It is clear that

$$(D_h p^\star_h, p^\star_h)_{L^2(\Omega)} = \sum_{\sigma\in\varepsilon} \bar{\lambda}_\sigma \left(\frac{u_E - u_W}{h}\right)^2 m(\chi_\sigma). \tag{47}$$

Using Young's inequality for second term on right side in (46) leads to

$$\begin{aligned} |(D_h p^\star_h, p_h - p^\star_h)_{L^2(\Omega)}| &= \left|\sum_{\sigma\in\mathcal{E}} \bar{\beta}_\sigma \frac{u_E - u_W}{h} \frac{u_N - u_S}{h} m(\chi_\sigma)\right| \\ \leq \sum_{\sigma\in\mathcal{E}_{int}} \frac{1}{2} &\left[\left(\frac{u_E - u_W}{h}\right)^2 + \left(\frac{\bar{\beta}_\sigma}{\bar{\lambda}_\sigma}\right)^2 \left(\frac{u_N - u_S}{h}\right)^2\right] \bar{\lambda}_\sigma m(\chi_\sigma), \end{aligned} \tag{48}$$

since $\phi^n_\sigma(u^n_{h,k}) = 0$, if $\sigma \in \partial\Omega$.
Using inequalities (36) we get

$$|(D_h p^\star_h, p_h - p^\star_h)_{L^2(\Omega)}| \leq \frac{1+\gamma}{2} \sum_{\sigma\in\mathcal{E}} \bar{\lambda}_\sigma \left(\frac{u_E - u_W}{h}\right)^2 m(\chi_\sigma) = \frac{1+\gamma}{2} (D_h p^\star_h, p^\star_h)_{L^2(\Omega)}. \tag{49}$$

It in turn implies

$$\frac{1}{2} Q(u^n_{h,k}) \geq \left(1 - \frac{1+\gamma}{2}\right) k(D p^\star_h, p^\star_h)_{L^2(\Omega)} \geq \bar{\lambda}_{\min} \frac{1-\gamma}{2} k ||u^n_{h,k}||^2_{1,\mathcal{T}}, \tag{50}$$

where $\bar{\lambda}_{\min} = \inf_{\sigma\in\mathcal{E}} \bar{\lambda}_\sigma \geq C > 0$.

Applying (42), (43) and (50) in (41) we get

$$\forall h > 0, \quad \forall u^n_{h,k} \in \mathcal{P}_0(\tau_h), \qquad \sum_{K\in\tau_h} \mathcal{L}_h(u^n_{h,k}) u^n_K \geq \alpha \left(||u^n_{h,k}||^2_{1,\mathcal{T}} + ||u^n_{h,k}||^2_{L^2(\Omega)}\right) \tag{51}$$

with $\alpha = \max(\bar{\lambda}_{\min}(1-\gamma)k, 1)$.

Theorem 3.1. For h sufficiently small, there exists unique solution $u_{h,k}$ given by scheme (14) with (22).

Proof of Theorem 3.1. Assume that u_K, $K \in \tau_h$ satisfies linear system (40) and $f = 0$. Using (51) and (40) we get

$$\alpha \left(||u^n_{h,k}||^2_{1,\mathcal{T}} + ||u^n_{h,k}||^2_{L^2(\Omega)}\right) \leq \sum_{K\in\tau_h} \mathcal{L}_h(u^n_{h,k}) u^n_K = \sum_{K\in\tau_h} f u^n_K = 0. \tag{52}$$

Due to relation (52), we know that $u_K^n = 0$, $\forall K \in \tau_h$. And it in turn implies that for each right side f exist unique solution.

6. Conclusion

The paper introduces the semi-implicit finite volume scheme for tensor anisotropic diffusion. This nine-point scheme was derived with the help of co-volume mesh. We also prove existence and uniqueness of its discrete solution. The main idea in this proof is a bounding of a gradient in tangential direction by using of a gradient in normal direction.

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