Spaces of faithful states on quantum structures

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Abstract

State spaces of quantum structures can be efficiently represented using hypergraphs. We introduce a new representation which allows to use rather general hypergraphs and preserves faithfulness of states.

The description of mathematical structures representing quantum mechanical systems can be a difficult task. It has been facilitated by the use of *Greechie diagrams*. These are hypergraphs in which vertices correspond to atoms and edges to maximal Boolean subalgebras. This technique was first published in [3]. Greechie diagrams allow to describe orthomodular lattices, orthomodular posets, and orthoalgebras provided that all chains are finite (see [2, 4, 9] for more details). Recently they were generalized to effect algebras [8]. Here we shall concentrate on the study of OMLs. This case appears to be the most difficult one, the results for other quantum structures follow from it.

A *state (=probability measure) on an OML L* is a mapping $s\colon L \to [0,1]$ such that $s(\mathbf{1}) = 1$ and $\forall a, b \in L,\ a \le b' : s(a \vee b) = s(a) + s(b)$. A *state on a hypergraph* $K = (\mathcal{V}, \mathcal{E})$ (where $\mathcal{V}$ is the set of vertices, $\mathcal{E}$ is the set of edges) is a mapping $s\colon \mathcal{V} \to [0,1]$ such that $\forall E \in \mathcal{E} :\ \sum_{v \in E} s(v) = 1$. By $\mathcal{S}(L)$ we denote the *state space*(=the set of all states) of an OML, resp. a hypergraph L.

The states on a quantum structure are in a one-to-one correspondence to states on its Greechie diagram. This allows to study state spaces efficiently [10]. However, a non-trivial question arises: Which hypergraphs are Greechie diagrams of (the desired class of) quantum structures? Although an answer is given for orthomodular lattices (OMLs) in [1], its application may be rather difficult. (Easier characterizations are known for more general structures—orthomodular posets and orthoalgebras.) In general, it is not easy to design a hypergraph with given state space properties and satisfying the conditions for Greechie diagrams. Even if a hypergraph is designed, the verification of these conditions may be difficult.

Another approach has been initiated in [5] and completed in [7]. It is based on a weaker correspondence between the hypergraph and the OML. The construction does not possess a one-to-one correspondence between elements of an

OML and some features of the hypergraph. Nevertheless, a one-to-one correspondence between states exists. This mapping is not only an affine homeomorphism as in [5]; it preserves also the ranges of states and the values at corresponding elements. This relation is expressed in terms of evaluation functionals.

The *evaluation functional* associated with $a \in L$ is the mapping $\mathbf{e}(a) \colon \mathcal{S}(L) \to [0,1],\ s \mapsto s(a)$. We use the notation $\mathbf{e}(L) := \{\mathbf{e}(a) : a \in L\}$. In order to extend the notion of evaluation functional to hypergraphs, we need to evaluate it not only on the vertices of the hypergraph, but also on all subsets of all edges. A precise definition is based on the notion of *semipasted family of Boolean algebras*, see [6, 7] for details.

Two OMLs L, K are called *functionally isomorphic* iff there is a bijection $\hat{}\colon \mathbf{e}(L) \to \mathbf{e}(K)$ and an affine homeomorphism $\tilde{}\colon \mathcal{S}(L) \to \mathcal{S}(K)$ such that $\hat{f}(\tilde{s}) = f(s)$ for all $f \in \mathbf{e}(L)$, $s \in \mathcal{S}(L)$. This notion is canonically extended to other quantum structures and to hypergraphs so that we can also speak of a functional isomorphism of an OML and a hypergraph. Following [5] and [7], given a hypergraph, we may construct a functionally isomorphic orthoalgebra, orthomodular poset, and even an orthomodular lattice:

Theorem 1 *[7] Every finite hypergraph is functionally isomorphic to a finite orthomodular lattice.*

Thus the state space can be modelled using any hypergraph and we can guarantee the existence of an OML with the desired properties. The only limitation is that we may refer only to properties preserved by the functional isomorphism. These include the convex structure, the ranges of states, etc.

One desirable property of states which is *not* preserved by the functional isomorphism is faithfulness. A state s is called *faithful* if $\forall a \neq 0 : s(a) \neq 0$. A functional isomorphism does not preserve this property. The construction of [7] results typically in an OML with non-zero elements on which all states vanish. Such elements cannot be distinguished from $\mathbf{0}$ by evaluation functionals. Starting from a faithful state on a hypergraph, it may happen that the corresponding state on an OML is not faithful. This is the motivation of a new approach. We want a stronger correspondence between the hypergraph and its corresponding OML so that faithfulness of states would be preserved: Two OMLs (or hypergraphs) L, K are called *faithfully functionally isomorphic* iff there is a bijection $\hat{}\colon \mathbf{e}(L \setminus \{\mathbf{0}\}) \to \mathbf{e}(K \setminus \{\mathbf{0}\})$ and an affine homeomorphism $\tilde{}\colon \mathcal{S}(L) \to \mathcal{S}(K)$ such that $\hat{f}(\tilde{s}) = f(s)$ for all $f \in \mathbf{e}(L \setminus \{\mathbf{0}\})$, $s \in \mathcal{S}(L)$.

Our intention is to keep a loose correspondence so that the restrictions on the hypergraph be as weak as possible. The optimal conclusion would be a variant of Theorem 1 with faithfulness preserved. However, this is not possible, as shown already in [5]: There are hypergraphs which are not faithfully functionally isomorphic to any OML. Here we announce a new result—a characterization of hypergraphs which are faithfully functionally isomorphic to OMLs. We shall need the following notions: A *component* of a hypergraph is a maximal connected subhypergraph. An odd graph is a hypergraph such that each edge has exactly 2 vertices and there is a loop (=cycle) of an odd order.

Theorem 2 *A hypergraph is faithfully functionally isomorphic to an orthomodular lattice iff at least one of the following conditions holds:*

- *There is no component which is an odd graph.*
- *There is a vertex v such that $s(v) = 0$ for each state s.*
- *There is a component which is* not *an odd graph and contains an element b such that $s(b) = 1/2$ for all states s.*

The use of the latter theorem is not as easy as Theorem 1, but it is much more comfortable than the original Greechie diagram approach of [1, 3]. It extends the contribution of [7] to the study of those properties of state spaces which are preserved by faithful functional isomorphisms, not only by ordinary functional isomorphisms.

Acknowledgements. This research was supported by the grant 201/07/1051 of the Czech Science Foundation.

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