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NUMERICAL SOLUTION TO TWO FACTOR MODEL FOR OPTION PRICING WITH STOCHASTIC VOLATILITY

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In this paper we discuss numerical solution of the two factor option pricing model represented by a second order parabolic partial differential equation. In spite of the standard Black-Scholes equation, not only the asset price but also volatility follow a stochastic differential equation of diffusive type.

We mainly deal with an application of the finite volume method to the corresponding convection-diffusion equation accompanied with Dirichlet boundary conditions. It seems that such boundary conditions are much more natural and easy to implement in algorithms for computing solution of the two factor option pricing models than ones used by Forsyth, Vetzal and Zvan [2].

Finally, the numerically computed examples for two-factor model are presented and discussed. For a special choice of parameters in the underlying stochastic differential equations it is possible to compare the behaviour of our numerical solution with the exact Black-Scholes solution of the European call option problem. We use such an approach for testing the numerical scheme.

K e y w o r d s: option pricing, stochastic volatility, finite volume method

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1 THE TWO FACTOR MODEL FOR OPTION PRICING

1.1 Governing equations

Let us consider the second order parabolic partial differential equation (PDE)

$$\frac{\partial V}{\partial t} + (r-D)S\frac{\partial V}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + (\kappa(\theta - v) - \lambda v)\frac{\partial V}{\partial v} - rV = 0\,, \qquad (1)$$

which is a well know model for option pricing with stochastic volatility [2, 5, 7]. In equation (1) the function $V(S(t), v(t), t)$ represents the value of the option as a function of S, v, t, where $S(t)$ is the market price of the underlying asset (for example stock), $v(t)$ is volatility of the underlying asset price and t is time, $t \in [0, T]$. The further parameters represent: σ — volatility of the volatility v, λ — the market price of risk, θ — the reversion level of volatility v, κ — the speed of reversion parameter for v, r — the riskfree interest rate and ρ — the coefficient of correlation. In this two factor model the time evolution of the asset price S and volatility v are solutions of the stochastic differential equations [2, 6]

$$\mathrm{d}S = \mu S\mathrm{d}t + \sqrt{v}S\mathrm{d}W_1\,, \qquad (2)$$

$$\mathrm{d}v = \kappa(\theta - v)\mathrm{d}t + \sigma\sqrt{v}\mathrm{d}W_2\,, \qquad (3)$$

where $\mathrm{d}S$ is the change of the stock price in a small time interval $\mathrm{d}t$, $\mathrm{d}v$ is the change of the stock volatility in $\mathrm{d}t$, μ is the expected rate of return on S and $\mathrm{d}W_i$ are standard Wiener processes, $i = 1, 2$ with correlation given by $E(\mathrm{d}W_1\mathrm{d}W_2) = \rho\mathrm{d}t$. If we want to solve PDE (1) uniquely we have to specify initial and boundary conditions. The value of the option at the expiry time $t = T$ is determined by the option payoff diagram. Then the value of the option $V(S, v, T)$ at expiry time is given by

$$V(S,v,T) = \begin{cases} \max(S-E, 0) & \text{for call option,} \\ \max(E-S, 0) & \text{for put option,} \end{cases} \qquad (4)$$

where E is the exercise (strike) price. Since we are dealing with $2D$ problem, we consider Dirichlet boundary conditions for $S = 0$, $S \to \infty$, $v \to 0$, $v \to \infty$. From the stochastic differential equation (2) it follows that if the price of the stock S is zero, then the change of the stock price $\mathrm{d}S$ through time interval $\mathrm{d}t$ is zero. By that we get the following condition

$$V(0, v, t) = 0\,, \quad S \to 0\,. \qquad (5)$$

In the case, when the price of stock rises, i.e. $S \to \infty$, we can expect that the owner of the call option will buy the stocks at the expiry date T. Moreover the strike price will be small in comparison with the stock price, so we get the boundary condition $V(S \to \infty, v, t) \to S$. However in the numerical approximation we are looking for the solution on the finite interval. So taking into account the

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dividends and the time value of the money the boundary condition has the following form

$$V(S,v,t) = Se^{-D(T-t)} - Ee^{-r(T-t)} \quad \text{for } S \to \infty \,. \tag{6}$$

In order to define Dirichlet boundary conditions for $v \to 0$ and $v \to \infty$ we use the explicit Black-Scholes formula for the value $c(S,\delta,t)$ of the European call option given by [4, 5, 7]

$$c(S,\delta,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)\,, \tag{7}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,\mathrm{d}y\,,$$
$$d_1 = \frac{\ln\frac{S}{E} + (r - D + \frac{\delta^2}{2})(T-t)}{\delta\sqrt{T-t}}\,,$$
$$d_2 = d_1 - \delta\sqrt{T-t}$$

and δ is a constant volatility of the stock. Let $\delta \to 0^+$. If $Se^{-D(T-t)} \geq Ee^{-r(T-t)}$ then $d_1, d_2 \to \infty$ and $c(S,\delta,t) = Se^{-D(T-t)} - Ee^{-r(T-t)}$. If $Se^{-D(T-t)} < Ee^{-r(T-t)}$ then $d_1, d_2 \to -\infty$ and $c(S,\delta,t) = 0$. Thus we get the boundary condition

$$V(S,v,t) = \max(Se^{-D(T-t)} - Ee^{-r(T-t)}, 0)\,,\ v \to 0\,. \tag{8}$$

In case $v \to \infty$ the equation is reduced to [2]

$$\frac{\partial V}{\partial t} + (r-D)S\frac{\partial V}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} - rV = 0\,, \tag{9}$$

which is nothing else than the Black-Scholes equation and we have

$$V(S,v,t) = c(S,v,t)\,,\ v \to \infty\,. \tag{10}$$

In the sequel we will consider equation (1) with the expiry condition (4) and boundary conditions (5), (6), (8) and (10).

1.2 Transformation of the model

Before numerical discretization, as it is usual in computational finance dealing with option pricing, we transform the model by introducing new variables x, y, τ given by $x = \ln(S/E)$, $y = v$ and $\tau = T - t$. Then $S = e^x$, $t = T - \tau$ and $V(S,v,t) = Eu(x,y,\tau)$. Using such transformation PDE (1) takes the form

$$\frac{\partial u}{\partial \tau} = \big(r - D - \frac{1}{2}y\big)\frac{\partial u}{\partial x} + \frac{1}{2}y\frac{\partial^2 u}{\partial x^2} + \rho\sigma y\frac{\partial^2 u}{\partial x \partial y}$$
$$+ \frac{1}{2}y\sigma^2\frac{\partial^2 u}{\partial y^2} + (\kappa(\theta - y) - \lambda y)\frac{\partial u}{\partial y} - ru\,. \tag{11}$$

In the same way we transform the expiry and boundary conditions, which get the form

$$u(x,y,0) = \max(e^x - 1, 0)\,, \tag{12}$$
$$u(x,y,\tau) = 0\,,\ x \to -\infty\,, \tag{13}$$
$$u(x,y,\tau) = e^{x-D\tau} - e^{-r\tau}\,,\ x \to \infty\,, \tag{14}$$
$$u(x,y,\tau) = \max(e^{x-D\tau} - e^{-r\tau}, 0)\,,\ y \to 0\,, \tag{15}$$
$$u(x,y,\tau) = \frac{c(Ee^x, y, \tau)}{E}\,,\ y \to \infty\,. \tag{16}$$

2 THE FINITE VOLUME METHOD

Before constructing the numerical scheme for PDE (11) we define the mesh $\mathbb{L}$ of the domain $\Omega \subset R^2$, which consists of a finite (control) volume p, such that $\overline{\Omega} = \bigcup_{p\in\mathbb{L}} p$.

Since we work with a rectangular domain and we usually construct a simple rectangular grid, we use mesh $\mathbb{L} = (p_{ij})_{i=1,\dots,N_x;j=1,\dots,N_y+1}$ of a domain $(X_L, X_R) \times (0, Y)$, satisfying the following assumptions. Let $N_x, N_y \in N$, $h_x > 0$, $h_y > 0$, $h_x, h_y \in R$ such that

$$\sum_{i=1}^{N_x} h_x = X_R - X_L\,,\ \frac{h_1}{2} + \sum_{i=2}^{N_y} h_y + \frac{h_{N_y+1}}{2} = Y\,,$$

where $h_1 = h_{N_y+1} = h_y$. Let

$$x_{\frac{1}{2}} = X_L\,,\quad x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + h_x\,,\ \text{for } i = 1,\dots,N_x,$$
$$y_{\frac{1}{2}} = \frac{-h_y}{2}\,,\ y_{j+\frac{1}{2}} = y_{j-\frac{1}{2}} + h_y\,,\ \text{for } j = 1,\dots,N_y+1\,,$$

such that

$$x_{N_x+\frac{1}{2}} = X_R\,,\quad y_{N_y+\frac{3}{2}} = Y + \frac{h_y}{2}\,.$$

Then

$$p_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\,.$$

Let $(x_i)_{i=1,\dots,N_x}$, $(y_j)_{j=1,\dots,N_y+1}$ be given by

$$x_i = x_{i-\frac{1}{2}} + \frac{1}{2}h_x \qquad \text{for } i = 1,\dots,N_x\,,$$
$$y_j = y_{j-\frac{1}{2}} + \frac{1}{2}h_y \qquad \text{for } j = 1,\dots,N_y+1\,.$$

and $k = \frac{T}{n}$ be a discrete time step, $\tau_n = nk$.

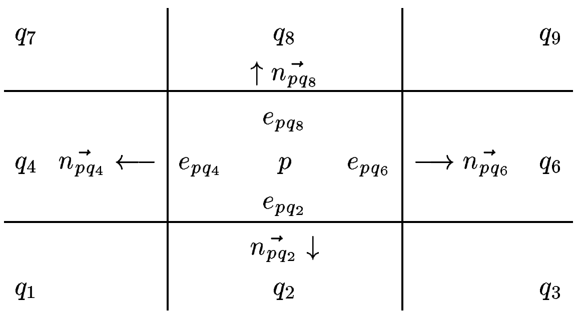

Fig. 1. The control volume p, set of its neighbours q_j, edges and normals e_{pq} and $\vec{n_{pq}}$.

2.1 Discretization of PDE

We can write equation (11) in the form

$$\frac{\partial u}{\partial \tau} + \vec{A} \cdot \nabla u = \nabla \cdot (B \nabla u) - ru\,, \tag{17}$$

where

$$B = \frac{1}{2} y \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}$$
$$\vec{A} = -\begin{pmatrix} r - D - \frac{1}{2}y - \frac{1}{2}\rho\sigma \\ \kappa(\theta - y) - \lambda y - \frac{1}{2}\sigma^2 \end{pmatrix} \tag{18}$$

represent the "diffusion matrix" and "velocity vector", respectively [1].

In order to define the finite volume numerical scheme we integrate equation (17) over each control volume $p \in \mathrm{L}$

$$\int_p \Big(\frac{\partial u}{\partial \tau} + \vec{A} \cdot \nabla u\Big) \mathrm{d}p = \int_p (\nabla \cdot (B \nabla u) - ru) \mathrm{d}p\,.$$

Let us denote by u_p^n the expected approximation of the solution u at time τ_n in the control volume p. This value can be viewed as an approximation of the mean value of u over volume p or as an approximation of the value $u(x_i, y_j, \tau_n)$ [1]. Let (x_p, y_p) be the coordinates of the center of the control volume p. Let e_{pq} denote the common boundary of the control volume p and q. Let (x_{pq}, y_{pq}) be the coordinates of the point of intersection of e_{pq} and the line connecting (x_p, y_p) and (x_q, y_q).

Now by backward Euler discretization for time derivative and using relation

$$\nabla \cdot (\vec{A} u) = \vec{A} \cdot \nabla u + (\nabla \cdot \vec{A}) u\,,$$

for the convective term we can write

$$\frac{1}{k}(u_p^n - u_p^{n-1}) m(p) + \int_p \nabla \cdot (\vec{A} u^n) \mathrm{d}p$$
$$= \int_p \nabla \cdot (B \nabla u^n) \mathrm{d}p + \int_p (\nabla \cdot \vec{A} - r) u^n \mathrm{d}p\,, \tag{19}$$

where $m(p) = h_x h_y$ is an area of the control volume p. Using divergence theorem

$$\frac{m(p)}{k}(u_p^n - u_p^{n-1}) + \sum_{q \in N_p} \int_{e_{pq}} \vec{A} \cdot \vec{n_{pq}} u^n \mathrm{d}\gamma$$
$$= \sum_{q \in N_p} \int_{e_{pq}} B \nabla u^n \cdot \vec{n_{pq}} \mathrm{d}\gamma + (\nabla \cdot \vec{A} - r) m(p) u_p^n\,, \tag{20}$$

where $\vec{n_{pq}}$ is a unit normal vector to e_{pq}, outward to p and N_p is a set of all neighbour volumes of the control volume p. In our case $N_p = \{q_1, \ldots, q_4, q_6, \ldots, q_9\}$ (see Fig. 1). Then we can write equation (20) in the form

$$\frac{m(p)}{k}(u_p^n - u_p^{n-1}) + \sum_{q \in N_p} Q_{pq} u_{p+}^n$$
$$= \sum_{q \in N_q} \int_{e_{pq}} B \nabla u^n \cdot \vec{n_{pq}} d\gamma + (\nabla \cdot \vec{A} - r) m(p) u_p^n\,. \tag{21}$$

where $Q_{pq} = \int_{e_{pq}} \vec{A} \cdot \vec{n_{pq}} \mathrm{d}\gamma$ and upwind technique is used to define

$$u_{p+}^n = \begin{cases} u_p^n & \text{for } Q_{pq} \geq 0\,, \\ u_q^n & \text{for } Q_{pq} < 0\,. \end{cases} \tag{22}$$

Since in our case $(\nabla \cdot \vec{A} - r) = \kappa + \lambda - r$ is a given constant, there it remains to approximate the term

$$\sum_{q \in N_P} \int_{e_{pq}} B \nabla u^n \cdot \vec{n_{pq}} \mathrm{d}\gamma\,. \tag{23}$$

We can approximate

$$S_2 := \int_{e_{pq_2}} B \nabla u^n \cdot \vec{n_{pq_2}} \mathrm{d}\gamma \approx \frac{1}{2} y_{pq_2} m(e_{pq_2})$$
$$\times \begin{pmatrix} \frac{\partial u^n}{\partial x}(x_{pq_2}, y_{pq_2}) + \rho\sigma \frac{\partial u^n}{\partial y}(x_{pq_2}, y_{pq_2}) \\ \rho\sigma \frac{\partial u^n}{\partial x}(x_{pq_2}, y_{pq_2}) + \sigma^2 \frac{\partial u^n}{\partial y}(x_{pq_2}, y_{pq_2}) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}^{\top},$$

where $m(e_{pq_2})$ is the length of the edge e_{pq_2}. Then

$$S_2 \approx -\frac{1}{2} y_{pq_2} h_x \Big(\rho\sigma \frac{\partial u^n}{\partial x}(x_{pq_2}, y_{pq_2})$$
$$+ \sigma^2 \frac{\partial u^n}{\partial y}(x_{pq_2}, y_{pq_2}) \Big)\,. \tag{24}$$

For the approximation of the partial derivative with respect to y we use

$$\frac{\partial u^n}{\partial y}(x_{pq_2}, y_{pq_2}) \approx \frac{u_p^n - u_{q_2}^n}{h_y}\,. \tag{25}$$

For the approximation of the partial derivative with respect to x we use

$$\frac{\partial u^n}{\partial x}(x_{pq_2}, y_{pq_2}) \approx \frac{1}{2} \left(\frac{u_{q_6}^n - u_{q_4}^n}{2h_x} + \frac{u_{q_3}^n - u_{q_1}^n}{2h_x} \right)\,. \tag{26}$$

Using (25) and (26) in (24) we derive the approximation

$$S_2 \approx -\frac{1}{2} y_{pq_2} h_x \Big(\frac{1}{4h_x} \rho\sigma (u_{q_6}^n - u_{q_4}^n + u_{q_3}^n - u_{q_1}^n)$$
$$+ \frac{1}{h_y} \sigma^2 (u_p^n - u_{q_2}^n) \Big)\,. \tag{27}$$

The remaining terms of the sum (23) can be derived in a similar way.

2.2 Discretization of initial and boundary conditions

Now we introduce a simple indexing of the control volumes in the admissible mesh Ł of the domain $(X_L, X_R) \times (0, Y)$. Let $u^n_{i,j}$, $i = 1, \dots, N_x$, $j = 1, \dots, N_y + 1$ be the approximation of the solution corresponding to u^n_p with volume center $(x_i, y_j) \in p$. This volume will be also denoted by p_{ij}.

Because the initial condition (12) does not depend on volatility and is not transformed in time, we can write it in the form

$$u^0_{i,j} = \max(e^{x_i} - 1, 0)\,, \qquad \text{for } i = 1, \dots, N_x\,,\ j = 1, \dots, N_y + 1\,. \tag{28}$$

In the same way we can write the approximation of Dirichlet boundary conditions (13) and (14)

$$u^n_{1,j} = 0\,, \tag{29}$$

$$u^n_{N_x,j} = e^{x_{N_x} - D\tau_n} - e^{-r\tau_n}\,, \tag{30}$$

for $j = 1, \dots, N_y + 1$ and conditions (15), (16)

$$u^n_{i,1} = \max(e^{x_i - D\tau_n} - e^{-r\tau_n}, 0)\,, \tag{31}$$

$$u^n_{i,N_y+1} = \frac{c(Ee^{x_i}, y_{N_y+1}, \tau_n)}{E}\,, \tag{32}$$

for $i = 1, \dots, N_x$ and c is given by (7).

2.3 The numerical scheme

Substitution (27) to (24) and calculating the remaining terms of the sum (23) gives for all neighbouring control volumes of the control volume $p \sim (i, j)$

$$\begin{aligned}
&- \frac{k}{8m(p)}\rho\sigma(y_{j-\frac12} + y_j)u^n_{i-1,j-1} - \frac{k}{2h_y^2}\sigma^2 y_{j-\frac12}u^n_{i,j-1} \\
&+ \frac{k}{8m(p)}\rho\sigma(y_{j-\frac12} + y_j)u^n_{i+1,j-1} - \Big(\frac{k}{8m(p)}\rho\sigma y_{j-\frac12} + \frac{k}{2h_x^2}y_j \\
&\qquad - \frac{k}{8m(p)}\rho\sigma y_{j+\frac12}\Big)u^n_{i-1,j} + \Big(1 + \frac{k}{h_x^2}y_j + \frac{k}{h_y^2}\sigma^2 y_j \\
&\qquad - k(\kappa + \lambda - r)\Big)u^n_{i,j} + \frac{k}{m(p)}\sum_{q \in N_{p_{ij}}} Q_{p_{ij}q}u^n_{p_{ij}+} \\
&- \Big(\frac{k}{8m(p)}\rho\sigma y_{j-\frac12} + \frac{k}{2h_x^2}y_j - \frac{k}{8m(p)}\rho\sigma y_{j+\frac12}\Big)u^n_{i+1,j} \\
&+ \frac{k}{8m(p)}\rho\sigma(y_j + y_{j+\frac12})u^n_{i-1,j+1} - \frac{k}{2h_y^2}\sigma^2 y_{j+\frac12}u^n_{i,j+1} \\
&\qquad - \frac{k}{8m(p)}\rho\sigma(y_j + y_{j+\frac12})u^n_{i+1,j+1} = u^{n-1}_{i,j}\,,
\end{aligned} \tag{33}$$

for $i = 2, \dots, N_x - 1$, and $j = 2, \dots, N_y$.

Let U^n be the vector of the approximate solution. Then the system of linear equations (33) together with the boundary conditions can be written in the matrix form

$$\mathbf{M}U^n = F\,, \tag{34}$$

which can be solved, e.g., by Gauss elimination efficiently.

3 DISCUSSION ON NUMERICAL EXPERIMENTS

3.1 The European call option

We present computational results for a two factor European call option model with parameters given in Table 1 (these data were taken from [2, 3]).

Table 1. Data used for computing values of the two factor European call option.

name	value
E — exercise price	60
r — the riskfree interest rate	0.05
τ — time to expiration	0.3
σ — the volatility of volatility	0.1
ρ — the coefficient of correlation	0.5, 0, -0.5
θ — the reversion level of the volatility	0.02
λ — the market price of the risk	0.0
κ — the mean reversion	2.0

In Table 2 we present computed values of the two factor European call option changing correlations between the evolution of the price and the volatility of the stock. In the first column there is the stock price S, in the second column we use three values of volatility v, the third column gives the values of the payoff diagram. In the 4th to 6th columns we have the computed values of $V(S, v, t)$ for three different coefficients of correlation and in the two last columns there are classical Black-Scholes values with fixed volatility $\delta = v$.

The differences between the explicit solution of the one factor model and our approximate solution of the two-factor model with stochastic volatility gives a reason for further research, for example, calibrating of the two factor model for a specific stock.

If we compare assumptions of the one factor Black-Scholes model and two-factor model with the stochastic volatility, namely stochastic differential equations which represent the evolution of the stock price S, we find a correspondence $\delta = \sqrt{v}$, where δ is a constant volatility of the Black-Scholes model and v is stochastic volatility of the two factor model. Then, if $\sigma \to 0$ and $\kappa \to \infty$, the values obtained by the two-factor model should converge to the values of the Black-Scholes model with $\delta = \sqrt{\theta}$. In Table 3 we present the computational results of the two-factor model at the stock price $S = 61.7649$, the volatility of the volatility $\sigma = 0.001$ and the reversion level of the volatility $\theta = 0.2$. As expected, with increasing speed of the reversion κ the values of the option converge to the explicit Black-Scholes solution for any level of volatility v.

Table 2. Values of the European call option depend on stock price S, stock volatility v and the coefficient of the correlation ρ.

S	v	payoff	European call option				
			$\rho=-0.5$	$\rho=0.$	$\rho=0.5$	$c(S,v,\tau)$	$c(S,\sqrt{v},\tau)$
55.64	0.1	0.	2.1805	2.1498	2.1996	0.2143	2.4651
61.77	0.1	1.77	5.3056	5.2250	5.2392	3.0537	5.6334
68.56	0.1	8.56	10.5112	10.4176	10.4028	9.4579	10.6494
55.64	0.2	0.	3.3755	3.4270	3.4971	1.1451	4.0288
61.77	0.2	1.77	6.6248	6.6488	6.6941	4.1787	7.3144
68.56	0.2	8.56	11.5161	11.5129	11.5291	9.7364	12.0493
55.64	0.3	0.	4.4146	4.4780	4.5476	2.2750	5.2461
61.77	0.3	1.77	7.7467	7.7884	7.8355	5.4270	8.6119
68.56	0.3	8.56	12.4993	12.5164	12.5388	10.4969	13.2419

Table 3. Values of the European call option obtained by two-factor model when the stock price $S = 61.7649$, volatility v, the volatility of the volatility $\sigma = 0.001$ with increasing speed of the reversion κ.

$S = 61.7649$ Explicit solution of B–S equation: 7.31444				
v	European call option			
	$\kappa=20$	$\kappa=50$	$\kappa=100$	$\kappa=1000$
0.05	6.71916	7.06955	7.19349	7.30255
0.10	6.88090	7.13334	7.22470	7.30561
0.15	7.06065	7.20648	7.26076	7.30916
0.20	7.31447	7.31448	7.31448	7.31448
0.25	7.55015	7.41841	7.36714	7.31980
0.30	7.70629	7.48694	7.40203	7.32334
0.35	7.84502	7.54799	7.43326	7.32652
0.40	7.97521	7.60558	7.46284	7.32955
0.50	8.22040	7.66106	7.49146	7.33538
0.60	8.45197	7.81977	7.61494	7.34106

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