Geometrical Diffusion in 3D-echocardiography

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Abstract

In the present paper we apply geometry driven diffusion equations to the processing of 2D and 3D echocardiographic images. In general, the blood - cardiac muscle interface is represented by a level surface of the image intensity function. This non-smooth image silhouette is moved in the direction of its inner normal vector field by the velocity proportional to its mean curvature. Such motion leads to a reasonable surface smoothing and thus to extraction of ventricular shape. We imbed the initial echocardiographic image to the 'geometrical scale space' by using the level set equation

$$u_t = |\nabla u| div(\frac{\nabla u}{|\nabla u|})$$

corresponding to the mean curvature flow of the isosurface. Evans-Spruck regularization of the level set equation is solved numerically by a kind of Rothe's method in time-scale combined with the finite element method for the space discretization.

1 Introduction

The aim of this paper is to present a robust technique for the processing of 3D (and 2D as well) echocardiographic images. The main goal is to extract a smooth shape of the left ventricle in a close as well as in an open phase from 3D greylevel image.

We use the ideas of nonlinear-geometrical scale space theory (see e.g. [29]). Thus, an image is considered as a real valued function $u_0(x)$, defined in a rectangular subdomain of R^N (N = 2 or 3), which represents the values of greylevel intensity function given in pixel/voxel structure of the image. The *scaling* (or *multiscale analysis*) then associates with $u(x, 0) = u_0(x)$ a 'sequence' of (simplified) images $u(x, t)$ depending on the abstract parameter $t > 0$, the scale. Then, the scaling of the initial image can be understood as its 'running' through the sequence of physical (biological, computer) filters, what causes the extracting of some relevant (e.g. for human perception or some further artificial intelligence operation) information from the recorded signal. Under some reasonable assumptions (see [1]), the family of nonlinear operators representing the scaling is given as a function $u(x, t)$ – the solution of a nonlinear partial differential equation of (degenerate) parabolic type. Such idea is so strong and usefull that one can provide the majority of classical as well as new image processing operations

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(filtering, restoration, enhancement, edge detection, segmentation, morphology etc.) by solving the evolutionary partial differential equations. This approach yields also the possibility to apply robust numerical techniques (finite element, finite volume or finite difference methods for space discretization and explicit, semiimlicit or implicit approximations in time-scale) to image analysis. One such access is presented in this paper. We solve numerically the regularization of the so called level set equation proposed by Osher and Sethian for describing the motion of curves and surfaces governed by the curvature. The motivation for the usage of level set equation for our problem of ventricular shape extraction is described in Section 3. In Section 4 we present the proposed numerical method and disscuss the computed results from mathematical and biomedical point of view. Our method of echocardiography segmentation is very general, successfull in both open and close ventricular phase, and thus gives the really new contribution to biomedical image analysis.

Finally, let us note that the term 'geometrical' in the title of the paper and in the geometrical scale space theory, too, means that, the image analysis would strongly respect the geometrical information contained in the image greylevel intensity function. Hence, the transparent work with the edges or silhouettes in the image (their motion, enhancement or smoothing) is highly desirable. The huge research is devoted to these questions in mathematical as well as computer vision and biomedical literature in the last decade and this paper also deals with an important application of those ideas.

2 State of the Art in 3D Bioimages Segmentation

Many techniques have been proposed in the literature to extract the ventricular surface in conditions of systole end/or diastole (thus the ventricle closed) starting from a small number of 2D images that represent different views of the ventricle. In this case the first step is to analyze the 2D images and identify manually, semi-automatically or completely automatically the edges of the ventricle using the techniques of edge detection and segmentation of 2D images. In order to reconstruct the surface in 3D a priori knowledge of the shape (that regularizes the problem) is then applied.

Azhari et al. propose a method for reconstructing the ventricular chamber from a scattered set of 2D echocardiographies by introducing a system of helicoidal coordinates to transform the data of the sections in the space into a single onedimensional function that is analyzed by means of Fourier techniques ([5]). In this way an analytical descriptive model of the three-dimensional geometry is defined, that can be used for spectral analysis and evaluation of the geometrical resemblance of the three-dimensional forms.

Other authors introduce a priori knowledge of the ventricular shape considering it as radially symmetric surface. Weaker constrain can be imposed by making the assumption of a vaguely ellipsoidal closed surface; the use of geometrical primitives such as superquadrics has been proposed in [6]. These functions can be seen as a generalization of traditional ellipsoids, that can be deformed by means of 6 parameters. This kind of representation offers a good overall view of the ventricular shape but neglects the details of the anatomical structure. To overcome the problem Terzopulos et al. introduced the possibility of locally deforming superquadrics ([32]). Finally, two kinds of deformation were unified in a single model that enables both a global control to roughly describe the shape, and a local one to represent the anatomical details correctly ([33], [26]).

Another way of representing the ventricular surface is based on the usage of orthogonal bases e.g. the spherical harmonics. A generalization of the decomposition of the function representing the ventricular surface in the orthogonal bases can be found in Galerkin's method for solving differential problems, in which the surface is subdivided into a mosaic of simple elements whose deformation can be expressed in analytical form ([27]).

Coppini et al. ([11]) consider the problem of reconstructing the ventricular chamber from the analysis of a set of 2D echocardiographies, making use of back-propagation neural networks. Moreover, an interesting regularization constrain is introduced for reconstructing the surface in 3D. On the basis of physical hypotheses, the ventricular surface is considered to be a thin, closed elastic surface, on which forces of an elastical kind are exerted by means of the edges. The ventricular surface is therefore identified with the solution of the forces balance problems or with the minimizer of the energy functional asssociated with the system.

The reconstruction of the ventricular chamber has been widely studied in recent years using TAC and Magnetic Resonance Images, too. In this case, some of the proposed methods provide a segmentation of 2D images manually or semi-automatically and then reconstruct the ventricular chamber by connecting the contiguous edges in the third dimension. Cohen introduces in ([12]) the concept of active deformable edges for the segmentation of the ventricles in 2D echocardiographic and 3D Magnetic Resonance images. His approach is a generalization of Kass, Witkin and Terzopulos deformable elastic edges model ([19]) in which an internal pressure is added to help the deformable contours to reach the interfaces bettween the segments.

A class of innovative and recent methods for the segmentation use the real valued evolving function through the 3D-image domain. Then, by selecting of a particular level of this function, (an implicit) level surface that segment the volume is obtained without the introduction of a priori knowledge of the topology. We can mention the work of Velho and Terzopulos ([34]) and Malladi at al. ([20]) based on this idea. The last one uses the level set approach of Osher and Sethian combined with deformable contour model of Kass, Witkin and Terzopulos, and the (implicit) level surfaces of the so called distance function are propagated with local speed influenced by both the curvature of the surface and the proximity of the edges to be detected.

These techniques, as well as other procedures of nonlinear image multiscale analysis, represented e.g. by the anisotropic diffusion equation of Perona and Malik ([28]), can be included in the general axiomatic framework given by Alvarez, Guichard, Lions and Morel ([1], [2]) mentioned also in the introduction. They are extremely valid, do not introduce topological assumptions on the resulting shape; therefore, they can be used for the segmentation of the blood-cardiac muscle interface in conditions of either open or closed ventricle, providing a good global representation and maintaining a good recognition of the details at the same time.

3 Geometrical Diffusion

For the motivation, let us consider the images collected in Figures 1 and 2. They are related to our testing data set. We have fourteen cubes (consisting of 256^3 voxels) representing echocardiographic images of the left heart ventricle in a different moments of cardiac cycle from systole to diastole phases. In Figure 1, the 2D cuts of the cube (intersecting in its center) for diastole are plotted.

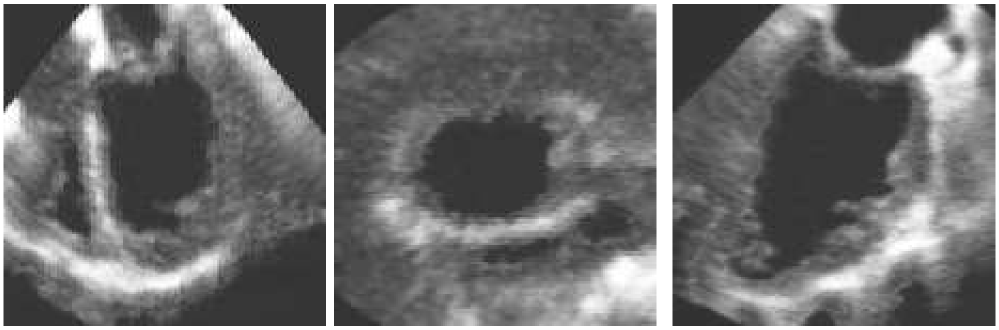


Figure 1.

In order to extract the ventricular shape and hence segment the 3D image, first we try to use 3D anisotropic diffusion model of Perona and Malik ([28], [9]) in an efficient numerical implementation based on [18]. The corresponding 2D cuts, results of such image processing operation, are plotted in Figure 2.

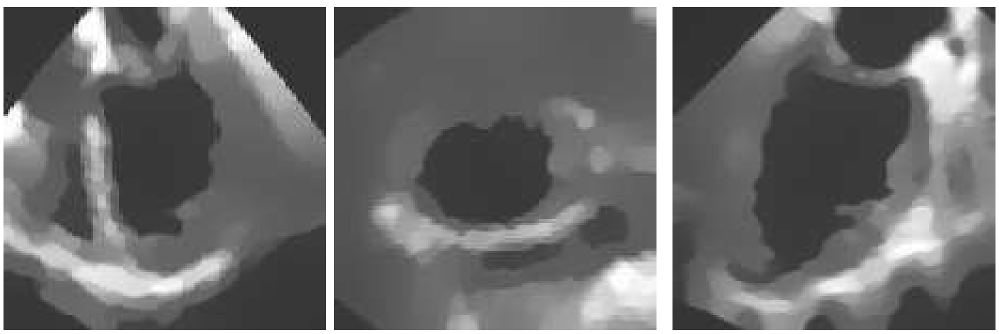


Figure 2.

Although, there is a large success in the edge enhancement and noise filtering, one can hardly use these results for the description of real *smooth* shape of the ventricle. The method conserves the nonrealistic fingers, incisions and peaks, origin of which is in the aquisition.

In rather general situations, the blood - cardiac muscle interface corresponds to an isosurface (isoline in 2D) of the greylevel image intensity function and hence it forms recognizable silhouette in the image. This phenomenon is rather transparently visible in Figures 1 and 2 as well. To remove this silhouette smoothless (from the original image or prefiltered by anisotropic diffusion image) it seems to be reasonable moving of such surface (curve) in the direction of its normal vector field with the velocity proportional to the mean curvature. The motions of convex and concave pieces are opposite due to the curvature sign, and the large fingers shrink much more faster than smoother parts due to the curvature dependence of the flow. Thus, locally in time, we can obtain a reasonable smoothing of the silhouette.

Following these intuitive considerations we use for the extraction of ventricular shapes from echocardiographies the so called curvature driven evolution of curves and surfaces leading to the corresponding geometrical diffusion equations. Let us give the mathematical formalization of the ideas described above (see also e.g. [16], [25]). We restrict ourselves to 2D case, the generalization to more dimensions is analogic and straightforward. Let $x(p,t) : S^1 \times [0,T] \to \mathbb{R}^2$ be the family of curves evolving in the plane by the equation

$$\frac{dx}{dt} = \beta(k)n, \tag{1}$$

where S^1 is the unit circle, k the curvature of the curve at time t at point $x(p,t)$, n the unit normal at that point and β is a nondecreasing real function. Without lost of generality we can assume the closedness and such smoothness of the curve, that all terms in (1) are well defined. The curve evolution equations of type (1) are used to describe various phenomena in physics, material sciences, computer vision, robotics and artificial intelligence from which came our motivation to use them also for echocardiography. In the vision theory, morphological shape analysis and especially the *affine invariant scale space* has special conceptual and practical importance ([1], [30]). It is natural generalization of the linear curve shortening flow, and is given by (1) with $\beta(k) = k^{1/3}$. The *active contour* models (*snakes*), related to edge detection, image segmentation and recognition, is another important field in which the geometrical equations (1) are widely used ([19]). In the context of multiphase thermomechanics with interfacial structure (free boundary problems) the plane curve evolution is a natural model for the *motion of phase interfaces.* The theory of Angenent and Gurtin ([4]) has also the form of equation (1).

There are two main approaches to solve the curvature driven evolution problems. First, the so called 'Lagrangean approach' consider the moving curve (or surface) itself as the main object of modelling and computing ([22], [21], [24], [14], [15]). In some situations, it is very efficient and computationaly fast method, but it can hardly handle the evolution through singularities (splitting and merging of the curves or surfaces, respectively, during the evolution). In spite of this, the 'Eulerian approach' handles implicitly the curvature driven motion passing the problem to the higher dimensional space. The *level set methods* of Osher and Sethian and *phase field models* ([8]) are the approaches of this type. So, we look for the function $u : \mathbb{R}^2 \to \mathbb{R}$, for which the moving curve is the same level line, at each time moment t. So we want to find the function u for which

$$u(x(t),t) = a \tag{2}$$

for every $t \in I$ and certain $a \in \mathbb{R}$. Let us differentiate the previous equality in time. We obtain

$$\frac{du(x(t),t)}{dt} = \nabla u . \frac{dx}{dt} + \frac{\partial u}{\partial t} = 0. \tag{3}$$

Considering the evolution equation (1) and the relation

$$n = -\frac{\nabla u}{|\nabla u|} \tag{4}$$

we obtain the following Hamilton-Jacobi partial differential equation

$$\frac{\partial u}{\partial t} = \beta(k)|\nabla u| \tag{5}$$

for the unknown function u. Due to the relation

$$k = div\left(\frac{\nabla u}{|\nabla u|}\right) \tag{6}$$

which holds for the curvature k of the isoline of u passing through point x, we obtain the *level set equation*

$$\frac{\partial u}{\partial t} = |\nabla u| \beta(div(\frac{\nabla u}{|\nabla u|})). \tag{7}$$

This equation moves all level lines of u by the curve evolution equation (1). Up to a constant and with $u : \mathbb{R}^3 \to \mathbb{R}$, this equation holds true also for the evolutionary isosurfaces. Thus, if the function u_0 represents the greylevel intensity function of processed image then the application of evolutionary equation (7) to u_0 as initial function leads to fulfiling of our intuitive requirement of reasonable silhouetess smoothing. So, as our geometric diffusion model we consider the equation (7) accompanied by zero Neumann boundary conditions and initial condition given by processed echocardiographic image. In this paper, we use the linear shape of the function β, namely

$$\beta(k) = k \quad or \quad \beta(k) = c + \delta k \tag{8}$$

with c and δ the real constants. However, the different - nonlinear shapes of β can have a large significance in considered image analysis, but it leads to more complicated numerical approximations.

In the end of this section, let us mention the special type of diffusion expressed by equation (7). If we fixed the coordinate system at point $x \in \mathbb{R}^2$ in the way that ξ - the first coordinate vector is identical with the tangent of the level line of u passing through x, and η orthogonal to ξ corresponds to its normal, then the differential operator $|\nabla u| div(\frac{\nabla u}{|\nabla u|}$ can be rewritten as $\frac{\partial^2 u}{\partial \xi^2}$. It means that the equation (7) diffuses the solution in the direction perpendicular to the silhouette and that there is no diffusion in the direction orthogonal to the level line. Thus, the equation (7) has the hyperbolic character in the directions of η-s - i.e. it is a degenerate parabolic, so its solution has to be considered in the viscosity sense of [13].

4 Computational method and disscusion on the results

As we have mentioned at the end of the previous section, the *level set equation* (7) is degenerate parabolic and hence complicated from the numerical point of view. Its viscosity solution ([10], [13], [16]) can be tracked numerically e.g. by special techniques based on a solution of Hamilton-Jacobi equation (5) ([25], [31]). We follow a totally different numerical approach. The motivation is to use standard numerical methods for solving parabolic PDEs, namely a finite element method for discretization in space and a kind of implicit (Rothe's) method in time. We solve a parabolic problem (in nondivergence form, however) which is close to the basic equation (7). For this purpose we use a special regularization depending on a small parameter ε used by Evans & Spruck in the proof of existence of a weak solution of *generalized mean curvature flow* ([16]). Their regularization is interpreted as a motion of a graph, with a slope proportional to $1/\varepsilon$, which is thus close to a cylinder with basis given by moving curve or surface. From [16], it is guaranteed that, for $\varepsilon \to 0$, solutions of the regularized problems tend to the viscosity solution of the *level set equation.*

We therefore solve numericaly the following initial - boundary value problem

$$\frac{1}{\sqrt{\varepsilon + |\nabla u|^2}} u_t - \beta(div(\frac{\nabla u}{\sqrt{\varepsilon + |\nabla u|^2}})) = 0 \quad \text{in} \quad I \times \Omega, \tag{9}$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega, \tag{10}$$

$$u(0,.) = u_0 \quad \text{in } \Omega, \tag{11}$$

where $1 > \varepsilon > 0$ is a (small) real number, $I = (0,T)$ is time-scale interval and $\Omega \subset \mathbb{R}^N$ (N=2,3).

In the linear cases (8) we propose to discretize the equation (9) by means of finite element method in space and use a kind of (semiimplicit) Rothe's method in time-scale treating the nonlinearities from the previous time step ([18], [7]). So, we choose discrete time-scale step τ and in each discrete time-scale moment $t_i = i\tau$ we solve the weak integral identity

$$(12) \qquad \int_\Omega \frac{(u_h^i - u_h^{i-1})\varphi_h}{\sqrt{\varepsilon + |\nabla u_h^{i-1}|^2}} + \tau \int_\Omega \frac{\nabla u_h^i . \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h^{i-1}|^2}} = 0, \ \forall \varphi_h \in X_h$$

for the unknown function $u_h^i \in X_h$ where X_h is suitable finite element space with grid size parameter h. The computational grid is given naturally by the pixel/voxel structure of the initial image. By choosing appropriate finite elements (e.g. linear) and using standard techniques, the previous integral identity reduces to the solving of linear systems of algebraic equations which can be done in many ways (e.g. using the iterative tridiagonal matrix solvers in order to save the memory in our very huge 3D problems). In particular cases, the convergence of such numerical scheme is studied in [17].

Applying algorithm (12) we try to obtain a realistic - smooth shape of the left heart ventricle. In Figures 3 - 7 we visualize the level surfaces which represent the boundary of the volume containing the blood in several discrete moments of cardiac cycle from our testing data set (there are documented 1st step - systole, 5th, 7th, 9th and 14th step - diastole). On the left sides of the figures, the unfiltered isosurfaces are plotted. The computational results are plotted on right sides of Figures 3 - 7. In the presented numerical experiments we use $\tau = 10^4$, $\varepsilon = 10^{-6}$, $h = 1/256$ and we have computed 21 time-scale steps on Cray C92.

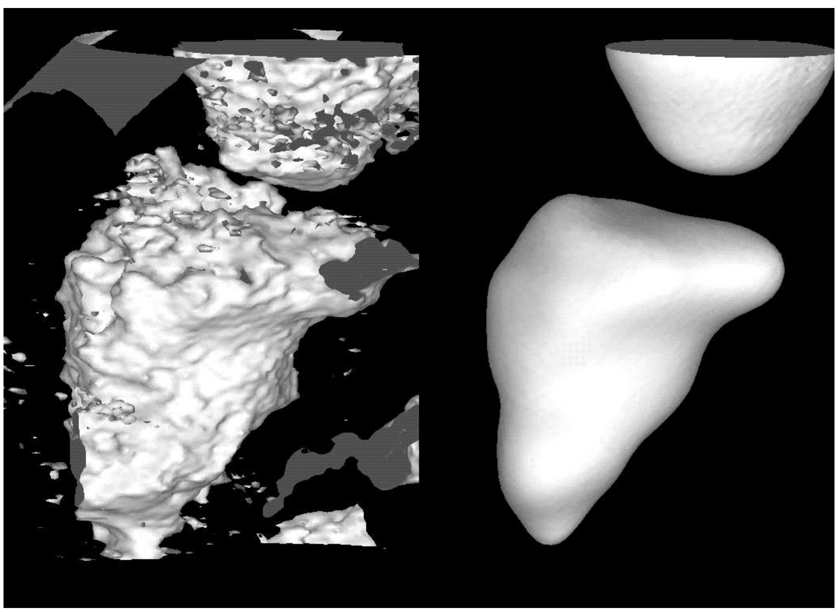


Figure 3.

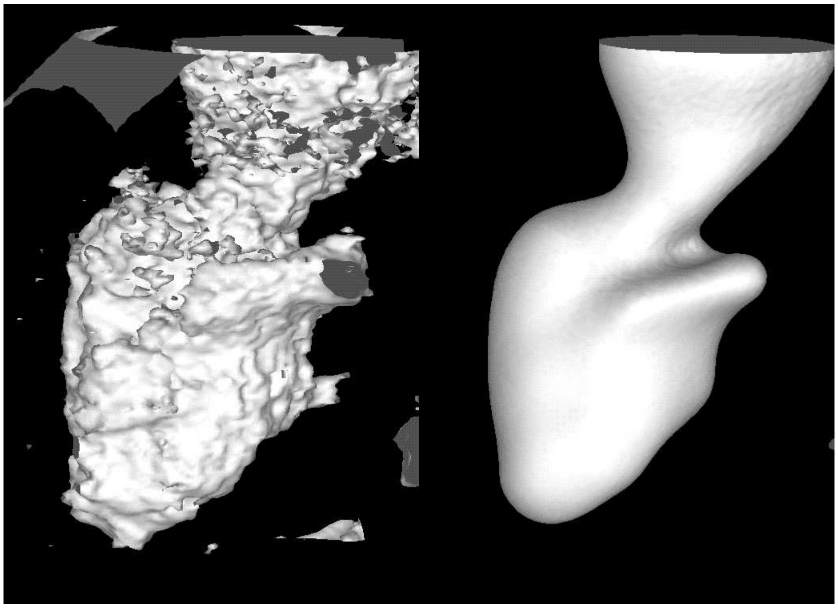


Figure 4.

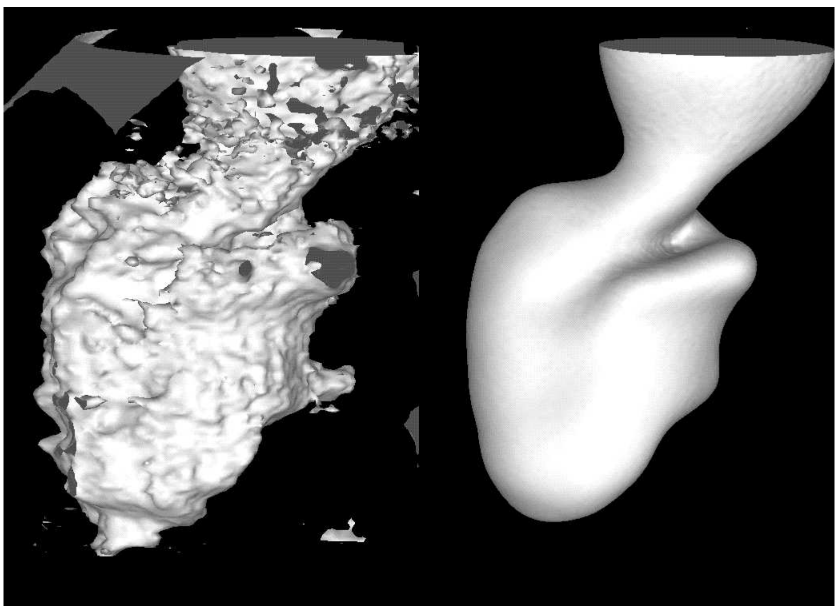


Figure 5.

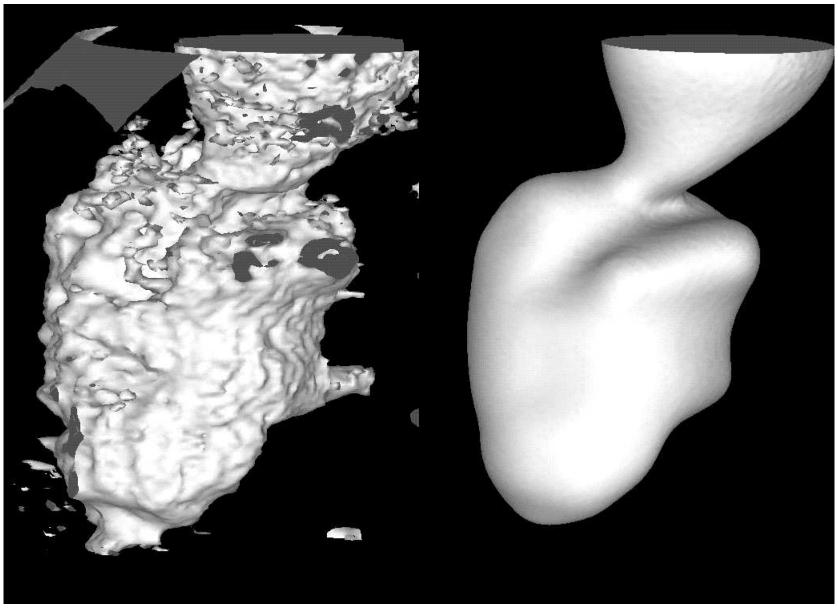


Figure 6.

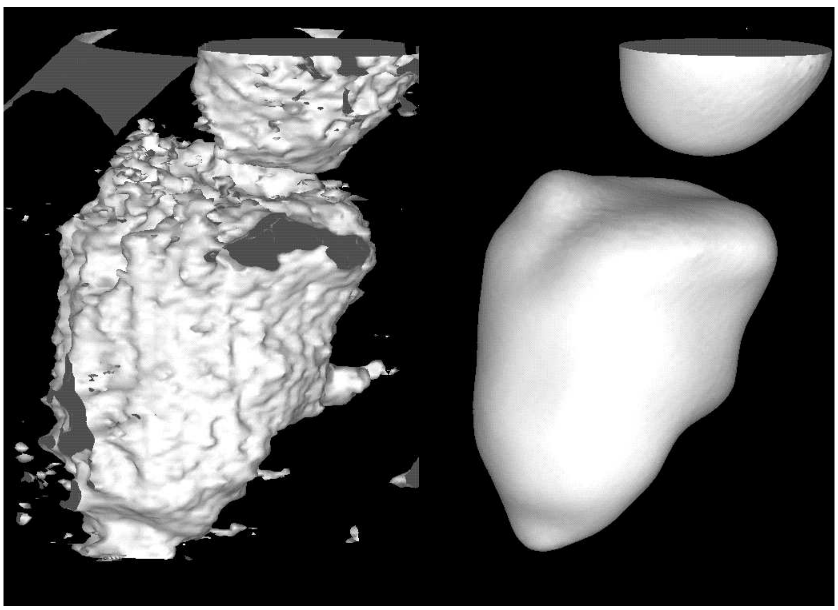


Figure 7.

In Figure 8 we present the smoothing effect of geometrical diffusion (7). In the left sides the cuts of unfiltered isosurfaces are plotted in the right the filtered ones. We can see the extinction of small structures due to the high curvature they have and remaining of the important information contained in the images.

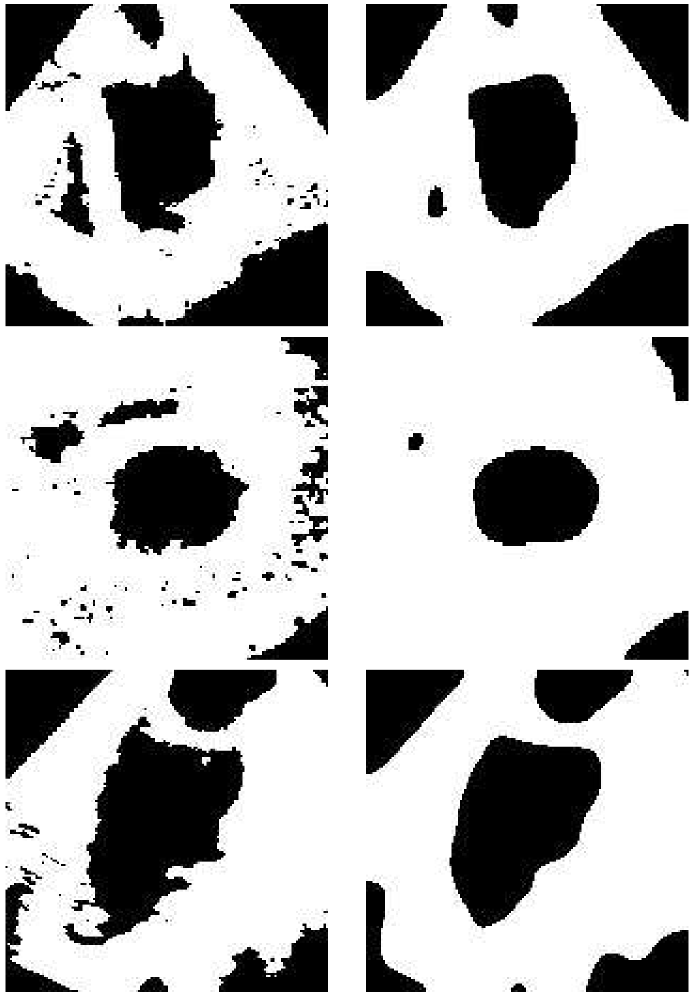
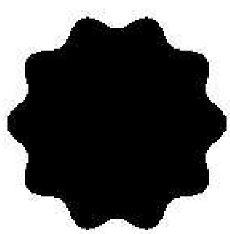


Figure 8.

However, there are many open questions e.g. the volume/area conservation as good as possible for as long time as possible. We use two phantoms, 2D and 3D, to study this phenomenon. During the scaling they converge to circle and sphere, respectively. The area/volume is conserved locally in time in good way, the errors in these quantities were less than 10 % in both cases until the circle/sphere like shape was obtained.



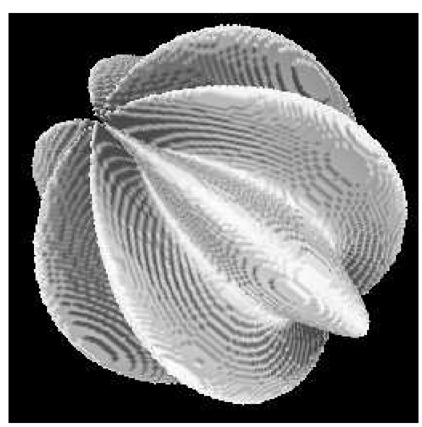


Figure 9.

Another difficult and interesting question is the problem of estimating the ammount of the blood contained in the ventricle in the open phases of cardiac cycle. The ventricle is connected with the atrium and it is difficult to divide this to objects in some reasonable and automatic way. One can use the operations of mathematical morphology, which can be included in the framework of partial differential equation (7). We consider $\beta(k) = \pm 1 + \delta k$, where δ is a small constant representing the addition of artifficial diffusion to the modeling of motion of isosurface in the normal direction by constant velocity. Equation (7) with $\beta(k) = 1$ provides the erosion of black shape on white background and its dilation (expansion) in the case $\beta(k) = -1$. We take the 2D image from Figure 10, then we make presmoothing by anisotropic diffusion combined with geometrical diffusion ($\beta(k) = k$) for very short time-scale to smooth the silhouette. Then we take the black and white image of the isoline and apply the erosion (Figure 11. left) and dilation (Figure 11. right) for the same time. We receive the final image (Figure 11. right) which in very good way correspond to the shape of 'closing' ventricle.

Figure 10.

Figure 11.

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