Inflow-Implicit/Outflow-Explicit Scheme for Solving Advection Equations

Karol Mikula and Mario Ohlberger

Abstract We present new method for solving non-stationary advection equations based on the finite volume space discretization and the semi-implicit discretization in time. Its basic idea is that outflow from a cell is treated explicitly while inflow is treated implicitly. Since the matrix of the system in this new I^2OE method is determined by the inflow fluxes it is an M-matrix yielding favourable solvability and stability properties. The method allows large time steps at a fixed spatial grid without losing stability and not deteriorating precision which makes it attractive for practical applications. Our new method is exact for any choice of a discrete time step on uniform rectangular grids in the case of constant velocity transport of quadratic functions in any dimension. We show that it is formally second order accurate in space and time for 1D advection problems with variable velocity and numerical experiments indicates its second order accuracy for smooth solutions in general.

Key words: advection equation, semi-implicit scheme, finite volume method
MSC2010: 35L04, 65M08, 65M12

1 Introduction

In this paper we present the inflow-implicit/outflow-explicit (I^2OE) method for solving variable velocity advection equations of the form

$$u_t + \mathbf{v} \cdot \nabla u = 0 \tag{1}$$

Karol Mikula
Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 81368 Bratislava, Slovakia, e-mail: mikula@math.sk

Mario Ohlberger
Institut für Numerische und Angewandte Mathematik, Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany e-mail: mario.ohlberger@uni-muenster.de

where $u \in \mathbb{R}^d \times [0,T]$ is the unknown function and $\mathbf{v}(x)$ is a vector field. The basic idea of our new method is that outflow from a cell is treated explicitly while inflow is treated implicitly. Such an approach is natural, since we know what is flowing out from a cell at an old time step $n-1$ but we leave the method to resolve a system of equations determined by the inflows to obtain a new value in the cell at time step n. Since the matrix of the system is determined by the inflow fluxes it is an M-matrix for Voronoi like grids and thus it has favourable discrete minimum-maximum properties. Consequently, the method allows large time steps at a fixed spatial grid without losing stability. Interestingly, the new I^{2}OE scheme is exact on rectangular grids for constant velocity transport of quadratic polynomials in any dimension and for any length of a time step. In general, it is second order accurate for smooth solutions, both for variable velocity and nonlinear advection problems [5]. A comparison with the second order Lax-Wendroff method for variable velocity shows good properties of the new scheme with respect to precision and CPU times. In [5], the I^{2}OE method was introduced in more general settings where $\mathbf{v} = \mathbf{v}(x,u,\nabla u)$. The semi-implicit forward-backward diffusion level set approach for motion in normal direction [4] is its special case. The variable and nonlinear velocity fields to which our method can be successfully applied arise in many applications, e.g. in level set methods and other transports with non-divergence free velocities and nonlinear conservation laws or in image segmentation by the active contours.

2 The inflow-implicit/outflow-explicit scheme

Let us consider equation (1) in a bounded polygonal domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, and time interval $[0,T]$. Let $\mathscr{Q}_h$ denote a primal polygonal partition of Ω. Let p be a finite volume (cell) of a corresponding dual Voronoi tessellation $\mathscr{T}_h$ with measure m_p and let e_{pq} be an edge between p and q, $q \in N(p)$, where $N(p)$ is a set of neighbouring finite volumes (i.e. $\bar{p} \cap \bar{q}$ has nonzero $(d-1)$-dimensional measure). Let c_{pq} be the length of e_{pq} and n_{pq} be the unit outer normal vector to e_{pq} with respect to p. We shall consider $\mathscr{T}_h$ to be an admissible mesh in the sense of [1], i.e., there exists a representative point x_p in the interior of every finite volume p such that the joining line between x_p and x_q, $q \in N(p)$, is orthogonal to e_{pq}. We denote by x_{pq} the intersection of this line segment with the edge e_{pq}. The length of this line segment is denoted by d_{pq}, i.e. $d_{pq} := |x_q - x_p|$. As we have build $\mathscr{T}_h$ based on the primal mesh $\mathscr{Q}_h$, we assume that the points x_p coincide with the vertices of $\mathscr{Q}_h$. Let us denote by u_p a (constant) value of the solution in a finite volume p computed by the scheme. For the solution representation inside the finite volume p we use either this value u_p or a reconstructed (but again constant) value denoted by $\overline{u}_p$. A constant value of the solution assigned to the edge e_{pq} (given again by a reconstruction) is denoted by $\overline{u}_{pq}$. Let us rewrite (1) in the formally equivalent form with conserving and non-conserving parts [2]

$$u_t + \nabla \cdot (\mathbf{v}u) - u\nabla \cdot \mathbf{v} = 0. \tag{2}$$

Integrating (2) over a finite volume p then yields

$$\int_p u_t\, dx + \int_p \nabla\cdot(\mathbf{v}u)\, dx - \int_p u\nabla\cdot\mathbf{v}\, dx = 0.$$

Applying the divergence theorem and using constant representations of the solution on the cell p, denoted by $\overline{u}_p$, and on the cell interfaces e_{pq}, denoted by $\overline{u}_{pq}$, we get

$$\int_p u_t\, dx + \sum_{q\in N(p)} \overline{u}_{pq}\int_{e_{pq}} \mathbf{v}\cdot n_{pq}\, ds - \overline{u}_p \sum_{q\in N(p)} \int_{e_{pq}} \mathbf{v}\cdot n_{pq}\, ds = 0.$$

If we denote the fluxes in the inward normal direction to the finite volume p by

$$\bar{v}_{pq} = -\int_{e_{pq}} \mathbf{v}\cdot n_{pq}\, ds, \tag{3}$$

we finally arrive at the equation

$$\int_p u_t\, dx + \sum_{q\in N(p)} \bar{v}_{pq}(\overline{u}_p - \overline{u}_{pq}) = 0. \tag{4}$$

The novelty of our scheme is to split the resulting fluxes into the corresponding inflow and outflow parts to the cell p. This is done by defining

$$a^{in}_{pq} = \max(\bar{v}_{pq}, 0), \quad a^{out}_{pq} = \min(\bar{v}_{pq}, 0). \tag{5}$$

We then approximate u_t by the time difference $\frac{u^n_p - u^{n-1}_p}{\tau}$, where τ is a uniform time step size, and take the inflow parts implicitly and the outflow parts explicitly in (4). This yields the following system of equations for the finite volume solution $u^n_p, p\in\mathscr{T}_h$ at the n-th discrete time step, representing the general I^{2}OE scheme:

$$m_p u^n_p + \tau \sum_{q\in N(p)} a^{in}_{pq}(\overline{u}^n_p - \overline{u}^n_{pq}) = m_p u^{n-1}_p - \tau \sum_{q\in N(p)} a^{out}_{pq}(\overline{u}^{n-1}_p - \overline{u}^{n-1}_{pq}). \tag{6}$$

The most natural choice for reconstructions $\overline{u}^n_p$ and $\overline{u}^n_{pq}$ at any time step n (i.e. old and new time steps) is given by $\overline{u}^n_p = u^n_p$, $\overline{u}^n_{pq} = \frac{1}{2}(u^n_p + u^n_q)$ and leads to the basic I^{2}OE scheme:

$$m_p u^n_p + \frac{\tau}{2} \sum_{q\in N(p)} a^{in}_{pq}(u^n_p - u^n_q) = m_p u^{n-1}_p - \frac{\tau}{2} \sum_{q\in N(p)} a^{out}_{pq}(u^{n-1}_p - u^{n-1}_q). \tag{7}$$

The equation (4) has the form of a discretization of a diffusion equation, where $\bar{v}_{pq}$ would represent the so-called transmissive coefficients (integrated diffusion fluxes divided by distances between cell centers). In standard forward diffusion all these coefficients are strictly positive which leads to a weighted averaging of the solution and the implicit schemes are natural in this case. On the other hand the negative coefficients would correspond to backward diffusion in which case information propagates outside the cell and explicit schemes are thus natural. In our case the sign of the coefficients is given by the inflow or outflow character of the cell boundary and the inflow-implicit/outflow-explicit approach is thus natural. It is also well-known that in the second order schemes for solving advection problems one can identify the "forward diffusion" part (like the first order upwinding) and the "backward diffusion" part given by the additional sharpening terms coming (sometimes surprisingly) from the second order Taylor's expansions, cf. the Lax-Wendroff scheme [3]. In our method this splitting arises naturally, gives second order accuracy and when treating it semi-implicitly it brings significant improvements in stability of computations.

Let us present the I^{2}OE scheme for 1D variable velocity equation $u_t + v(x)u_x = 0$, which will be used in numerical computations of Section 4. Let p_i be the cell with the spatial index i, length h, center point x_i, left border $x_{i-\frac{1}{2}}$ and right border $x_{i+\frac{1}{2}}$. Let us denote u_i^n the value of the numerical solution at time step n and $\overline{u}_i^n, \overline{u}_{i-\frac{1}{2}}^n$ the reconstructed values. We define

$$a_{i-\frac{1}{2}}^{in} = \max(v(x_{i-\frac{1}{2}}),0), \;\; a_{i-\frac{1}{2}}^{out} = \min(v(x_{i-\frac{1}{2}}),0),$$
$$a_{i+\frac{1}{2}}^{in} = \max(-v(x_{i+\frac{1}{2}}),0), \;\; a_{i+\frac{1}{2}}^{out} = \min(-v(x_{i+\frac{1}{2}}),0),$$

and if we use the reconstructions $\overline{u}_i^n = u_i^n$, $\overline{u}_{i-\frac{1}{2}}^n = \frac{1}{2}(u_i^n + u_{i-1}^n)$ in both new and old time steps, the basic one-dimensional I^{2}OE scheme has the following form

$$\begin{aligned} u_i^n &+ \frac{\tau}{2h} a_{i-\frac{1}{2}}^{in}(u_i^n - u_{i-1}^n) + \frac{\tau}{2h} a_{i+\frac{1}{2}}^{in}(u_i^n - u_{i+1}^n) = u_i^{n-1} \\ &- \frac{\tau}{2h} a_{i-\frac{1}{2}}^{out}(u_i^{n-1} - u_{i-1}^{n-1}) - \frac{\tau}{2h} a_{i+\frac{1}{2}}^{out}(u_i^{n-1} - u_{i+1}^{n-1}) . \end{aligned} \tag{8}$$

The scheme (8) requires to solve a tridiagonal system in every time step which is done by using the standard tridiagonal solver (also called the Thomas algorithm). In practice, the I^{2}OE scheme allows to use much larger time steps without losing L_∞-stability than given by a standard CFL condition for explicit schemes, cf. Section 3. However, the "backward diffusion" (outflow) explicit part is not necessarily always dominated by the implicit part in the basic form of the scheme (8). Some oscillations (not unboundedly growing in time) may arise e.g. on coarse grids or in solutions tending to a shock. One possibility is to leave the method with oscillations and remove them at the end of computations using e.g. some edge preserving filters. Another approach is to supress the oscillations during the computation. In our scheme, one can use an averaging (by a larger stencil) in the reconstruction of $\overline{u}_p^{n-1}$, similarly to the FBD schemes from [4], or to modify the "backward diffusion" part on the right hand side of (8) by using the standard limiters, for details see [5].

Theorem 1. *Let us consider the equation* (1) *in 1D with constant velocity v and* I^{2}OE *scheme* (8) *on uniform grid. If the initial condition is given by a second order polynomial, then the scheme gives the exact solution for any choice of time step.*

Proof. The initial condition has the form $u_0(x) = ax^2 + bx + c$ and the exact solution is given by $u(x,\tau) = u^0(x - v\tau)$. For $v > 0$ the scheme (8) takes the form

$$u_i^n + \frac{\tau v}{2h}(u_i^n - u_{i-1}^n) = u_i^{n-1} - \frac{\tau(-v)}{2h}(u_i^{n-1} - u_{i+1}^{n-1}) \tag{9}$$

One can easily check that if we plug the exact values in grid points x_i, x_{i-1}, x_{i+1} at time steps $n = 1$ and $n - 1 = 0$, namely

$$u_i^{n-1} = ax_i^2 + bx_i + c, \quad u_{i+1}^{n-1} = a(x_i + h)^2 + b(x_i + h) + c, \tag{10}$$
$$u_i^n = a(x_i - v\tau)^2 + b(x_i - v\tau) + c, \quad u_{i-1}^n = a(x_i - h - v\tau)^2 + b(x_i - h - v\tau) + c,$$

into the scheme (9), we get true identity, and the same we obtain for $v < 0$. □

It is also possible to make similar considerations as above in higher dimensional case for uniform rectangular grids and constant velocity vector field . One can plug a general 2D or 3D quadratic polynomial as initial condition and the corresponding exact solution at time τ into the I^{2}OE scheme (7), use a symbolic computational software like the Mathematica, and check that the scheme is exact in such situations.

Theorem 2. *Let us consider the equation* (1) *in 1D with variable velocity* $v(x) \geq 0$ *(or* $v(x) \leq 0$*) and the* I^{2}OE *scheme* (8) *on a uniform grid. Then the scheme is formally second order and the consistency error is of order* $\mathcal{O}(h^2) + \mathcal{O}(\tau h) + \mathcal{O}(\tau^2)$.

Proof. We write our transport equation as $\partial_t u + f(v, \partial_x u) = 0$ with $f(v, \partial_x u) := v(x)\partial_x u$ and let $v(x) \geq 0$. We will use notations $u^n := u(t^n)$, $f^n := f(v, \partial_t u^n)$. The Taylor expansion in time yields

$$u^n = u^{n-1} + \tau\partial_t u^{n-1} + \frac{\tau^2}{2}\partial_t^2 u^{n-1} + \mathcal{O}(\tau^3), \quad u^{n-1} = u^n - \tau\partial_t u^n + \frac{\tau^2}{2}\partial_t^2 u^n + \mathcal{O}(\tau^3).$$

Subtracting these two equations we derive relation

$$u^n - u^{n-1} = \frac{\tau}{2}(\partial_t u^n + \partial_t u^{n-1}) + \frac{\tau^2}{4}(\partial_t^2 u^{n-1} - \partial_t^2 u^n) + \mathcal{O}(\tau^3). \tag{11}$$

We can see that the second term on the right hand side is also $\mathcal{O}(\tau^3)$ and using the equation $\partial_t u + f(v, \partial_x u) = 0$, we get for the first term of the right hand side

$$I = \frac{\tau}{2}(\partial_t u^n + \partial_t u^{n-1}) = -\frac{\tau}{2}(f^n + f^{n-1}). \tag{12}$$

Using the notation $f_i := f(x_i) = v(x_i)\partial_x u(x_i)$, by the Taylor expansion in space we have (for $v(x) \geq 0$)

$$f_{i-1/2}^n = f_i^n - \frac{h}{2}\partial_x f_i^n + \mathcal{O}(h^2), \quad f_{i+1/2}^{n-1} = f_i^{n-1} + \frac{h}{2}\partial_x f_i^{n-1} + \mathcal{O}(h^2) \tag{13}$$

or (for $v(x) \leq 0$)

$$f_{i-1/2}^{n-1} = f_i^{n-1} - \frac{h}{2}\partial_x f_i^{n-1} + \mathscr{O}(h^2), \quad f_{i+1/2}^{n} = f_i^{n} + \frac{h}{2}\partial_x f_i^{n} + \mathscr{O}(h^2)\,. \tag{14}$$

We continue (for $v(x) \geq 0$) and using (12)-(13) we derive

$$I_i = -\frac{\tau}{2}(f_i^n + f_i^{n-1}) = -\frac{\tau}{2}\Big(f_{i-1/2}^{n} + f_{i+1/2}^{n-1} + \frac{h}{2}(\partial_x f_i^n - \partial_x f_i^{n-1}) + \mathscr{O}(h^2)\Big).$$

The second term in the brackets on the right hand side is of order $\mathscr{O}(\tau h)$ and we shall analyse the first one. We know that

$$\partial_x u_{i-1/2}^{n} = \frac{1}{h}(u_i^n - u_{i-1}^n) + \mathscr{O}(h^2), \quad \partial_x u_{i+1/2}^{n-1} = \frac{1}{h}(u_{i+1}^{n-1} - u_i^{n-1}) + \mathscr{O}(h^2)$$

and resubstituting for $f_{i-1/2}^{n} = v_{i-1/2}\partial_x u_{i-1/2}^{n}$ and $f_{i+1/2}^{n-1} = v_{i+1/2}\partial_x u_{i+1/2}^{n-1}$ we get

$$I_i = -\frac{\tau}{2}\Big(v_{i-1/2}\frac{1}{h}(u_i^n - u_{i-1}) + v_{i+1/2}\frac{1}{h}(u_{i+1}^{n-1} - u_i^{n-1})\Big) + \mathscr{O}(\tau^2 h) + \mathscr{O}(\tau h^2). \tag{15}$$

From (11) and (15) we finally get

$$\begin{aligned} u_i^n - u_i^{n-1} = &-\frac{\tau}{2}\Big(\frac{v_{i-1/2}}{h}(u_i^n - u_{i-1}^n) + \frac{v_{i+1/2}}{h}(u_{i+1}^{n-1} - u_i^{n-1})\Big) \\ &+ \mathscr{O}(\tau^2 h) + \mathscr{O}(\tau h^2) + \mathscr{O}(\tau^3) \end{aligned}$$

where we recognize the scheme (8) for $v(x) \geq 0$, cf. also (9), and dividing by τ we get the consistency error of the I^2OE scheme stated in the theorem. □

3 Numerical experiments

First, let us consider 1D equation (1) with $v(x) \equiv 1$ in interval $\Omega = (-1,1)$ and time interval $I = (0,T)$, $T = 1$. Let the initial condition u_0 be given by a quadratic polynomial $u_0(x) = 1 - \frac{1}{2}(x^2 - x)$. The exact solution is given $u(x,t) = u_0(x - vt)$. We solve this problem numerically using the exact Dirichlet boundary conditions and compare the results of the I^2OE method (8), the standard Lax-Wendroff and explicit up-wind schemes [3] with the exact solution. In all experiments we used increasing number n of finite volumes discretizing Ω, $h = 2/n$, and we consider various choices of time step τ and corresponding number of time steps NTS. In Table 1 we report the errors in $L_2(I, L_2)$ norm for all the methods. As one can see, the I^2OE method is exact for any relation between space and time step, see Theorem 1, and one can use extremely large (e.g. just one time step $\tau = T$) without any deterioration of the numerical result. Here the errors are comparable to machine precision, they are not exact zeros because we have to solve a tridiagonal system in every time step yielding some rounding errors which, however, do not propagate even in a long run. The Lax-Wendroff method, as the second order, is exact for any quadratic initial function whenever it is stable, i.e. $\tau \leq h$. For Courant numbers

Table 1 Report on the $L_2(I,L_2)$ errors of the I^2OE method, the Lax-Wendroff scheme, and the explicit up-wind scheme for the initial quadratic polynomial and for various choices of time step. We note that all the methods are exact for $\tau = h$.

n	$\tau = h/2$	NTS	I^2OE	Lax-Wendroff	Up-wind
20	0.05	20	$3.7\ 10^{-16}$	$5.1\ 10^{-17}$	$1.83\ 10^{-2}$
40	0.025	40	$8.0\ 10^{-16}$	$7.5\ 10^{-17}$	$8.99\ 10^{-3}$
80	0.0125	80	$1.1\ 10^{-15}$	$8.3\ 10^{-17}$	$4.45\ 10^{-3}$
160	0.00625	160	$2.4\ 10^{-15}$	$9.9\ 10^{-17}$	$2.22\ 10^{-3}$
n	$\tau = 2h$	NTS	I^2OE	Lax-Wendroff	Up-wind
20	0.2	5	$2.1\ 10^{-16}$	$1.1\ 10^{-11}$	$5.02\ 10^{-2}$
40	0.1	10	$2.1\ 10^{-16}$	$1.4\ 10^{-9}$	0.641
80	0.05	20	$3.9\ 10^{-16}$	0.466	$3.8\ 10^{+3}$
160	0.025	40	$5.7\ 10^{-16}$	$1.6\ 10^{+16}$	$1.3\ 10^{+12}$
160	$\tau = 10h = 0.125$	8	$2.5\ 10^{-15}$	–	–
160	$\tau = 40h = 0.5$	2	$1.7\ 10^{-15}$	–	–
160	$\tau = 80h = 1$	1	$2.6\ 10^{-15}$	–	–

larger than 1, one can see instabilities in the third and 4th rows of Table 1, when $\tau = 2h$ and grid is refined. The explicit upwind scheme is the first order and exact for any initial data only if the relation $\tau = h$ is fulfilled. Its first order accuracy can be seen for $\tau = h/2$, and oscillations occur soon for $\tau > h$ as documented in Table 1.

Next, let us consider an example with variable velocity field $v(x) = -\sin(x)$ and let the initial profile be given by $u_0(x) = \sin(x)$, $\Omega = (-1,1)$ and $I = (0,T)$, $T = 1$. The exact solution can be derived by the method of characteristics and is given as $u(x,t) = u_0(\frac{2}{\pi}\text{arctg}(e^{\pi t}\text{tg}(\frac{\pi x}{2})))$. We compare the precision and CPU-time of the I^2OE and the Lax-Wendroff scheme [3]. In the solutions a strong peak is formed at $T = 1$, see Figure 1. Both schemes are stable with slight overshoot and undershoot in the result by the Lax-Wendroff scheme on coarser grids. No overshoot or undershoot is observed for the I^2OE scheme, cf. Figure 1. Figure 2 shows log-log plots of CPU time versus error of the schemes. We can see superior behavior of the I^2OE scheme in this example with considerable speed-up when using larger time steps up to 4-8 times exceeding the CFL condition, which must be respected in the Lax-Wendroff scheme. In this case both schemes are second order accurate which holds true for any time step size of the I^2OE scheme.

Further 1D and 2D numerical experiments are reported in [5] showing the second order convergence of the I^2OE method for any choice of the time steps. This is the main advantage of the new scheme when comparing with standard explicit second order methods, or, when using limiters, in comparison with the so-called high resolution methods for solving advection equations.

Acknowledgements The work of the first author was supported by the grants APVV-0351-07 and VEGA 1/0269/09.

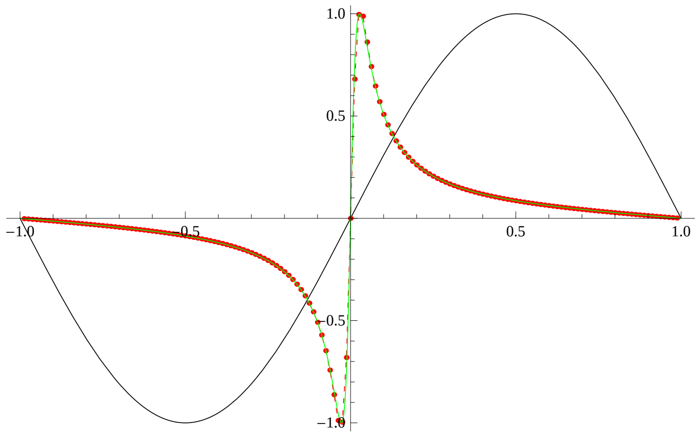

Fig. 1 The result of the I^2OE scheme (up, red points) at time $T = 1$, computed with $n = 160$ and $\tau = h$. By green line we plot the exact solution at T and by black line the initial condition.

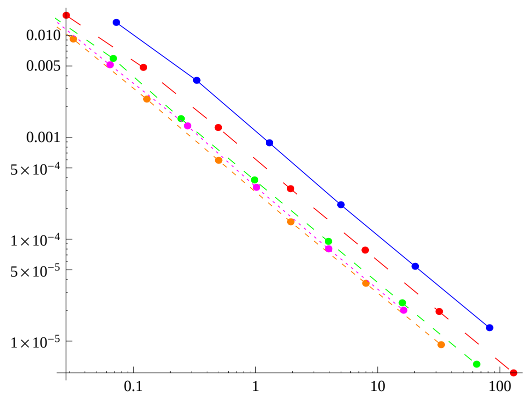

Fig. 2 CPU versus $L_2(I,L_2)$-error for the Lax-Wendroff method (blue solid line) and for the I^2OE scheme with CFL=1 (red large dashing, $\tau = h$), CFL=2 (green medium dashing, $\tau = 2h$), CFL=4 (orange small dashing, $\tau = 4h$) and CFL=8 (magenta tiny dashing, $\tau = 8h$) for the experiment from Fig. 1. The plots indicate that I^2OE scheme is about 4–times faster in order to get the same $L_2(I,L_2)$-error.

References

1. Eymard, R., Gallouet, T., & Herbin R.: The finite volume methods, Handbook for Numerical Analysis, 2000, Vol. 7 (Ph. Ciarlet, J. L. Lions, eds.), Elsevier.
2. Frolkovič, P., Mikula, K.: Flux-based level set method: a finite volume method for evolving interfaces, Applied Numerical Mathematics, Vol. 57, No. 4 (2007) pp. 436-454.
3. LeVeque R.J., Finite Volume Methods for Hyperbolic Problems, Cambridge Texts in Applied Mathematics. Cambridge University Press, 2002.
4. Mikula, K., Ohlberger, M.: A new level set method for motion in normal direction based on a semi-implicit forward-backward diffusion approach, SIAM J. Scientific Computing, Vol. 32 , No. 3 (2010) pp. 1527-1544.
5. Mikula, K., Ohlberger, M.: A New Inflow-Implicit/Outflow-Explicit Finite Volume Method for Solving Variable Velocity Advection Equations, Preprint 01/10 - N, Angewandte Mathematik und Informatik, Universität Münster, June 2010, pp.1-20

The paper is in final form and no similar paper has been or is being submitted elsewhere.