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NEW SECOND ORDER UP-WIND SCHEME FOR OBLIQUE DERIVATIVE BOUNDARY VALUE PROBLEM*

MATEJ MEDĽA† AND KAROL MIKULA‡

Abstract. This work is devoted to solving the Laplace equation with an oblique derivative prescribed as a boundary condition on a non-uniform logically rectangular grids. Laplace equation is solved using a finite volume method and we use new up-wind type discretization for the oblique derivative. In order to approximate Laplace equation on non-uniform 3D meshes,the normal derivative is split into the tangential derivative on finite volume faces and derivative in the direction of the vector connecting representative points of finite volumes. New second order up-wind discretization of the oblique derivative, based on linear reconstruction of solution on 3D grid, is presented. A gradient is used for a better approximation of unknown value on the boundary of finite volume. Since both, up-wind and finite volume method, are second order, the whole scheme is second order.

Key words. Laplace equation, finite volume method, up-wind, second order scheme

AMS subject classifications. 35J05, 35J25, 65N08

1. Introduction.

The Laplace equation is a partial differential equation which describes a variety of physical phenomena. This work deals with finding solution of the Laplace equation with the oblique derivative prescribed on the part of the boundary

$$-\Delta T(x) = 0, \quad x \in \Omega \subset R^3, \tag{1.1}$$

$$v(x) \cdot \nabla T(x) = g(x), \quad x \in \Gamma, \tag{1.2}$$

$$T(x) = T_{Dir}(x), x \in \partial\Omega - \Gamma, \tag{1.3}$$

where Γ is the part of the boundary with derivative in the direction v prescribed. Finding a solution on a complex computational domain can be difficult. That's why we use numerical methods. This work deals with finding the solution of (1.1) on non-uniform hexahedron grids using the finite volume method [4]. The problem (1.1) has application in physical geodesy, in determination of the Earth gravity field, and is called the Geodetic Boundary Value Problem [8, 3, 1], where $T(x)$ is a so called disturbing potential, which is a real Earth gravity potential minus a so called normal gravity potential of a rotating ellipsoid. The computational domain is a volume over the Earth's topography. A bottom boundary has a gravity disturbances prescribed and the remaining boundary has the disturbing potential prescribed by using information from satellite missions and global geopotential models. Because of compatibility of these two boundary conditions a part of the bottom boundary near side boundaries has also the disturbing potential prescribed.

The paper is organized as follows. In Section 2 we describe our method for approximation of the Laplace equation. In Section 3 we describe approximation of the oblique derivative boundary condition. And in Section 4 we discuss numerical experiments.

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†Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 813 68 Bratislava (medla@math.sk).

‡Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 813 68 Bratislava (mikula@math.sk).

2. Approximation of the Laplace equation. Let us have Laplace equation on a three dimensional domain Ω with Dirichlet boundary conditions

$$-\Delta T(x) = 0, x \in \Omega \tag{2.1}$$
$$T(x) = T_{Dir},\ x \in \partial\Omega. \tag{2.2}$$

We divide the domain Ω into the regular hexahedron grid. Vertices of hexahedron represent the representative points of finite volumes constructed later. Representative points are denoted by $x_{i,j,k}$. Hexahedron finite volumes are constructed around inner (those that do not lie on the boundary $\partial\Omega$) representative points. Let $p, q, r \in \{-1, 0, 1\}$ and let N_{int} denote set of all (p, q, r),$|p| + |q| + |r| = int$. Vertices of finite volumes are denoted by $x^{p,q,r}_{i,j,k}$, where $(p, q, r) \in N_3$, see Figure 2.1. Vertex $x^{p,q,r}_{i,j,k}$ is constructed in such way that is located in the center of eight neighbouring representative points, i.e.,

$$x^{p,q,r}_{i,j,k} = \frac{1}{8} \sum_{(l,m,n)\in B(p,q,r)} x_{i+l,j+m,k+n}, \tag{2.3}$$

where $B(p, q, r) = \{(p, q, r),(p, q, 0),(p, 0, r),(p, 0, 0),(0, q, r),(0, q, 0),(0, 0, r),(0, 0, 0)\}$. The finite volume associated with the representative point $x_{i,j,k}$ is denoted by $V_{i,j,k}$.

By integrating the equation (2.1) over the finite volume V_{ijk} we obtain

$$\int_{V_{i,j,k}} -\Delta T dx = 0. \tag{2.4}$$

Using Green's theorem we obtain

$$\int_{\partial V_{i,j,k}} -\nabla T \cdot \mathbf{n} d\tau = 0. \tag{2.5}$$

Considering that the finite volume $V_{i,j,k}$ has neighbouring volumes $V_{i+p,j+q,k+r}$, $(p, q, r) \in N_1$with non-zero measure of the common boundary, and $e^{p,q,r}_{i,j,k}$ is the boundary between volumes $V_{i,j,k}$ and $V_{i+p,j+q,k+r}$, we can rewrite the equation (2.5) to the form

$$-\sum_{(p,q,r)\in N_1} \int_{e^{p,q,r}_{i,j,k}} \nabla T \cdot \mathbf{n} d\tau = 0. \tag{2.6}$$

Unknown values $T_{i,j,k}$ are considered in points $x_{i,j,k}$.

Unit vector $\mathbf{s}^{p,q,r}_{i,j,k}$, which is pointing from the neighbouring point $x_{i,j,k}$ to the point $x_{i+p,j+q,k+r}$, is given by

$$\mathbf{s}^{p,q,r}_{i,j,k} = \frac{x_{i+p,j+q,k+r} - x_{i,j,k}}{|x_{i+p,j+q,k+r} - x_{i,j,k}|}, \tag{2.7}$$

where $|x|$ is Euclidian norm of a vector x. Let us introduce new operations on the set N_1

$$\oplus(p, q, r) = \begin{cases} (p, 1, 1), & p, \neq 0 \\ (1, q, 1), & q, \neq 0 \\ (1, 1, r), & r, \neq 0 \end{cases}, \quad \ominus(p, q, r) = \begin{cases} (p, -1, -1), & p \neq, \\ (-1, q, -1), & q \neq 0 \\ (-1, -1, r), & r \neq 0 \end{cases}$$

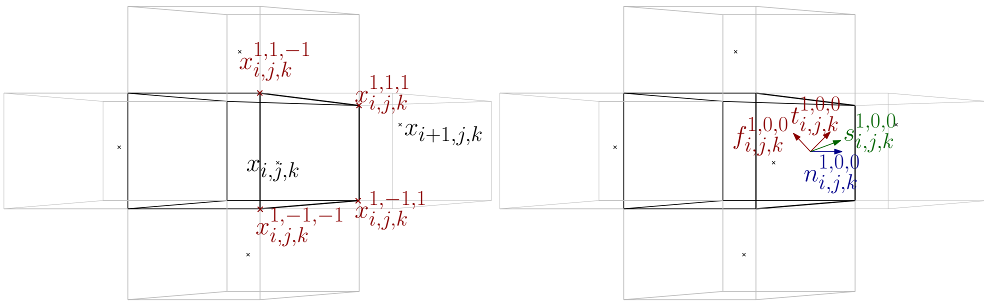

FIG. 2.1. *Finite volume*

$$\boxplus(p,q,r) = \begin{cases} (p,1,-1), & p \neq 0 \\ (1,q,-1), & q \neq 0 \\ (1,-1,r), & r \neq 0 \end{cases}, \quad \boxminus(p,q,r) = \begin{cases} (p,-1,1), & p \neq 0 \\ (-1,q,1), & q \neq 0 \\ (-1,1,r), & r \neq 0 \end{cases}$$

Thanks to our structure of finite volumes, the faces of finite volumes are given by four vertices. These vertices are used to compute tangent vectors. The first tangent vector $\mathbf{t}^{p,q,r}_{i,j,k}$ to the boundary between $V_{i,j,k}$ and $V_{i+p,j+q,k+r}$ is given by

$$\mathbf{t}^{p,q,r}_{i,j,k} = \frac{x^{\oplus(p,q,r)}_{i,j,k} - x^{\ominus(p,q,r)}_{i,j,k}}{|x^{\oplus(p,q,r)}_{i,j,k} - x^{\ominus(p,q,r)}_{i,j,k}|}, \tag{2.8}$$

where $x^{\oplus(p,q,r)}_{i,j,k}$ and $x^{\ominus(p,q,r)}_{i,j,k}$ are vertices of $e^{p,q,r}_{i,j,k}$, those connecting line is diagonal of the face $e^{p,q,r}_{i,j,k}$. The second tangent vector $\mathbf{f}^{p,q,r}_{i,j,k}$ is given by other two vertices of $e^{p,q,r}_{i,j,k}$,

$$\mathbf{f}^{p,q,r}_{i,j,k} = \frac{x^{\boxplus(p,q,r)}_{i,j,k} - x^{\boxminus(p,q,r)}_{i,j,k}}{|x^{\boxplus(p,q,r)}_{i,j,k} - x^{\boxminus(p,q,r)}_{i,j,k}|}. \tag{2.9}$$

The normal vector to the boundary of the finite volume is then defined by

$$\mathbf{n}^{p,q,r}_{i,j,k} = \mathbf{t}^{p,q,r}_{i,j,k} \times \mathbf{f}^{p,q,r}_{i,j,k}. \tag{2.10}$$

where $\mathbf{n}^{p,q,r}_{i,j,k}$ is the outer normal relative to the finite volume $V_{i,j,k}$. See Figure 2.1.

Since the vector $\mathbf{s}^{pqr}_{ijk}$ can be expressed as a linear reconstruction of $\mathbf{n}^{pqr}_{ijk}$, $\mathbf{t}^{pqr}_{ijk}$, $\mathbf{f}^{pqr}_{ijk}$, it holds

$$\begin{aligned} \nabla T \cdot \mathbf{s}^{pqr}_{ijk} &= \nabla T \cdot (\beta^{pqr}_{ijk}\mathbf{n}^{pqr}_{ijk} + \alpha^{pqr}_{ijk}\mathbf{t}^{pqr}_{ijk} + \gamma^{pqr}_{ijk}\mathbf{f}^{pqr}_{ijk}) \\ &= \beta^{pqr}_{ijk}\nabla T \cdot \mathbf{n}^{pqr}_{ijk} + \alpha^{pqr}_{ijk}\nabla T \cdot \mathbf{t}^{pqr}_{ijk} + \gamma^{pqr}_{ijk}\nabla T \cdot \mathbf{f}^{pqr}_{ijk}, \end{aligned} \tag{2.11}$$

where coefficients α^{pqr}_{ijk}, β^{pqr}_{ijk} and γ^{pqr}_{ijk} are given by solving a linear system of equations

$$\mathbf{s}^{pqr}_{ijk} = \beta^{pqr}_{ijk}\mathbf{n}^{pqr}_{ijk} + \alpha^{pqr}_{ijk}\mathbf{t}^{pqr}_{ijk} + \gamma^{pqr}_{ijk}\mathbf{f}^{pqr}_{ijk}. \tag{2.12}$$

Therefore, for the derivative in the direction of normal we get

$$\nabla T \cdot \mathbf{n}^{pqr}_{ijk} = \frac{1}{\beta^{pqr}_{ijk}}(\nabla T \cdot \mathbf{s}^{pqr}_{ijk} - \alpha^{pqr}_{ijk}\nabla T \cdot \mathbf{t}^{pqr}_{ijk} - \gamma^{pqr}_{ijk}\nabla T \cdot \mathbf{f}^{pqr}_{ijk}). \tag{2.13}$$

The equation (2.13) is approximated by

$$(2.14)\quad \frac{1}{\beta^{pqr}_{ijk}}(\nabla T\cdot \mathbf{s}^{pqr}_{ijk} - \alpha^{pqr}_{ijk}\nabla T\cdot \mathbf{t}^{pqr}_{ijk} - \gamma^{pqr}_{ijk}\nabla T\cdot \mathbf{f}^{pqr}_{ijk}) \approx$$

$$\frac{1}{\beta^{pqr}_{ijk}}\frac{T_{ijk}-T_{i+p,j+q,k+r}}{d^{pqr}_{ijk}} - \frac{\alpha^{pqr}_{ijk}}{\beta^{pqr}_{ijk}}\frac{T^{\oplus(p,q,r)}_{i,j,k}-T^{\ominus(p,q,r)}_{i,j,k}}{|x^{\oplus(p,q,r)}_{i,j,k}-x^{\ominus(p,q,r)}_{i,j,k}|} - \frac{\gamma^{pqr}_{ijk}}{\beta^{pqr}_{ijk}}\frac{T^{\boxplus(p,q,r)}_{i,j,k}-T^{\boxminus(p,q,r)}_{i,j,k}}{|x^{\boxplus(p,q,r)}_{i,j,k}-x^{\boxminus(p,q,r)}_{i,j,k}|},$$

where $T^{\oplus(p,q,r)}_{i,j,k}$ are values in points $x^{\oplus(p,q,r)}_{i,j,k}$ and d^{pqr}_{ijk} is the distance between $x^{p,q,r}_{i,j,k}$ and $x_{i,j,k}$.

The equation (2.6) can be rewritten using the equation (2.14) to form

$$(2.15)\quad -\sum_{(p,q,r)\in N_1} m(e^{pqr}_{ijk})\Bigg(\frac{1}{\beta^{pqr}_{ijk}}\frac{T_{ijk}-T_{i+p,j+q,k+r}}{d^{pqr}_{ijk}} - \frac{\alpha^{pqr}_{ijk}}{\beta^{pqr}_{ijk}}\frac{T^{\oplus(p,q,r)}_{i,j,k}-T^{\ominus(p,q,r)}_{i,j,k}}{|x^{\oplus(p,q,r)}_{i,j,k}-x^{\ominus(p,q,r)}_{i,j,k}|} - \frac{\gamma^{pqr}_{ijk}}{\beta^{pqr}_{ijk}}\frac{T^{\boxplus(p,q,r)}_{i,j,k}-T^{\boxminus(p,q,r)}_{i,j,k}}{|x^{\boxplus(p,q,r)}_{i,j,k}-x^{\boxminus(p,q,r)}_{i,j,k}|}\Bigg)=0,$$

where $m(e^{pqr}_{ijk})$ is areas of the face e^{pqr}_{ijk}. For the finite volumes, that are adjacent to the boundary finite volumes, the value $T_{i+p,j+q,k+r}$ is given by the Dirichlet boundary condition (2.2). Values $T^{\oplus(p,q,r)}_{i,j,k}$ are unknowns not given in representative points, but unknowns in points $x^{\oplus(p,q,r)}_{i,j,k}$, vertices of finite volume. They lie in the center of corresponding representative points (2.3). So values $T^{\oplus(p,q,r)}_{i,j,k}$ are approximated by

$$(2.16)\quad T^{\oplus(p,q,r)}_{i,j,k} = T(x^{\oplus(p,q,r)}_{i,j,k}) = \frac{1}{8}\sum_{(l,m,n)\in B(\oplus(p,q,r))} T_{i+l,j+m,k+n},$$

and values $T^{\ominus(p,q,r)}_{i,j,k}$, $T^{\boxplus(p,q,r)}_{i,j,k}$, $T^{\boxminus(p,q,r)}_{i,j,k}$ in the equation (2.15) can be expressed similarly.

Equation (2.15) is given for every inner finite volume $V_{i,j,k}$ with the unknown value $T_{i,j,k}$. Therefore, we have as many equations as unknowns, and we get the linear system, which can be solved, e.q. by BiCGStab method [9]. Numerical experiment for solving the problem (2.1)-(2.2) is presented in Section 4.

3. Approximation of the oblique derivative boundary condition. Let us have the Laplace equation (2.1) on the domain Ω with prescribed derivative in the direction v, pointing outward from Ω, on the part of the domain boundary Γ and the Dirichlet boundary condition (2.2) on the rest of the boundary. The oblique derivative boundary condition is thus given by

$$(3.1)\quad v(x)\cdot\nabla T(x) = g(x), x\in\Gamma.$$

The computational domain is divided by finite volumes as in the previous section. However the finite volumes are constructed also around representative points on the boundary Γ. Vertices common to boundary finite volumes and inner finite volumes are located in the center of representative points, defined by (2.3). Other vertices of boundary finite volumes are obtained by mirroring of the former ones through Γ. The set of added finite volumes is denoted by O.

We understand the equation (3.1) as advection equation, see [4], we integrate it over the finite volume $V_{i,j,k} \in O$:

$$\int_{V_{i,j,k}} v \cdot \nabla T dx = \int_p g dx. \tag{3.2}$$

Since

$$v \cdot \nabla T = \nabla \cdot (vT) - T\nabla \cdot v, \tag{3.3}$$

we can rewrite the equation (3.2) into the form

$$\int_{V_{i,j,k}} \nabla \cdot (vT) dx - \int_{V_{i,j,k}} T\nabla \cdot v dx = \int_{V_{i,j,k}} g dx. \tag{3.4}$$

Since the value T is considered constant on the finite volume, we can take out T in the second integral

$$\int_{V_{i,j,k}} \nabla \cdot (vT) dx - T_{i,j,k} \int_{V_{i,j,k}} \nabla \cdot v dx = \int_{V_{i,j,k}} g dx. \tag{3.5}$$

Using Green's theorem

$$\int_{\partial V_{i,j,k}} T v \cdot \mathbf{n} ds - T_{i,j,k} \int_{\partial V_{i,j,k}} v \cdot \mathbf{n} ds = \int_{V_{i,j,k}} g dx. \tag{3.6}$$

Suppose g is constant on the finite volume and T is constant on its faces $e^{p,q,r}_{i,j,k}$,then the equation (3.6) can be rewritten as

$$\sum_{(p,q,r)\in N_1} T^{p,q,r}_{i,j,k} \int_{e^{p,q,r}_{i,j,k}} v \cdot \mathbf{n} ds - T_{i,j,k} \sum_{(p,q,r)\in N_1} \int_{e^{p,q,r}_{i,j,k}} v \cdot \mathbf{n} ds = |V_{i,j,k}| g, \tag{3.7}$$

where $T^{p,q,r}_{i,j,k}$ is the value on the boundary $e^{p,q,r}_{i,j,k}$ and $|V_{i,j,k}|$ is a 3D measure of the finite volume $V_{i,j,k}$.

The up-wind principle will be used in the sequel. Let us define the integrated flux over $e^{p,q,r}_{i,j,k}$ by

$$v^{p,q,r}_{i,j,k} = \int_{e^{p,q,r}_{i,j,k}} v \cdot \mathbf{n} ds. \tag{3.8}$$

If $v^{p,q,r}_{i,j,k} > 0$, $e^{p,q,r}_{i,j,k}$ is an outflow face. Thus $T^{p,q,r}_{i,j,k}$ should be computed by using the information from inside of the finite volume, $T^{p,q,r}_{i,j,k} := T_{i,j,k} + \nabla T_{i,j,k} \cdot (x^{p,q,r}_{i,j,k} - x_{i,j,k})$, where $\nabla T_{i,j,k}$ is an approximation of gradient in finite volume $V_{i,j,k}$. If $v^{p,q,r}_{i,j,k} < 0$, $e^{p,q,r}_{i,j,k}$ represents an inflow face, thus $T^{p,q,r}_{i,j,k}$ is computed using information from the neighbouring finite volume. Hence $T^{p,q,r}_{i,j,k} := T_{i+p,j+q,k+r} + \nabla T_{i+p,j+q,k+r} \cdot (x^{p,q,r}_{i,j,k} - x_{i+p,j+q,k+r})$, see Figure 3.1.

Let us split the set N_1 for (i,j,k) into $N_1^{in}(i,j,k)$ and $N_1^{out}(i,j,k)$, $N_1^{in}(i,j,k)$ are indexes of neighbours for which $v^{p,q,r}_{i,j,k} < 0$ and $N_1^{out}(i,j,k)$ are indexes of neighbours for which $v^{p,q,r}_{i,j,k} > 0$. Then the equation (3.7) can be rewritten in the form

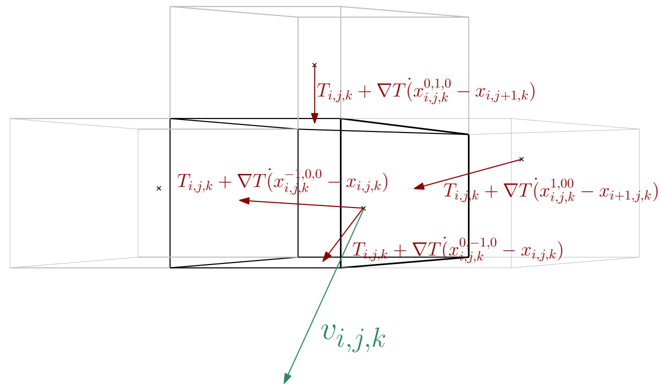

FIG. 3.1. *Neighbor finite volume*

$$\begin{aligned}(3.9)\quad &\sum_{(p,q,r)\in N_1^{in}(i,j,k)} (T_{i+p,j+q,k+r} + \nabla T_{i+p,j+q,k+r}\cdot(x^{p,q,r}_{i,j,k} - x_{i+p,j+q,k+r}))v^{p,q,r}_{i,j,k}\\ &+ \sum_{(p,q,r)\in N_1^{out}(i,j,k)} (T_{i,j,k} + \nabla T_{i,j,k}\cdot(x^{p,q,r}_{i,j,k} - x_{i,j,k}))v^{p,q,r}_{i,j,k}\\ &- \sum_{(p,q,r)\in N_1^{in}(i,j,k)} T_{i,j,k}v^{p,q,r}_{i,j,k} - \sum_{(p,q,r)\in N_1^{out}(i,j,k)} T_{i,j,k}v^{p,q,r}_{i,j,k} = |V_{i,j,k}|g,\end{aligned}$$

where $v^{p,q,r}_{i,j,k}$ is approximated by

$$(3.10)\qquad v^{p,q,r}_{i,j,k} = |e^{p,q,r}_{i,j,k}|(v\cdot \mathbf{n}^{p,q,r}_{i,j,k}),$$

where $|e^{p,q,r}_{i,j,k}|$ is a 2D measure of the face $e^{p,q,r}_{i,j,k}$. By using functions $\max(0, v^{p,q,r}_{i,j,k})$ and $\min(0, v^{p,q,r}_{i,j,k})$ we can rewrite (3.9) as

$$\begin{aligned}(3.11)\quad &\sum_{(p,q,r)\in N_1} \Big[(T_{i+p,j+q,k+r} + \nabla T_{i+p,j+q,k+r}\cdot(x^{p,q,r}_{i,j,k} - x_{i+p,j+q,k+r}))\min(0, v^{p,q,r}_{i,j,k})\\ &+ (T_{i,j,k} + \nabla T_{i,j,k}\cdot(x^{p,q,r}_{i,j,k} - x_{i,j,k}))\max(0, v^{p,q,r}_{i,j,k}) - T_{i,j,k}v^{p,q,r}_{i,j,k}\Big] = |V_{i,j,k}|g.\end{aligned}$$

The gradient on the finite volume $V_{i,j,k}$ can be expressed using derivatives in three linear independent directions. Let us denote these directions $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and define them by

$$(3.12)\qquad \mathbf{p} = \frac{x_{i+p,j,k} - x_{i,j,k}|}{|x_{i+p,j,k} - x_{i,j,k}|},\ \mathbf{q} = \frac{x_{i,j+q,k} - x_{i,j,k}}{|x_{i,j+q,k} - x_{i,j,k}|},\ \mathbf{r} = \frac{x_{i,j,k+r} - x_{i,j,k}}{|x_{i,j,k+r} - x_{i,j,k}|},$$

where values of $p, q, r \in \{-1, 1\}$ are chosen according to up-wind direction. Approximation of derivatives in these directions are

$$(3.13)\qquad \frac{\partial T}{\partial \mathbf{p}} \approx \frac{T_{i+p,j,k} - T_{i,j,k}}{|x_{i+p,j,k} - x_{i,j,k}|},\ \frac{\partial T}{\partial \mathbf{q}} \approx \frac{T_{i,j+q,k} - T_{i,j,k}}{|x_{i,j+q,k} - x_{i,j,k}|},\ \frac{\partial T}{\partial \mathbf{r}} \approx \frac{T_{i,j,k+r} - T_{i,j,k}}{|x_{i,j,k+r} - x_{i,j,k}|},$$

and since

$$\begin{aligned}\frac{\partial T}{\partial \mathbf{p}} &= \nabla T \cdot \mathbf{p} = \frac{\partial T}{\partial x}p_x + \frac{\partial T}{\partial y}p_y + \frac{\partial T}{\partial z}p_z,\\ \frac{\partial T}{\partial \mathbf{q}} &= \nabla T \cdot \mathbf{q} = \frac{\partial T}{\partial x}q_x + \frac{\partial T}{\partial y}q_y + \frac{\partial T}{\partial z}q_z,\\ \frac{\partial T}{\partial \mathbf{r}} &= \nabla T \cdot \mathbf{r} = \frac{\partial T}{\partial x}r_x + \frac{\partial T}{\partial y}r_y + \frac{\partial T}{\partial z}r_z,\end{aligned} \tag{3.14}$$

we obtain a system of linear equations for unknowns $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$, $\frac{\partial T}{\partial z}$. By solution of (3.14) we get

$$\begin{aligned}\frac{\partial T}{\partial x} &= -\frac{-p_z q_y \frac{\partial T}{\partial \mathbf{r}} + p_y q_z \frac{\partial T}{\partial \mathbf{r}} - q_z \frac{\partial T}{\partial \mathbf{p}} r_y + p_z \frac{\partial T}{\partial \mathbf{q}} r_y + q_y \frac{\partial T}{\partial \mathbf{p}} r_z - p_y \frac{\partial T}{\partial \mathbf{q}} r_z}{p_z q_y r_x - p_y q_z r_x - p_z q_x r_y + p_x q_z r_y + p_y q_x r_z - p_x q_y r_z},\\ \frac{\partial T}{\partial y} &= -\frac{p_z q_x \frac{\partial T}{\partial \mathbf{r}} - p_x q_z \frac{\partial T}{\partial \mathbf{r}} + q_z \frac{\partial T}{\partial \mathbf{p}} r_x - p_z \frac{\partial T}{\partial \mathbf{q}} r_x - q_x \frac{\partial T}{\partial \mathbf{p}} r_z + p_x \frac{\partial T}{\partial \mathbf{q}} r_z}{p_z q_y r_x - p_y q_z r_x - p_z q_x r_y + p_x q_z r_y + p_y q_x r_z - p_x q_y r_z},\\ \frac{\partial T}{\partial z} &= -\frac{-p_y q_x \frac{\partial T}{\partial \mathbf{r}} + p_x q_y \frac{\partial T}{\partial \mathbf{r}} - q_y \frac{\partial T}{\partial \mathbf{p}} r_x + p_y \frac{\partial T}{\partial \mathbf{q}} r_x + q_x \frac{\partial T}{\partial \mathbf{p}} r_y - p_x \frac{\partial T}{\partial \mathbf{q}} r_y}{p_z q_y r_x - p_y q_z r_x - p_z q_x r_y + p_x q_z r_y + p_y q_x r_z - p_x q_y r_z},\end{aligned} \tag{3.15}$$

and thus

$$\nabla T_{i,j,k} = \frac{\mathbf{p} \times \mathbf{q} \frac{\partial T}{\partial \mathbf{r}} + \mathbf{q} \times \mathbf{r} \frac{\partial T}{\partial \mathbf{p}} + \mathbf{r} \times \mathbf{p} \frac{\partial T}{\partial \mathbf{q}}}{\det(\mathbf{p}, \mathbf{q}, \mathbf{r})}, \tag{3.16}$$

where

$$\det(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \det \begin{pmatrix} p_x & p_y & p_z \\ q_x & q_y & q_z \\ r_x & r_y & r_z \end{pmatrix}. \tag{3.17}$$

If we substitute derivatives in directions **p**, **q**, **r** in (3.16) by the approximations (3.13) we get

$$\nabla T_{i,j,k} \approx \frac{\mathbf{p} \times \mathbf{q} \frac{T_{i+p,j,k} - T_{i,j,k}}{|x_{i+p,j,k} - x_{i,j,k}|} + \mathbf{q} \times \mathbf{r} \frac{T_{i,j+q,k} - T_{i,j,k}}{|x_{i,j+q,k} - x_{i,j,k}|} + \mathbf{r} \times \mathbf{p} \frac{T_{i,j,k+r} - T_{i,j,k}}{|x_{i,j,k+r} - x_{i,j,k}|}}{\det(\mathbf{p}, \mathbf{q}, \mathbf{r})}. \tag{3.18}$$

Now we have to decide, which values $p \in \{-1, 1\}$, $q \in \{-1, 1\}$, $r \in \{-1, 1\}$ to use in (3.18). To that goal we use a kind of up-wind approach again. The preferred choice of neighbouring value is given by the face with larger inflow flux. If there is no inflow face, we use outflow neighbour with smallest outflow flux. We define a function

$$I(a, b, v_1, v_2) = \begin{cases} a, & v_1 \le v_2 \\ b, & v_1 > v_2, \end{cases} \tag{3.19}$$

and using it in (3.18) we get

$$\nabla T_{i,j,k} = \left(\mathbf{p} \times \mathbf{q} \frac{I(T_{i+1,j,k}, T_{i-1,j,k}, v_{i,j,k}^{1,0,0}, v_{i,j,k}^{-1,0,0}) - T_{i,j,k}}{|I(x_{i+1,j,k}, x_{i-1,j,k}, v_{i,j,k}^{1,0,0}, v_{i,j,k}^{-1,0,0}) - x_{i,j,k}|} + \mathbf{q} \times \mathbf{r} \frac{I(T_{i,j+1,k}, T_{i,j-1,k}, v_{i,j,k}^{0,1,0}, v_{i,j,k}^{0,-1,0}) - T_{i,j,k}}{|I(x_{i,j+1,k}, x_{i,j-1,k}, v_{i,j,k}^{0,1,0}, v_{i,j,k}^{0,-1,0}) - x_{i,j,k}|} + \mathbf{r} \times \mathbf{p} \frac{I(T_{i,j,k+1}, T_{i,j,k-1}, v_{i,j,k}^{0,0,1}, v_{i,j,k}^{0,0,-1}) - T_{i,j,k}}{|I(x_{i,j,k+1}, x_{i,j,k-1}, v_{i,j,k}^{0,0,1}, v_{i,j,k}^{0,0,-1}) - x_{i,j,k}|} \right) \Big/ \det(\mathbf{p}, \mathbf{q}, \mathbf{r}), \tag{3.20}$$

where vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are

$$\mathbf{p} = \frac{I(x_{i+1,j,k}, x_{i-1,j,k}, v_{i,j,k}^{1,0,0}, v_{i,j,k}^{-1,0,0}) - x_{i,j,k}}{\left| I(x_{i+1,j,k}, x_{i-1,j,k}, v_{i,j,k}^{1,0,0}, v_{i,j,k}^{-1,0,0}) - x_{i,j,k} \right|}, \quad \mathbf{q} = \frac{I(x_{i,j+1,k}, x_{i,j-1,k}, v_{i,j,k}^{0,1,0}, v_{i,j,k}^{0,-1,0}) - x_{i,j,k}}{\left| I(x_{i,j+1,k}, x_{i,j-1,k}, v_{i,j,k}^{0,1,0}, v_{i,j,k}^{0,-1,0}) - x_{i,j,k} \right|}, \quad \mathbf{r} = \frac{I(x_{i,j,k+1}, x_{i,j,k-1}, v_{i,j,k}^{0,0,1}, v_{i,j,k}^{0,0,-1}) - x_{i,j,k}}{\left| I(x_{i,j,k+1}, x_{i,j,k-1}, v_{i,j,k}^{0,0,1}, v_{i,j,k}^{0,0,-1}) - x_{i,j,k} \right|}. \tag{3.21}$$

Substituting (3.20) into the (3.11), we get equations for boundary finite volumes $V_{i,j,k} \in O$. Due to construction of our scheme the equations for these finite volumes may require two neighbouring finite volumes in directions of $\mathbf{q}$ and $\mathbf{r}$. For those, which do not have such neighbours, we have to prescribe Dirichlet boundary conditions, which is also in accordance with the compatibility of boundary conditions mentioned in the introduction. All these equations together with equations from the discretization of Laplace equation form a numerical scheme for solving the problem (1.1).

4. Numerical experiment. The computational domain is a segment of 3D space where the bottom boundary is a perturbed sphere, see Figure 4.1, left. In order to test the numerical scheme we constructed the most coarse grid by using a surface evolution of the bottom boundary, see [2, 7]. Then a refined grids are constructed by adding new representative points in-between representative points of previous grid using the equation (2.3). The exact solution is taken as $T(\mathbf{x}) = \frac{1}{|\mathbf{x}-(0.1,0.2,0.3)|}$ and the exact solution values are prescribed on the Dirichlet part of the boundary. Five experiments are presented. In Table 4.1 errors in L_2-norm, maximum norm, and the experimental order of convergence (EOC) for the case with Dirichlet boundary condition only is presented, i.e., solving (2.1)-(2.2). Next tables show errors and EOC for the case with oblique derivative prescribed on the bottom boundary Γ, see Figure 4.1, right. In Tables 4.2 and 4.3 the vector v is in the direction of the $\nabla T(\mathbf{x})$. The Table 4.2 shows results of classical first order up-wind scheme and Table 4.3 shows results for our second order up-wind scheme. In Tables 4.4 and 4.5 the vector v is not in the direction of the gradient $T(\mathbf{x})$, in this case $\nabla T(\mathbf{x})$ is rotated alternately by an angle of $\pi/6$ around x, y, z axes to get the vector v, see Figure 4.1, right. The Table 4.4 shows results for the first order up-wind scheme and the Table 4.5 shows results for the up-wind scheme presented in this paper.

h_{max}	$\|\|e_{h_{max}}\|\|_{L_2}$	EOC	$\|\|e_{h_{max}}\|\|_{max}$	EOC_{max}
0.125851	9.15713e-05		0.000477755	
0.0642099	2.0156e-05	2.24924	0.000163837	1.59036
0.0324264	4.71736e-06	2.12571	5.50703e-05	1.59586
0.0162958	1.14219e-06	2.06128	1.66108e-05	1.74192
0.00817221	2.81252e-07	2.0306	4.82242e-06	1.792

TABLE 4.1

Results of our scheme with Dirichlet boundary conditions only.

h_{max}	$\|\|e_{h_{max}}\|\|_{L_2}$	EOC	$\|\|e_{h_{max}}\|\|_{max}$	EOC_{max}
0.125851	0.000802119		0.00643173	
0.0642099	0.000496918	0.711546	0.00490152	0.403746
0.0324264	0.000290903	0.783734	0.00328474	0.585871
0.0162958	0.000166066	0.814758	0.00207933	0.664534
0.00817221	9.37399e-05	0.82858	0.00127585	0.707698

TABLE 4.2

Results of our scheme with oblique derivative boundary conditions approximated by classical first order upwind, oblique vector v is in direction of gradient.

h_{max}	$\|\|e_{h_{max}}\|\|_{L_2}$	EOC	$\|\|e_{h_{max}}\|\|_{max}$	EOC_{max}
0.125851	0.00014071		0.00142455	
0.0642099	3.85122e-05	1.92546	0.000571048	1.35842
0.0324264	1.30241e-05	1.58694	0.000220138	1.39526
0.0162958	4.14275e-06	1.66473	7.53294e-05	1.55855
0.00817221	1.25519e-06	1.73012	2.40939e-05	1.65165

TABLE 4.3

Results of our scheme with oblique derivative boundary conditions approximated by our second order upwind, oblique vector v is in direction of gradient.

h_{max}	$\|\|e_{h_{max}}\|\|_{L_2}$	EOC	$\|\|e_{h_{max}}\|\|_{max}$	EOC_{max}
0.125851	0.000724601		0.00655075	
0.0642099	0.00053919	0.439194	0.00621504	0.0781763
0.0324264	0.000367634	0.560583	0.0049306	0.338871
0.0162958	0.000231581	0.671677	0.00336302	0.55608
0.00817221	0.000139933	0.729914	0.00214585	0.651011

TABLE 4.4

Results of our scheme with oblique derivative boundary conditions approximated by classical first order upwind, oblique vector v is not in direction of gradient.

h_{max}	$\|\|e_{h_{max}}\|\|_{L_2}$	EOC	$\|\|e_{h_{max}}\|\|_{max}$	EOC_{max}
0.125851	0.000210131		0.00176395	
0.0642099	6.52277e-05	1.7384	0.0007382	1.29445
0.0324264	2.42012e-05	1.45127	0.000324416	1.20347
0.0162958	7.85634e-06	1.63513	0.00013536	1.27035
0.00817221	2.47186e-06	1.67546	5.0292e-05	1.43457

TABLE 4.5

Results of our scheme with oblique derivative boundary conditions approximated by our second order upwind, oblique vector v is not in direction of gradient.

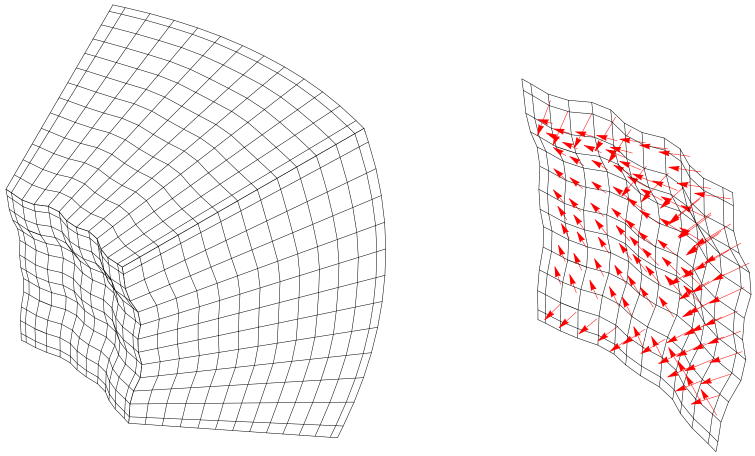

FIG. 4.1. *Left: computational domain, Right: oblique derivative directions*

4.1. Conclusion. In the paper we present new numerical method for solving the Laplace equation with oblique derivative prescribed on a part of the boundary. As one can see in Tables, the experimental order of convergence is approaching two and the errors are much smaller then these obtained by the classical up-wind scheme both in L_2 and maximum norms. In the future we plan to apply this method to real Geodetic BVPs.

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