Slowed Anisotropic Diffusion

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Abstract. A generalization of the regularized (in the sense of Catté, Lions, Morel and Coll) Perona-Malik *anisotropic diffusion* equation is proposed for image analysis. We present a numerical method for solving the above nonlinear degenerate diffusion problem, together with existence and convergence results. Numerical experiments are disscussed.

1 Introduction and motivation for the model equation

Let $u(t,x)$ be a function (representing the greylevel intensity function in image multiscale analysis - [1], [11]) satisfying the partial differential equation

$$\partial_t u - \nabla.(g(|\nabla G_\sigma * u|)\nabla\beta(x,u)) = f(\beta(x,u_0) - \beta(x,u)) \qquad (1)$$

for $t \in I \equiv [0,T]$, $x \in \Omega \subset \mathbb{R}^N$ ($N = 2$ or 3 in practice of image analysis), where Ω is a bounded Lipschitz domain with unit normal vector ν. The equation is coupled with homogeneous Neumann boundary and initial conditions in the form

$$\partial_\nu \beta(x,u) = 0 \text{ on } I \times \partial\Omega, \quad u(0,x) = u_0(x) \text{ in } \Omega. \qquad (2)$$

We assume that
(H1) $\beta(x,s)$ is nondecreasing Lipschitz continuous in s, $\beta(x,0) = 0$,
(H2) g is Lipschitz continuous function, $g(0) = 1$ and $0 < g(s) \to 0$ for $s \to \infty$,
(H3) $G_\sigma \in C^\infty(\mathbb{R}^N)$ is a compactly supported smoothing kernel
($\int_{\mathbb{R}^N} G_\sigma(x)dx = 1$, $G_\sigma(x) \to \delta_x$ - Dirac measure at point x, for $\sigma \to 0$),
(H4) f is Lipschitz continuous, nondecreasing function, $f(0) = 0$,
(H5) $u_0 \in L_2(\Omega)$ (represents a processed image).

The problem (1)-(2) is a generalization of the regularized (in the sense of Catté, Lions, Morel and Coll) Perona - Malik *anisotropic diffusion* equation widely used in image smoothing and edge detection. Previous papers on the topic were only dealing with the linear case $\beta(x,s) \equiv s$ ([10], [3], [6], [2]).

Due to the properties of g suggested in [10], image analysis depends in a special way on ∇u (edge indicator) and the equation diffuses the image selectively conserving the edge positions. However, one can see that if $g(s)s$ is decreasing then the Perona-Malik equation can behave locally like backward heat equation, which is a well known ill-posed problem. Consequently, there are serious mathematical troubles with the existence and uniqueness of a solution. Using convolution with Gaussian kernel, Catté, Lions, Morel and Coll explicitly introduced a *presmoothing* (implicitely included in numerical schemes solving *anisotropic diffusion* in original form). This slight modification allowed them to prove the existence and uniqueness of the solution. From the practical point of view, it keeps all advantages of the Perona-Malik model.

In the present paper, we add a *new nonlinearity* represented by the function β which makes the image multiscale analysis locally dependent on the values of intensity function u and on the position in the image x. The motivation for our generalization of the Catté, Lions, Morel and Coll regularization is the following. The setting of the treshold for edges (by a choice of g) and denoising of the image (by explicit/implicit presmoothing) improve some sets of edges. On the other hand, it destroys the details which are under the treshold or undistinguished from the noise in some scale. If such details are contained in certain ranges of (unnoisy) greylevels or image regions, then they can be conserved by a special choice of β. Generally, in the points where the slope $\beta'_s(x,u)$ is small, the diffusion process is slowed down and vice versa. So, if a different speed of diffusion process is desirable in different parts of the image or for different values of intensity function, the model equation (1)-(2) is a very natural tool. Briefly, the 'gradient term' is multiplied by $\beta'_s(x,u)$ and it stays in place of the diffusion coefficient now. Moreover, if the 'contrast' or 'enthalpy' function β is constant in certain subintervals of intensity range $[0,1]$ (and correspondingly β'_s is equal 0 which stops the diffusion) we obtain *degenerate cases* which are interesting in practice (see Figures 1 and 2). In a sense, it is similar to the multiphase (latent) heat transfer process when there are no changes on temperature for some ranges of energies. However, due to the possible degeneracies described above, the mathematical and computational treatment of (1)-(2) needs a nonstandard technique. Its numerical approximation is based on a special implicit time discretization developed and applied in [4], [5], [8], [9] to solve Stefan-like problems in enthalpy formulation, flow in porous media, anisotropic mean curvature flow and affine invariant convex curve evolution. Together with the methods developed in [3] and [6] we obtain the existence of a variational solution of (1)-(2) and the convergence of approximations.

2 Approximation scheme and numerical experiments

Approximation scheme 2.1: Let $n \in \mathbb{N}$ and $\tau = \frac{T}{n}$ be a discrete scale step. On each discrete scale level $t_i = i\tau$, $i = 1, \ldots, n$ we look for the solution θ_i, $\partial_\nu \theta_i = 0$ on $\partial\Omega$, of the linear elliptic partial differential equation

$$\mu_i(\theta_i - \beta(x, u_{i-1})) - \tau\nabla(g(|\nabla G_\sigma * u_{i-1}|)\nabla\theta_i) = \tau f(\beta(x, u_0) - \beta(x, u_{i-1})) \qquad (3)$$

where $\mu_i \in L_\infty(\Omega)$ is a relaxation function connected with θ_i by the convergence condition

$$\frac{1}{2}\tau^d \le \mu_i \le \min\{\frac{\beta_n^{-1}(x, \beta(x, u_{i-1}) + \alpha(\theta_i - \beta(x, u_{i-1}))) - u_{i-1}}{\theta_i - \beta(x, u_{i-1})}, K\}, \qquad (4)$$

where $\alpha \in (0,1)$ (α close to 1), $0 < K$ (large) and $d \in (0,1)$ are the parameters of the method and $\beta_n(x,s) := \beta(x,s) + \tau^d s$. The function u_i (image in scale t_i approximation) is obtained by the algebraic correction

$$u_i := u_{i-1} + \mu_i(\theta_i - \beta(x, u_{i-1})). \tag{5}$$

The approximation scheme 2.1 is not explicit with respect to μ_i, θ_i. However, it is a powerfull theoretical and practical technique for solving nonlinear degenerate parabolic equations ([4], [8]). We can find μ_i and θ_i using coupled iterations in (3) and in the right hand side of (4) - for details we refer to [7], [8], [4]. In the numerical implementation one can also put $\alpha = 1$ which simplifies the formulas. In some situations it is sufficient to put simply $\mu_i = \min\{1/\beta'_s(x, u_{i-1}), K\}$ and then use the inversion $u_i := \beta_n^{-1}(x, \theta_i)$. The linear PDE (3) can be solved by standard robust numerical methods - finite elements, finite volumes or finite differences. The pixel-voxel structure is a natural basis for the computational grid. Space discretization leads to linear systems of equations which have to be solved in each discrete scale step (see e.g. [2], [12]).

From theoretical point of view, using the function u_i determined by Approximation schemes 2.1, we construct Rothe's (step) functions $\overline{u}^{(n)}(t) = u_i$, for $t_{i-1} < t \le t_i$, $i = 1, \ldots, n$, $\overline{u}^{(n)}(0) = u_0$ which are considered as approximations of a solution of (1)-(2). Using the ideas of [5] and [6] we can prove ([7]) that $\overline{u}^{(n)} \rightharpoonup u$ in $L_2(I, L_2)$ for $n \to \infty$, where u is a variational solution of the problem (1)-(2).

We present two numerical experiments demonstrating the features of *slowed anisotropic diffusion*. Approximation scheme 2.1 is used and the results are compared with the multiscale analysis based on classical *anisotropic diffusion* equation ([3], [6]). We use $g(s) = 1/(1+s^2)$, $f \equiv 0$ and the convolution is realized as a weighted average on some neighborhood pixels (on which the support of kernel is located). Together with initial noisy images, the computed solutions after 10 discrete scale steps are plotted. The choice of β's stops the diffusion where we want to keep some fine details in the image (otherwise destroyed). E.g. in Figure 2, only the colours on Flora's face (detail of Botticelli's painting Primavera) contain a 'damage'. Using a proper choice of β which is constant for lower (dark) greylevels and linear for upper range of u, the face is selectively smoothed and flowers arround are kept.

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Fig. 1. Restoration of the noisy image (left, 200x200 pixels) by *anistropic diffusion* (middle) in comparison with *slowed anisotropic diffusion* (right); $\beta(x,s) = 0$ for $s \in [0, 0.5]$, $\beta(x,s) = s$ for $s \in (0.5, 1]$.

Fig. 2. Restoration of the real image (left, 570x350 pixels) by *anistropic diffusion*(middle) in comparison with *slowed anisotropic diffusion* (right); $\beta(x,s) = 0$ for $s \in [0, 0.39]$, $\beta(x,s) = s$ for $s \in (0.39, 1]$.

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