Gradient Evaluation on a Quadtree Based Finite Volume Grid

Krivá Zuzana, Handlovičová Angela and Mikula Karol

Abstract Many problems described by nonlinear PDEs need good approximations of gradients on finite volumes. Using finite volume methods, this can be difficult task if discretization of a computational domain does not fulfill the classical orthogonality property. Such a situation can occur, e.g., during coarsening in image processing using quadtree grids. We present a construction of an adjusted quadtree grid for which the connection of representative points of two adjacent finite volumes is perpendicular to their common boundary. On the other hand, for such an adjusted grid, the intersection of representative points connection with a finite volume boundary is not a middle point of their common edge. In this paper we present a new method of gradient evaluation for such a situation.

1 The computational grid

In this section we introduce our finite volume computational grid, its construction and its properties. Our purpose is to build the grid using large elements for regions with homogeneous values of a solution function - in our experiment representing image intensities. To this purpose we first build a graded quadtree, i.e. the quadtree, in which the difference in a level between adjacent cells is constrained, in our case to one. Grids associated with such trees are often used in order to produce procedures that are easier to implement. Moreover, in our case it is an inevitable requirement to be able to adjust the quadtree to the consistent finite volume grid. The consistent grid possesses the important property that the connection of two representative points of two adjacent finite volumes is perpendicular to their common boundary, which is

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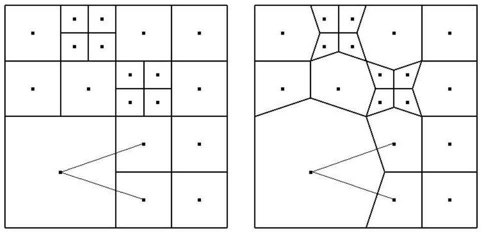

Fig. 1 An example of the original quadtree grid together with the representative points of its elements (on the left). This grid is transformed into the consistent one (on the right).

an important fact when we use the classical finite volume discretization [2]. An example of a quadtree and a corresponding consistent grid is displayed in Fig.1.

Building the quadtree. Let us suppose that our data is given on a regular non-adaptive square grid (which corresponds e.g. to the pixel structure of an image). First we build the quadtree by merging the elements with similar values from the smaller cells to the larger cells, i.e. from leaves to the root. The old values are either unchanged, or replaced by averaging the values from the processed area. During this process, the information about successful or unsuccessful merging is stored in a binary field with the size corresponding to the image. Moreover, this information is stored in such a way that it enables us to create a graded quadtree with a prescribed ratio of elements. It can be also used as a stopping criterion during *traversing* the quadtree and to test the configurations of elements - the leaves of the quadtree.

As we have already mentioned, in order to simplify creating the linear system matrix, where access to neighbors is needed, and to enable creating the consistent grid, we require that the ratio of sides of two adjacent squares is 1 : 1, 1 : 2 or 2 : 1. The used technique of building the quadtree adaptive grids is described in [4]. It uses the following *coarsening criterion*: the cells are merged if a difference in their intensities is below a prescribed tolerance ε.

Adjustment to the quadtree based consistent grid. The quadtree grid (Fig. 1 left) is *inconsistent* in the sense, that we cannot find the unique representative points of the adjacent grid elements - finite volumes - such that the connection of their representative points is perpendicular to their common boundary. The adaptive grid fulfilling this condition is called *consistent* and it is an *admissible* mesh in the sense of [2]. However, the basic quadtree grid can be adjusted to a consistent one procedurally: we must adjust the shape, if two adjacent finite volumes p and q are of different size. If we denote the length of a common edge in the original quadtree by h and we shift the "hanging node" by $v = \frac{h}{3}$ (e.g. in Fig. 2 we shift X to X'), then the connection of representative points is perpendicular to the shifted common boundary. This fact (and also the fact that $\frac{BX'}{PQ} = \frac{2}{3}$) follows from the similarity of triangles $\triangle AQP$ and $\triangle XX'B$ with the ratio of their adjacent sides 1:3. The area of p is also evaluated procedurally - it depends on a configuration of its neighbors.

Notations. Let every finite volume p of measure $|p|$ have a representative point X_p lying in its center or in the center of the original square for an adjusted element of

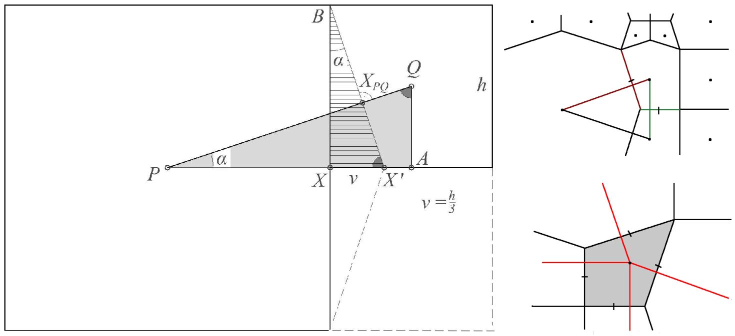

Fig. 2 Adjustment to the consistent grid. $|XX'| = v = \frac{1}{3}h$. XB=$\frac{2}{3}$PA, hence $\frac{BX'}{PQ} = \frac{2}{3}$. Examples of the shapes where the intersection of the connection of representative points and a common edge σ is not the midpoint of σ.

the consistent grid. The common interface of p and q is a line segment - an edge σ_{pq} with a nonzero measure in $\boldsymbol{R}$ denoted by $|\sigma_{pq}|$ and $d_{pq} = |X_q - X_p|$ is the distance of representative points. Let us denote by X_σ auch a point of σ_{pq}, which represents the *intersection of the line segment* X_pX_q *and* σ_{pq}. In our consistent grid, X_pX_q is perpendicular to σ, but the intersection X_σ is not the midpoint of σ in the general case. Let us denote by X^*_σ the *midpoint* of the edge σ. By $\mathscr{E}_p$ we denote the set of all edges σ of p. When we speak about a unit outer normal vector to $\sigma \in \mathscr{E}_p$, we denote it by $\mathbf{n}_{p\sigma}$.

2 Approximation of the gradient on the consistent grid

Our method for evaluation of gradients on finite volumes is based on [3]. Such a method works locally in that sense that we consider also representative points on finite volume edges, but not values at the corners. Then, with a help of these points we only need access to neighbors sharing a common edge, which is important when working on adaptive grids.

When solving PDEs where nonlinearities depend on the solution gradient, the method from [3] works as follows:

1. for edges σ of a finite volume p we define representative points X^*_σ - their midpoints, it must hold $X^*_\sigma = X_\sigma$,
2. with a help of these points, we evaluate the norm of gradient on p locally using the consequence of the Stokes formula, see (3)-(4),
3. discrete equation for the finite volume p is derived locally,
4. values of solution in X^*_σ are obtained by using conservation principle.

In the consistent adaptive grid $X^*_\sigma \neq X_\sigma$ in general. Such a situation occurs on edges containing a hanging node in the original quadtree grid. The most critical shape in this sense is the sharp element where X_σ is not the midpoint on any of the edges (Fig.2 right).

Let us suppose the linear approximation of the solution over the finite volume p. At $X \in p$ any linear function can be written as

$$u(X) = u(X_p) + \nabla u \cdot (X - X_p) = u_p + \nabla u \cdot (X - X_p). \tag{1}$$

If $X = X_\sigma$ it holds

$$u_\sigma - u_p = \nabla u \cdot (X_\sigma - X_p), \tag{2}$$

where u_σ, u_p represent values of the solution at points X_σ and X_p. The gradient of the linear function is a constant vector in $\boldsymbol{R}^2$, thus also over a control volume p. It will be denoted by ∇u. Then it holds

$$\begin{aligned}\nabla u &= \frac{1}{|p|}\int_p \nabla u dX = \frac{1}{|p|}\int_{\partial p} u \mathbf{n}_p dS = \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p}\int_\sigma (u_p + \nabla u \cdot (X - X_p))\mathbf{n}_{p\sigma} dS \\ &= \frac{1}{|p|} u_p \sum_{\sigma\in\mathscr{E}_p} |\sigma|\mathbf{n}_{p\sigma} + \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p} |\sigma|\nabla u\cdot (X^*_\sigma - X_p)\mathbf{n}_{p\sigma}. \end{aligned} \tag{3}$$

The term $\sum_{\sigma\in\mathscr{E}_p} |\sigma|\mathbf{n}_{p\sigma} = \mathbf{0}$ and the expression $|\sigma|\nabla u(X^*_\sigma - X_p)\mathbf{n}_{p\sigma}$ represents the precise integration of a linear function over the edge σ. Thus we have

$$\nabla u = \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p} |\sigma|\nabla u\cdot (X^*_\sigma - X_p)\mathbf{n}_{p\sigma}. \tag{4}$$

On the edges, where $X_\sigma \neq X^*_\sigma$, we can express

$$X^*_\sigma - X_p = (X_\sigma - X_p) + (X^*_\sigma - X_\sigma). \tag{5}$$

Then ∇u can be split into two parts

$$\nabla u = \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p} |\sigma|\nabla u\cdot (X_\sigma - X_p)\mathbf{n}_{p\sigma} + \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p} |\sigma|\nabla u\cdot (X^*_\sigma - X_\sigma)\mathbf{n}_{p\sigma}. \tag{6}$$

The part of ∇u given by the first term of (6) will be denoted as $(\nabla u)^A$ and due to (2) it can be evaluated as

$$(\nabla u)^A = \frac{1}{|p|}\sum_{\sigma\in\mathscr{E}_p} |\sigma|(u_\sigma - u_p)\mathbf{n}_{p\sigma}. \tag{7}$$

The second term of (6) is a *correction* of $(\nabla u)^A$ and it depends on the unknown gradient.

2.1 Evaluation of the gradients with corrections

In the following text we use subscripts in two ways: if they represent derivatives, we use x or y and if they represent the vector components, we use 1 or 2. Let us denote the correction vector $(X^*_\sigma - X_\sigma)$ by $\mathbf{c}_\sigma = ((c_\sigma)_1, (c_\sigma)_2)$. We will work with $(\nabla u)^A = ((u_x)^A, (u_y)^A)$, $\mathbf{n}_{p\sigma} = ((n_{p\sigma})_1, (n_{p\sigma})_2)$ and the unknown vector $\nabla u = (u_x, u_y)$. Now (6) can be rewritten into the form

$$(u_x, u_y) = ((u_x)^A, (u_y)^A) + \frac{1}{|p|} \sum_{\sigma \in \mathscr{E}_p} |\sigma| ((c_\sigma)_1 u_x + (c_\sigma)_2 u_y)((n_{p\sigma})_1, (n_{p\sigma})_2). \quad (8)$$

We see that (8) represents the linear system of two equations with two unknowns u_x and u_y which can be adjusted to the following form:

$$\begin{aligned} u_x(1 - \frac{1}{|p|} \sum_{\sigma \in \mathscr{E}_p} |\sigma| (c_\sigma)_1 (n_{p\sigma})_1) \quad &+ \quad u_y(-\frac{1}{|p|} \sum_{\sigma \in \mathscr{E}_p} |\sigma| (c_\sigma)_2 (n_{p\sigma})_1) \quad = (u_x)^A, \\ u_x(-\frac{1}{|p|} \sum_{\sigma \in \mathscr{E}_p} |\sigma| (c_\sigma)_1 (n_{p\sigma})_2) \quad &+ \quad u_y(1 - \frac{1}{|p|} \sum_{\sigma \in \mathscr{E}_p} |\sigma| (c_\sigma)_2 (n_{p\sigma})_2) \quad = (u_y)^A. \end{aligned}$$

We rewrite the system into such a form that we can see that the coefficient matrix denoted by B depends only on the shape of a grid element, but not on its size (level). Let us denote: $\mathbf{N}_{p\sigma} = \frac{|\sigma| \mathbf{n}_{p\sigma}}{l}$ and $\mathbf{C}_\sigma = \frac{\mathbf{c}_\sigma}{l}$, where l is the edge length of the square in the non adjusted quadtree. We have:

$$\begin{aligned} u_x \left(1 - \frac{l^2}{|p|} \sum_{\sigma \in \mathscr{E}_p} (C_\sigma)_1 (N_{p\sigma})_1 \right) \quad &+ \quad u_y \left(-\frac{l^2}{|p|} \sum_{\sigma \in \mathscr{E}_p} (C_\sigma)_2 (N_{p\sigma})_1 \right) \quad = (u_x)^A, \\ u_x \left(-\frac{l^2}{|p|} \sum_{\sigma \in \mathscr{E}_p} (C_\sigma)_1 (N_{p\sigma})_2 \right) \quad &+ \quad u_y \left(1 - \frac{l^2}{|p|} \sum_{\sigma \in \mathscr{E}_p} (C_\sigma)_2 (N_{p\sigma})_2 \right) = (u_y)^A. \end{aligned} \quad (9)$$

The elements of the coefficient matrix in (9) can be evaluated procedurally traversing the quadtree, or we can construct B using its properties mentioned later. B can be also precalculated in advance for every shape (there is only limited number of shapes in the consistent quadtree grid) - we can store B^{-1} and evaluate $\nabla u = B^{-1} (\nabla u)^A$.

Example 1. Let us take the consistent quadtree grid built over a uniform grid with 32×32 elements (Fig.3 left). We inspect specific functions defined on $[-1.25, 1.25] \times [-1.25, 1.25]$: we consider the norm of the gradient evaluated analytically, the norm of $(\nabla u)^A$ and ∇u obtained by solving (9). First let us take the function $u_1(X) = \frac{1}{2}(x^2 + y^2)$. We take the sharp marked element (Fig.3 left) with the representative point $(x_p, y_p) = (0.742, -0.89)$. First u_σ is set to the exact value evaluated using $u(X)$. The approximated gradient - the vector $(\nabla u)^A$ evaluated without correction is equal to $(-1,711, -1.801)$. After correction using (9) it is equal to $(0.860, -1.03)$, while the analytical gradient at this point has the value (x_p, y_p)

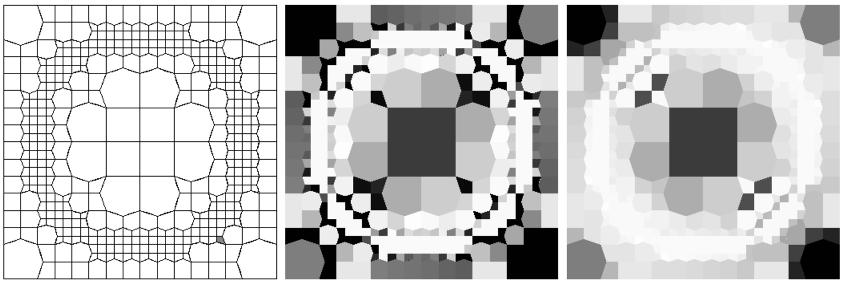

Fig. 3 Example 1. Left: the consistent quadtree grid with the inspected element. For $u_2(X) = \frac{1}{3}(x^3+y^3)$ we compare $(\nabla u)_p$ and $(\nabla u)_p^A$ with the values of the gradient evaluated analytically in (x_p, y_p). The values are scaled with darker values representing larger differences. Middle: $||(\nabla u)^A| - |\nabla u_{exact}||$. Right: $||\nabla u| - |\nabla u_{exact}||$.

given above. In practical tasks, u_σ is obtained by an interpolation. Thus we consider also that u_σ is obtained by a linear interpolation between u_p and u_q, its neighbor. It is interesting that in such case the approximated gradient of the quadratic function $u_1(X)$ obtained by (9) is equal to the analytical one. Secondly, let us take the function $u_2(X) = \frac{1}{3}(x^3+y^3)$, the selected volume as in the previous case and u_σ obtained by a linear interpolation. The analytical value of the gradient is $(0.551, 0.807)$, using (7) we get $(\nabla u)^A = (0.813, 0.987)$ and using (9) $\nabla u = (0.5572, 0.813)$. Fig.3 depicts differences of norms of $(\nabla u)_p^A$ and analytical gradient evaluated in the representative points of grid elements (x_p, y_p) (middle) and the norms of $(\nabla u)_p$ obtained by (9) and the analytical gradient (right) for the function $u_2(X)$ in (x_p, y_p). At the end we explored L_2 norms of errors $|\nabla u| - |\nabla u_{exact}|$ evaluated on four consistent adaptive grids obtained by consequent refinement of the grid from Fig.3: every finite volume of a corresponding quadtree grid was divided into four subvolumes and afterwards the grid was adjusted to the consistent one. We have obtained following results: 0.0619, 0.0173, 0. 0051 and 0.00158.

Properties of the coefficient matrix B. The nonzero corrections occur only if one of the edgepoints of σ is the shifted node. Let the edge vector σ be oriented from the shifted node to the quadtree corner. It can be shown that:

1. on the aligned edge σ, the correction $\mathbf{c}_\sigma$ can be expressed like $\mathbf{c}_\sigma = \frac{\sigma}{10}$, on the vertical or horizontal edge $\mathbf{c}_\sigma = \frac{\sigma}{4}$,
2. $\sum_{\sigma\in\mathscr{E}_p} (C_\sigma)_1 (N_{p\sigma})_1 = -\sum_{\sigma\in\mathscr{E}_p} (C_\sigma)_2 (N_{p\sigma})_2$,
3. $\sum_{\sigma\in\mathscr{E}_p} (C_\sigma)_1 (N_{p\sigma})_2 = \sum_{\sigma\in\mathscr{E}_p} (C_\sigma)_2 (N_{p\sigma})_1$,
4. It holds that the matrix B is regular ($det(B) > 0$) and the system (9) has always a unique solution. It can be proved using properties 1, 2 and 3.

3 Numerical solution of the regularized Perona-Malik equation on the consistent adaptive grid

In this section we present one experiment - solution of the regularized Perona-Malik equation [1] on a rectangular domain $\Omega \subset \boldsymbol{R}^2$ discretized with help of a consistent adaptive grid. The scaling interval $I = [0, T]$ is discretized into scale steps with $t^n = t^n + \tau$, τ is the scale step size, on the boundaries we keep the zero Neumann boundary conditions. So we solve the problem

$$\partial_t u - \nabla.(g(|\nabla G_s * u|)\nabla u) = 0, \quad \text{in } Q_T \equiv I \times \Omega, \tag{10}$$

where $g(s) = \frac{1}{1+Ks^2}$, $K > 0$ is the Perona-Malik function slowing down the diffusion in the vicinity of edges and $G_s(x)$ is the smoothing kernel. In our algorithm we realize the convolution $\nabla(G_s * u) = G_s * \nabla u$ by solving the linear heat equation. We apply one or several steps of the adaptive scheme for time T_s corresponding to s to both x and y coordinates of the gradient, then we evaluate the norm of the gradients and apply the Perona-Malik function g to get the diffusion coefficient denoted by $g_p^{s,n-1}$.

Let us denote by u_σ^n the value of the solution in X_σ at the time step t^n. The derivative in the direction $\mathbf{n}_{p\sigma}$ is approximated by $\nabla u^n \cdot \mathbf{n}_{p\sigma} \approx \frac{\left(u_\sigma^n - u_p^n\right)}{d_{p\sigma}}$. The diffusion coefficient $g_p^{s,n-1}$ is constant all over p, thus the flux over σ can be approximated by

$$F_{p\sigma}^n = g_p^{s,n-1} \frac{|\sigma|}{d_{p\sigma}} \left(u_\sigma^n - u_p^n\right). \tag{11}$$

A good way to evaluate $\frac{|\sigma|}{d_{p\sigma}}$ is to consider the neighbor q sharing σ with p. Then we can express (11) with a help of the transmissivity coefficient $T_{pq} = \frac{|\sigma|}{d_{pq}}$ and the ratio of $d_{p\sigma}$ and $d_{q\sigma}$, where $d_{p\sigma}$ and $d_{q\sigma}$ are distances of representative points from X_σ. If $\sigma \perp X_p X_q$ in the non adjusted grid, $\frac{d_{p\sigma}}{d_{q\sigma}} = 1$, otherwise, $\frac{d_{p\sigma}}{d_{q\sigma}} = \frac{4}{1}$ or $\frac{1}{4}$. For T_{pq} it holds that if one edgepoint of σ is a hanging node in the nonadjusted quadtree, then $T_{pq} = \frac{2}{3}$, otherwise $T_{pq} = 1$. The approximated flux (11) can be expressed as

$$F_{p\sigma}^n = T_{pq} \left(1 + \frac{d_{q\sigma}}{d_{p\sigma}}\right) g_p^{s,n-1} \left(u_\sigma^n - u_p^n\right). \tag{12}$$

Now we solve the linear system, where the set of equations for all finite volumes p

$$\left(u_p^n - u_p^{n-1}\right) |p| = \tau \sum_{\sigma \in \mathscr{E}_p} F_{p\sigma}^n \tag{13}$$

is accompanied by a set of equations for every u_σ^n, $\sigma \in \mathscr{E}_p$, obtained from the relationship $F_{p\sigma}^n = -F_{q\sigma}^n$ resulting in

Fig. 4 Numerical experiment. The artificial noisy image, the filtered image and the fixed adaptive grid.

$$u_\sigma^n = \frac{d_{q\sigma} g_p^{s,n-1} u_p^n + d_{p\sigma} g_q^{s,n-1} u_q^n}{d_{q\sigma} g_p^{s,n-1} + d_{p\sigma} g_q^{s,n-1}}.$$

We present here a numerical experiment where we begin with a regular grid and continue to use it until the decrease of elements is sufficient. Then we run the adaptive algorithm on the same adaptive grid. Advantage of this approach is that for the fixed adaptive grid we can store all necessary information, e.g. configurations of neighbors, matrix B, etc. We consider the image of the size 128×128 disturbed by the additive noise. We performed 13 scale steps with $\tau = 1$, with $K = 1000$ in the Perona-Malik function g and the time of presmoothing $T_s = 0.6$. The number of grid elements was reduced to $\frac{1}{3}$ after 5 scale steps, and then we continued on the fixed grid. The parameter ε used in the coarsening criterion is set to 0.01. Fig.4 shows the data itself, the filtered data and the adaptive grid fixed after 5 scale steps.

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