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APPROXIMATION OF NONLINEAR PARABOLIC EQUATIONS USING A FAMILY OF CONFORMAL AND NON-CONFORMAL SCHEMES

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Abstract. We consider a family of space discretisations for the approximation of nonlinear parabolic equations, such as the regularised mean curvature flow level set equation, using semi-implicit or fully implicit time schemes. The approximate solution provided by such a scheme is shown to converge thanks to compactness and monotony arguments. Numerical examples show the accuracy of the method.

1. **Introduction.** Nonlinear parabolic equations are involved in different physical or engineering frameworks. For example, the porous medium equation $u_t - \Delta u^m = 0$, the Stefan problem $u_t - \Delta\varphi(u) = 0$ arise in the framework of fluid flows within porous media. Important improvements in the approximation of their solutions have been obtained, using finite volume methods. Indeed, such methods are well suited to the conservative form of these equations.

More surprising is the success of finite volume methods for the approximation of some nonlinear problems, under the more general form $u_t - F(u, \nabla u, D^2 u) = 0$. For example, in [16], a few algorithms are proposed for the approximation of motion by mean curvature equation, including finite volume methods, whereas the equation, namely

$$u_t - |\nabla u|\mathrm{div}\,(\nabla u/|\nabla u|) = 0, \tag{1}$$

is not in the divergence form. In such cases, finite difference methods have more intensively been used. The mathematical framework which is under consideration for the analysis of the convergence of these finite difference schemes relies on the notion of viscosity solution and monotonous scheme. Such a monotonous behaviour does not seem straightforward in the framework of finite volume schemes. Indeed, in a recent paper [8], we study a finite volume method for the approximation of the motion by mean curvature equation in a regularised sense. The principles, used in [8] for the mathematical analysis of the convergence of the finite volume scheme, completely differ from that of the viscosity solutions [5, 3], and do not allow for handling the case of the non-regularised motion by mean curvature equation

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(nevertheless, this case is handled in some numerical examples provided at the end of this paper). A regularised sense, as detailed below, must be used for the proof of convergence of the method.

In the present paper, our aim is to propose a more general framework of approximation methods for some nonlinear parabolic equations in non-conservative form.

$$\nu(u,\nabla u)u_t - \operatorname{div}(\mu(|\nabla u|)\nabla u) = f, \text{ a.e. in } \Omega\times]0,T[\tag{2}$$

with the initial condition

$$u(x,0) = u_0(x), \text{ for a.e. } x \in \Omega, \tag{3}$$

and the boundary condition

$$u(x,t) = 0, \text{ for a.e. } (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{4}$$

under the following hypotheses (called hypotheses (H) in the following) on the real functions μ, ν, the initial data u_0, the right hand side f, and on the domain Ω:

1. Ω is a finite bounded connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$ (where $\mathbb{N}^\star$ denotes the set $\mathbb{N} \setminus \{0\}$),
2. $u_0 \in H_0^1(\Omega)$,
3. $f \in L^2(\Omega\times]0,T[)$ for all $T > 0$,
4. $\nu \in C^0(\mathbb{R} \times \mathbb{R}^d; [\nu_{\min}, \nu_{\max}])$, with given $\nu_{\max} \geq \nu_{\min} > 0$,
5. $\mu \in C^0(\mathbb{R}_+; [\mu_{\min}, \mu_{\max}])$, with given $\mu_{\max} \geq \mu_{\min} > 0$, is a Lipschitz continuous (non-strictly) decreasing function, and $(x\mu(x))' \geq \alpha$ for a.e. $x \in \mathbb{R}^+$ for a given $\alpha > 0$.

Remark 1. We could as well consider bounded functions $\nu(x,t,s,\xi)$ for $(x,t,s,\xi) \in \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^d$, measurable with respect to (x,t), continuous with respect to s and ξ.

It is worth noticing that the functions μ and ν given by

$$\begin{aligned} &\mu(s) = \max(1/\sqrt{s^2+a^2}, 1/b), \ \forall s \in \mathbb{R}_+, \\ &\nu(z,\xi) = \mu(|\xi|), \ \forall z \in \mathbb{R}, \ \forall \xi \in \mathbb{R}^d, \end{aligned} \tag{5}$$

for given reals $0 < a \leq b$, satisfy (H4-5) with $\alpha = a^2/b^3$ (this corresponds to the regularised level set equation [5]). Let us now give the precise mathematical sense that we consider for a solution to Problem (2)-(3)-(4) under Hypotheses (H).

Definition 1.1. (Weak solution of (2)-(3)-(4)) Under hypotheses (H), we say that u is a weak solution of (2)-(3)-(4) if, for all $T > 0$,

1. $u \in L^2(0,T;H_0^1(\Omega))$ and $u_t \in L^2(\Omega\times]0,T[)$ (hence $u \in C^0(0,T;L^2(\Omega))$),
2. $u(\cdot,0) = u_0$,
3. the following holds

$$\int_0^T\int_\Omega (\nu(u,\nabla u)u_t v + \mu(|\nabla u|)\nabla u\cdot\nabla v)\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega f\ v\mathrm{d}x\mathrm{d}t, \forall v \in L^2(0,T;H_0^1(\Omega)). \tag{6}$$

In the spirit of [8], where we prove the convergence of a finite volume scheme for the approximation of a weak solution of (2)-(3)-(4) in the sense of Definition 1.1, we develop in this paper a series of new features:

1. We consider a more general framework for the space discretisations, including conformal and non-conformal finite element methods and finite volume methods inspired by multipoint flux approximation [1].

2. In [8], the discrete norms involved in the scheme as arguments of functions μ and ν do not correspond to the exact L^2 norm of an approximate gradient (this imposes to separately prove the strong convergence of this approximate norm, and of an approximate gradient), whereas we consider in this paper a family of schemes such that exact norms of the discrete gradients are used, which allows to directly prove the strong convergence of the gradient from the convergence of its norm. Hence we can more easily consider in this paper the framework of a function $\nu(u, \nabla u)$ instead of $\nu(u, |\nabla u|)$, since we prove the strong convergence of the discrete gradient used in the discretisation of $\nu(u, \nabla u)$.
3. We present numerical schemes which can resume to 9-point stencil finite volume scheme (see Section 3.2, where the local elimination of interface unknowns is possible).
4. The proof that the discrete gradient and its norm are strongly convergent relies on Hypothesis (H5) instead of Leray-Lions method [12] (see (41)).

The main result of this paper is Theorem (5.5), which states the strong convergence of the discrete schemes to a solution of (6). It is proved thanks to the following property. Let F be the function defined by

$$\forall s \in \mathbb{R}_+,\ \ F(s) = \int_0^s z\mu(z)\mathrm{d}z \in \left[\mu_{\min}\frac{s^2}{2}, \mu_{\max}\frac{s^2}{2}\right]. \tag{7}$$

Then, for any sufficiently regular function u, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(|\nabla u(x,t)|)\mathrm{d}x = \int_\Omega \mu(|\nabla u(x,t)|)\nabla u(x,t)\cdot\nabla u_t(x,t)\mathrm{d}x\mathrm{d}t. \tag{8}$$

Therefore, assuming that this function u is solution of (2) with $f = 0$ for the sake of simplicity, we get, by taking $v = u_t$ in (6), that $\nabla u \in C^0([0,T]; L^2(\Omega))$ and

$$\int_0^T\int_\Omega \nu(u,\nabla u)u_t(x,t)^2\mathrm{d}x\mathrm{d}t + \int_\Omega F(|\nabla u(x,T)|)\mathrm{d}x = \int_\Omega F(|\nabla u_0(x)|)\mathrm{d}x. \tag{9}$$

The discrete equivalent of this property is shown in Lemma 4.1 for the fully-implicit scheme (using that $x \mapsto x\mu(x)$ is strictly increasing), and in Lemma 4.3 for the semi-implicit scheme (using that μ is decreasing). Note that the hypothesis that $x \mapsto x\mu(x)$ is strictly increasing is used in both schemes for the proof of the strong convergence of the discrete approximate of the gradient. Unfortunately, although it is possible to extend some of these properties to the case $\mu(x) = 1/x$, the convergence study provided in this paper does not hold in this framework.

Remark 2. Note that, thanks to the convergence result proved in this paper, we also prove the existence of a weak solution u of (2)-(3)-(4) in the sense of Definition 1.1, which satisfies, for all $T > 0$:

1. $u \in L^2(0,T; H^1_0(\Omega))$ and $u_t \in L^2(\Omega\times]0,T[)$ (hence $u \in C^0(0,T; L^2(\Omega))$),
2. $u(\cdot, 0) = u_0$,
3. $\mathrm{div}\,(\mu(|\nabla u|)\nabla u) \in L^2(\Omega\times]0,T[)$,
4. $\nu(u,\nabla u)u_t - \mathrm{div}\,(\mu(|\nabla u|)\nabla u) = f$ a.e. in $\Omega\times]0,T[$.

This paper is organised as follows. In Section 2, we present a family of discretisation tools, examples of which (case of rectangular or simplicial meshes) are given in Section 3. Then in Section 4, we show some estimates that are used on one hand in the proof of the existence of at least one solution to the fully implicit scheme, and of the existence and uniqueness of the solution to the semi-implicit scheme, on the other hand, in the convergence proof provided in Section 5. Finally, numerical results are given in Section 6.

2. The family of discrete schemes.

We now introduce the tools used for prescribing the space discretisation.

Definition 2.1 (Space discretisation)**.** Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. A space discretisation of Ω is defined by $\mathcal{D} = (H_{\mathcal{D}}, P_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where

1. We denote by $H_{\mathcal{D}}$ a finite dimension vector space on $\mathbb{R}$ (the component of any $u \in H_{\mathcal{D}}$ being the degrees of freedom of u).
2. We denote by $P_{\mathcal{D}} : H_0^1(\Omega) \to H_{\mathcal{D}}$ a linear operator (the interpolation operator).
3. We denote by $\nabla_{\mathcal{D}} : H_{\mathcal{D}} \to L^2(\Omega)^d$ a linear operator (reconstruction of the gradient), such that $\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}$ is a norm on $H_{\mathcal{D}}$, denoted by $\|u\|_{\mathcal{D}}$.
4. We denote by $\Pi_{\mathcal{D}} : H_{\mathcal{D}} \to L^2(\Omega)$ a linear function (reconstruction of the function). Therefore, we classically denote by $\|\Pi_{\mathcal{D}}\|_{\mathcal{L}(H_{\mathcal{D}}, L^2(\Omega))} = \sup\{\|\Pi_{\mathcal{D}} u\|_{L^2(\Omega)}$, $u \in H_{\mathcal{D}}$ with $\|u\|_{\mathcal{D}} = 1\}$.

We can notice that the operators used in Definition 2.1 are quite general, and provided by a large variety of discretisation schemes.

Remark 3. The easiest examples of such a discretisation are the conformal Lagrange finite element methods: let us assume that a finite family $(x_i)_{i\in I}$ of points of Ω is given, and that, for all $i \in I$, a function $\varphi_i \in H_0^1(\Omega)$ is defined such that $\varphi_i(x_i) = 1$ and $\varphi_i(x_j) = 0$ if $i \neq j$. We then denote $H_{\mathcal{D}} = \mathbb{R}^I$, the reconstruction operator is defined for $u \in H_{\mathcal{D}}$ by $\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i \varphi_i$, the interpolation operator is defined for $u \in H_0^1(\Omega)$ by $(P_{\mathcal{D}} u)_i = \frac{1}{|B(x_i,r)|}\int_{B(x_i,r)} u(x)\mathrm{d}x$ for some $r > 0$ with $B(x_i, r) \subset \Omega$. Then $\nabla_{\mathcal{D}}$ is defined by $\nabla_{\mathcal{D}} u = \sum_{i\in I} u_i \nabla\varphi_i$. This example will not be further considered in this paper, since we prefer focusing on non-conformal methods including finite volume ones.

Remark 4. The examples which are provided in Section 3 can be seen as non-conformal finite elements, since they provide an external approximation of the continuous problem using a discrete variational formulation. But they also can be seen as finite volume methods, since the discrete variational formulation leads to discrete balance equations in a partition of the domain, and since the reconstruction of the solution is piecewise constant in the mesh. Therefore the elements of the mesh can at the same time be called "finite elements" or "finite volumes".

Let us now turn to space-time discretisations.

Definition 2.2 (Space-time discretisation)**.** Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ and let $T > 0$ be given. We say that $(\mathcal{D}, \tau)$ is a space-time discretisation of $\Omega\times]0, T[$ if $\mathcal{D}$ is a space discretisation of Ω in the sense of Definition 2.1 and if there exists $N_T \in \mathbb{N}$ with $T = N_T\tau$, where $\tau > 0$ is time step. We denote by $H_{\mathcal{D},\tau}$ (resp. $H^\star_{\mathcal{D},\tau}$) the set of all $u = (u^n)_{n=0,\dots,N_T}$ (resp. $u = (u^n)_{n=1,\dots,N_T}$) with $u^n \in H_{\mathcal{D}}$ for all $n = 0, \dots, N_T$ (resp. $n = 1, \dots, N_T$). We denote for all $u \in H^\star_{\mathcal{D},\tau}$ (resp. $u \in H_{\mathcal{D},\tau}$), by $\Pi_{\mathcal{D},\tau} u \in L^2(\Omega\times]0, T[)$ and $\nabla_{\mathcal{D},\tau} u \in L^2(\Omega\times]0, T[)^d$ (resp. $\widetilde{\Pi}_{\mathcal{D},\tau} u \in L^2(\Omega\times]0, T[)$ and $\widetilde{\nabla}_{\mathcal{D},\tau} u \in L^2(\Omega\times]0, T[)^d$) the functions defined by $\Pi_{\mathcal{D},\tau} u(x, t) = \Pi_{\mathcal{D}} u^n(x)$ and $\nabla_{\mathcal{D},\tau} u(x, t) = \nabla_{\mathcal{D}} u^n(x)$ (resp. $\widetilde{\Pi}_{\mathcal{D},\tau} u(x, t) = \Pi_{\mathcal{D}} u^{n-1}(x)$ and $\widetilde{\nabla}_{\mathcal{D},\tau} u(x, t) = \nabla_{\mathcal{D}} u^{n-1}(x)$) for a.e. $x \in \Omega$ and all

$t \in](n-1)\tau, n\tau]$. We finally denote, for all $u \in H_{\mathcal{D},\tau}$, by $D_\tau u$ the function defined by $D_\tau u(x,t) = \Pi_{\mathcal{D}}(u^n - u^{n-1})(x)/\tau$ for a.e. $(x,t) \in \Omega \times](n-1)\tau, n\tau[$,

We finally define $P_{\mathcal{D},\tau} \ : \ L^2(0,T;H^1_0(\Omega)) \to H^\star_{\mathcal{D},\tau}$,

by $(P_{\mathcal{D},\tau} v)^n = P_{\mathcal{D}}\left(\frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} v(\cdot,t)\mathrm{d}t\right)$, for $n = 1,\ldots,N_T$.

Let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times]0,T[$.

We now define two numerical schemes. The fully implicit scheme is defined by

$$\begin{aligned} & u \in H_{\mathcal{D},\tau}, \quad u^0 = P_{\mathcal{D}} u_0, \\ & \int_0^T \int_\Omega \big(\nu(\Pi_{\mathcal{D},\tau} u, \nabla_{\mathcal{D},\tau} u) D_\tau u \Pi_{\mathcal{D},\tau} v + \mu(|\nabla_{\mathcal{D},\tau} u|)\nabla_{\mathcal{D},\tau} u \cdot \nabla_{\mathcal{D},\tau} v\big)\mathrm{d}x\mathrm{d}t \\ & = \int_0^T \int_\Omega f \ \Pi_{\mathcal{D},\tau} v \ \mathrm{d}x\mathrm{d}t, \ \forall v \in H^\star_{\mathcal{D},\tau}. \end{aligned} \tag{10}$$

The semi-implicit scheme is defined by

$$\begin{aligned} & u \in H_{\mathcal{D},\tau}, \quad u^0 = P_{\mathcal{D}} u_0, \\ & \int_0^T \int_\Omega \big(\nu(\widetilde{\Pi}_{\mathcal{D},\tau} u, \widetilde{\nabla}_{\mathcal{D},\tau} u) D_\tau u \Pi_{\mathcal{D},\tau} v + \mu(|\widetilde{\nabla}_{\mathcal{D},\tau} u|)\nabla_{\mathcal{D},\tau} u \cdot \nabla_{\mathcal{D},\tau} v\big)\mathrm{d}x\mathrm{d}t \\ & = \int_0^T \int_\Omega f \ \Pi_{\mathcal{D},\tau} v \ \mathrm{d}x\mathrm{d}t, \ \forall v \in H^\star_{\mathcal{D},\tau}. \end{aligned} \tag{11}$$

Definition 2.3 (Admissible sequence of space discretisations)**.** Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ (where $\mathbb{N}^\star$ denotes the set $\mathbb{N}\setminus\{0\}$). We say that $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an admissible sequence of space discretisations of Ω if the following conditions are fulfilled:

1. there exists $C > 0$ such that
$$\|\Pi_{\mathcal{D}_m}\|_{\mathcal{L}(H_{\mathcal{D}_m}, L^2(\Omega))} \le C, \ \forall m \in \mathbb{N}, \tag{12}$$
and
$$\|\nabla_{\mathcal{D}_m} P_{\mathcal{D}_m} v\|_{L^2(\Omega)^d} \le C\|\nabla v\|_{L^2(\Omega)^d}, \ \forall m \in \mathbb{N}, \ \forall v \in H^1_0(\Omega), \tag{13}$$
2. the following consistency property holds
$$\lim_{m\to\infty} \big(\|v - \Pi_{\mathcal{D}_m} P_{\mathcal{D}_m} v\|_{L^2(\Omega)} + \|\nabla v - \nabla_{\mathcal{D}_m} P_{\mathcal{D}_m} v\|_{L^2(\Omega)^d}\big) = 0, \ \forall v \in H^1_0(\Omega), \tag{14}$$
3. the following compactness property holds: for all sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in H_{\mathcal{D}_m}$ such that there exists $C > 0$ with $\|u_m\|_{\mathcal{D}_m} \le C$ for all $m \in \mathbb{N}$, then there exists u in $L^2(\Omega)$ such that, up to a sub-sequence, $\Pi_{\mathcal{D}_m} u_m$ converges to u in $L^2(\Omega)$,
4. for all sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in H_{\mathcal{D}_m}$ such that there exists $C > 0$ with $\|u_m\|_{\mathcal{D}_m} \le C$ for all $m \in \mathbb{N}$, and such that there exists $u \in L^2(\Omega)$ such that $\Pi_{\mathcal{D}_m} u_m$ converges to u in $L^2(\Omega)$, then $\nabla_{\mathcal{D}_m} u_m$ converges to ∇u for the weak topology of $L^2(\mathbb{R})^d$, prolonging by 0 all functions outside Ω.

Remark 5. The required compactness property 3 can result from different analysis frameworks. In the case of the conformal finite elements (see Remark 3) it is a consequence of Rellich's theorem. In the case of the examples below, which are inspired by the finite volume framework, this compactness property is a result of the discrete functional analysis proposed in [7].

Remark 6. It results from the above definition that, if a sequence $(u_m)_{m\in\mathbb{N}}$ with $u_m \in H_{\mathcal{D}_m}$ is such that there exists $C > 0$ with $\|u_m\|_{\mathcal{D}_m} \le C$ for all $m \in \mathbb{N}$, and that there exists u in $L^2(\Omega)$ such that u_m converges to u in $L^2(\Omega)$, then $u \in H^1_0(\Omega)$.

Definition 2.4 (Admissible sequence of space-time discretisations)**.** Let Ω be an open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ and let $T > 0$. We say that $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ is an admissible sequence of space-time discretisations of $\Omega\times]0,T[$ if $(\mathcal{D}_m, \tau_m)$ is a space-time discretisation of $\Omega\times]0,T[$ in the sense of Definition 2.2 for all $m \in \mathbb{N}$, if $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an admissible sequence of space discretisations of Ω in the sense of Definition 2.3, and if $(\tau_m)_{m\in\mathbb{N}}$ converges to 0.

Remark 7. It results from the above definition that, for all $v \in L^2(0,T;H_0^1(\Omega))$, thanks to dominated convergence, $\nabla_{\mathcal{D}_m,\tau_m} P_{\mathcal{D}_m,\tau_m} v$ converges to ∇v in $L^2(\Omega\times]0,T[)^d$ and $\Pi_{\mathcal{D}_m,\tau_m} P_{\mathcal{D}_m,\tau_m} v$ converges to v in $L^2(\Omega\times]0,T[)$.

The next section is devoted to the presentation of precise examples of space discretisations, and to the detailed expression of Schemes (10) and (11) in these cases.

3. **Examples of non-conformal space discretisations.** Since the main applications which are considered are devoted to image processing, we first focus on non conformal rectangular finite elements on rectangular domains, and then on non conformal simplicial finite elements on polygonal domains. All these non conformal finite element methods can also be seen as finite volume methods.

3.1. **A first scheme on rectangular domains.** We consider the particular case where $\Omega =]a_1, b_1[\times \ldots \times]a_d, b_d[$ is an open rectangle in $\mathbb{R}^d$.

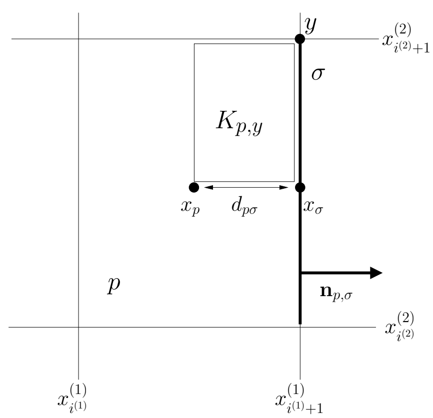

FIGURE 1. Notations for rectangular meshes of Section 3.1 and Section 3.2

Definition 3.1. A space discretisation in the sense of Definition 2.1 is defined by the following way (see Figure 1).

1. A rectangular discretisation of Ω is defined by the increasing sequences $a_i = x_0^{(i)} < x_1^{(i)} < \ldots < x_{n^{(i)}}^{(i)} = b_i$, $i = 1, \ldots, d$.
2. We denote by

$$\mathcal{M} = \left\{]x^{(1)}_{i^{(1)}}, x^{(1)}_{i^{(1)}+1}[\times \ldots \times]x^{(d)}_{i^{(d)}}, x^{(d)}_{i^{(d)}+1}[,\ 0 \le i^{(1)} < n^{(1)},\ \ldots,\ 0 \le i^{(d)} < n^{(d)} \right\}$$

the set of the control volumes. The elements of $\mathcal{M}$ are denoted $p, q, \ldots$. We denote by x_p the centre of p. For any $p \in \mathcal{M}$, let $\partial p = \overline{p} \setminus p$ be the boundary of p; let $|p| > 0$ denote the measure of p and let h_p denote the diameter of p and $h_{\mathcal{D}}$ denote the maximum value of $(h_p)_{p\in\mathcal{M}}$.

3. We denote by $\mathcal{E}_p$ the set of all the faces of $p \in \mathcal{M}$, by $\mathcal{E}$ the union of all $\mathcal{E}_p$, and for all $\sigma \in \mathcal{E}$, we denote by $|\sigma|$ its $(d-1)$-dimensional measure. For any $\sigma \in \mathcal{E}$, we define the set $\mathcal{M}_\sigma = \{p \in \mathcal{M}, \sigma \in \mathcal{E}_p\}$ (which has therefore one or two elements), we denote by $\mathcal{E}_p$ the set of the faces of $p \in \mathcal{M}$ (it has $2\,d$ elements) and by x_σ the centre of σ. We then denote by $d_{p\sigma} = |x_\sigma - x_p|$ the orthogonal distance between x_p and $\sigma \in \mathcal{E}_p$ and by $\mathbf{n}_{p,\sigma}$ the normal vector to σ, outward to p.
4. We denote by $\mathcal{V}_p$ the set of all the vertexes of $p \in \mathcal{M}$ (it has 2^d elements), by $\mathcal{V}$ the union of all $\mathcal{V}_p$, $p \in \mathcal{M}$. For $y \in \mathcal{V}_p$, we denote by $K_{p,y}$ the rectangle whose faces are parallel to those of p, and whose the set of vertexes contains x_p and y. We denote by $\mathcal{V}_\sigma$ the set of all vertexes of $\sigma \in \mathcal{E}$ (it has 2^{d-1} elements), and by $\mathcal{E}_{p,y}$ the set of all $\sigma \in \mathcal{E}_p$ such that $y \in \mathcal{V}_\sigma$ (it has d elements).
5. We define the set $H_\mathcal{D}$ of all $u \in \mathbb{R}^\mathcal{M} \times \mathbb{R}^\mathcal{E}$, with $u_\sigma = 0$ for $\sigma \subset \partial\Omega$ and $n = 1, \ldots, N_T$.
6. We denote, for all $u \in H_\mathcal{D}$, by $\Pi_\mathcal{D} u \in L^2(\Omega)$ the function defined by the constant value u_p a.e. in $p \in \mathcal{M}$.
7. We denote, for all $v \in H_0^1(\Omega)$, by $P_\mathcal{D} v \in H_\mathcal{D}$ the element defined by $(P_\mathcal{D} v)_p = \frac{1}{|p|}\int_p v(x)\mathrm{d}x$ for all $p \in \mathcal{M}$, and by $(P_\mathcal{D} v)_\sigma = \frac{1}{|\sigma|}\int_\sigma v(x)\mathrm{d}s(x)$ for all $\sigma \in \mathcal{E}$.
8. For $u \in H_\mathcal{D}$, $p \in \mathcal{M}$ and $y \in \mathcal{V}_p$, we denote by
$$\nabla_{p,y} u = \frac{2}{|p|} \sum_{\sigma \in \mathcal{E}_{p,y}} |\sigma|(u_\sigma - u_p)\mathbf{n}_{p,\sigma} = \sum_{\sigma \in \mathcal{E}_{p,y}} \frac{u_\sigma - u_p}{d_{p\sigma}} \mathbf{n}_{p,\sigma}, \tag{15}$$
and by $\nabla_\mathcal{D} u$ the function defined a.e. on Ω by $\nabla_{p,y} u$ on $K_{p,y}$.

We then have the following result.

Lemma 3.2. *Let $\Omega =]a_1, b_1[\times \ldots \times]a_d, b_d[$ be an open rectangle in $\mathbb{R}^d$. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$, with $\mathcal{D}_m = (H_{\mathcal{D}_m}, P_{\mathcal{D}_m}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ be defined by Definition 3.1, where $h_m = \max(x^{(j)}_{i^{(j)}+1} - x^{(j)}_{i^{(j)}})$ tends to 0 as $m \to \infty$, with $\theta_m = (x^{(j)}_{i^{(j)}+1} - x^{(j)}_{i^{(j)}})/(x^{(k)}_{i^{(k)}+1} - x^{(k)}_{i^{(k)}})$ remains bounded.*

Then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an admissible sequence of space discretisations of Ω in the sense of Definition 2.3.

Proof. Let us recall the result, proved in [7]: for a discretization $\mathcal{D}$ in the sense of Definition 3.1, then the expression $\|u\|_\mathcal{D}$, defined by
$$\|u\|_\mathcal{D}^2 = \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma - u_p)^2,\ \forall u \in H_\mathcal{D},$$
is a norm on $H_\mathcal{D}$ such that the discrete Poincaré inequality
$$\|\Pi_\mathcal{D} u\|_{L^2(\Omega)} \le C\|u\|_\mathcal{D},\ \forall u \in H_\mathcal{D},$$
holds, where C only depends on the bound on θ. Moreover, one has that
$$\|\nabla_\mathcal{D} u\|_{L^2(\Omega)^d} = \|u\|_\mathcal{D},\ \forall u \in H_\mathcal{D}.$$
Then the (12) is an immediate consequence of the above relations. Inequality (13) is an immediate consequence of the inequality
$$((P_\mathcal{D} v)_p - (P_\mathcal{D} v)_\sigma)^2 \le C\frac{\mathrm{diam}(p)}{|\sigma|}\|\nabla v\|^2_{L^2(\Omega)^d},$$
proved p777 in [6]. Property (14) is a straightforward for any $v \in C_c^\infty(\Omega)$ by consistency. It therefore holds for all $v \in H_0^1(\Omega)$ by density and continuity of $P_\mathcal{D}$ on $H_0^1(\Omega)$. The compactness property (point 3 of Definition 2.3) is proved in [7]. □

Let us write the schemes (10) and (11) in this case. We first choose for test function $v \in H^{\star}_{\mathcal{D},\tau}$, the function such that $v_p^n = 1$ for a given $p \in \mathcal{M}$ and $n = 1,\dots,N_T$, and all other components equal to 0. We get

$$|p|\nu_p^m(u_p^n - u_p^{n-1}) - \tau \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|\mu_{p,\sigma}^m}{d_{p\sigma}}(u_\sigma^n - u_p^n) = f_p^n, \tag{16}$$

where we set

$$2^d\nu_p^m = \sum_{y\in\mathcal{V}_p} \nu(u_p^m, \nabla_{p,y}u^m) \text{ and } 2^{d-1}\mu_{p,\sigma}^m = \sum_{y\in\mathcal{V}_\sigma} \mu(|\nabla_{p,y}u^m|), \tag{17}$$

with $m = n$ for (10) and $m = n-1$ for (11), and

$$f_p^n = \int_{(n-1)\tau}^{n\tau}\int_p f(x,t)\mathrm{d}x\mathrm{d}t.$$

We then choose for test function $v \in H^{\star}_{\mathcal{D},\tau}$, the function such that $v_\sigma^n = 1$ for a given interior face σ common to both control volumes $p, q \in \mathcal{M}$ and $n = 1,\dots,N_T$, and all other components equal to 0. We obtain

$$\frac{\mu_{p,\sigma}^m}{d_{p\sigma}}(u_\sigma^n - u_p^n) + \frac{\mu_{q,\sigma}^m}{d_{q\sigma}}(u_\sigma^n - u_q^n) = 0.$$

The above expression allows, in the case of Scheme (11), for eliminating u_σ^n with respect to u_p^n and u_q^n. It is then easy to derive an L^∞ estimate in this case, which resumes to L^∞ stability if $f = 0$. Indeed, after the elimination of u_σ^n, consider the maximum value of u_p^n. Thanks to the sign of the coefficients, we get that this maximum value must be comprised between u_p^{n-1} and that of all u_q^n, for q neighbour of p. Therefore, we obtain that it must be bounded by the maximum values of all u_p^{n-1} and the boundary value 0. The same reasoning holds for the minimum value, hence providing the L^∞ estimate.

3.2. **A second scheme on rectangular domains.** We again consider the particular case where $\Omega =]a_1, b_1[\times\ldots\times]a_d, b_d[$ is an open rectangle in $\mathbb{R}^d$.

Definition 3.3. A space discretisation in the sense of Definition 2.1 is now defined by the following method (see again Figure 1).

1-4. identical to [1-4] section 3.1.

5. We define the set $H_{\mathcal{D}}$ of all $u = ((u_p)_{p\in\mathcal{M}}, (u_{\sigma,y})_{\sigma\in\mathcal{E},y\in\mathcal{V}_\sigma})$, with $u_{\sigma,y} = 0$ for $\sigma \subset \partial\Omega$, $y \in \mathcal{V}_\sigma$ and $n = 1,\dots,N_T$.
6. We denote, for all $u \in H_{\mathcal{D}}$, by $\Pi_{\mathcal{D}}u \in L^2(\Omega)$ the function defined by the constant value u_p a.e. in $p \in \mathcal{M}$.
7. We denote, for all $v \in H_0^1(\Omega)$, by $P_{\mathcal{D}}v \in H_{\mathcal{D}}$ the element defined by $(P_{\mathcal{D}}v)_p = \frac{1}{|p|}\int_p v(x)\mathrm{d}x$ for all $p \in \mathcal{M}$, and by $(P_{\mathcal{D}}v)_{\sigma,y} = \frac{1}{|\sigma|}\int_\sigma v(x)\mathrm{d}s(x)$ for all $\sigma \in \mathcal{E}$ and $y \in \mathcal{V}_\sigma$.
8. For $u \in H_{\mathcal{D}}$, $p \in \mathcal{M}$ and $y \in \mathcal{V}_p$, we denote by

$$\nabla_{p,y}u = \frac{2}{|p|}\sum_{\sigma\in\mathcal{E}_{p,y}} |\sigma|(u_{\sigma,y} - u_p)\mathbf{n}_{p,\sigma} = \sum_{\sigma\in\mathcal{E}_{p,y}} \frac{u_{\sigma,y} - u_p}{d_{p\sigma}}\mathbf{n}_{p,\sigma}, \tag{18}$$

and by $\nabla_{\mathcal{D}}u$ the function defined a.e. on Ω by $\nabla_{p,y}u$ on $K_{p,y}$.

Remark 8. This definition differs from that of section (3.1) by the use of 2^{d-1} different unknowns $u_{\sigma,y}$ at the interface σ instead of only one u_σ.

We then have the following result, the proof of which is similar to that of Lemma 3.2.

Lemma 3.4. *Let $\Omega =]a_1, b_1[\times \ldots \times]a_d, b_d[$ be an open rectangle in $\mathbb{R}^d$. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$, with $\mathcal{D}_m = (H_{\mathcal{D}_m}, P_{\mathcal{D}_m}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ be defined by Definition 3.3, where $h_m = \max(x^{(j)}_{i^{(j)}+1} - x^{(j)}_{i^{(j)}})$ tends to 0 as $m \to \infty$, with $\theta_m = (x^{(j)}_{i^{(j)}+1} - x^{(j)}_{i^{(j)}})/(x^{(k)}_{i^{(k)}+1} - x^{(k)}_{i^{(k)}})$ remains bounded.*

Then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an admissible sequence of space discretisations of Ω in the sense of Definition 2.3.

Let us write the schemes (10) and (11) in this case. For a given $p \in \mathcal{M}$ and $n = 1, \ldots, N_T$, we get

$$|p|\nu^m_p(u^n_p - u^{n-1}_p) - \tau \sum_{\sigma\in\mathcal{E}_p}\sum_{y\in\mathcal{V}_\sigma} \frac{|\sigma|\mu(|\nabla_{p,y}u^m|)}{2^{d-1}d_{p\sigma}}(u^n_{\sigma,y} - u^n_p) = f^n_p, \tag{19}$$

where we set

$$2^d\nu^m_p = \sum_{y\in\mathcal{V}_p} \nu(u^m_p, \nabla_{p,y}u^m),$$

with $m = n$ for (10) and $m = n - 1$ for (11), and

$$f^n_p = \int_{(n-1)\tau}^{n\tau}\int_p f(x,t)\mathrm{d}x\mathrm{d}t.$$

For a given interior σ common to $p, q \in \mathcal{M}$, $y \in \mathcal{V}_\sigma$ and $n = 1, \ldots, N_T$, we have

$$\frac{\mu(|\nabla_{p,y}u^m|)}{d_{p\sigma}}(u^n_{\sigma,y} - u^n_p) + \frac{\mu(|\nabla_{q,y}u^m|)}{d_{q\sigma}}(u^n_{\sigma,y} - u^n_q) = 0. \tag{20}$$

Again, the above expression allows to eliminate $u^n_{\sigma,y}$ in the case of Scheme (11), and an L^∞ estimate is derived, following the same reasoning as the one which is described at the end of the previous section.

3.3. **A scheme applying on simplicial meshes.** This scheme has a few common points with the scheme presented in Section 3.2, although we now consider that Ω be an open bounded polyhedron in $\mathbb{R}^d$.

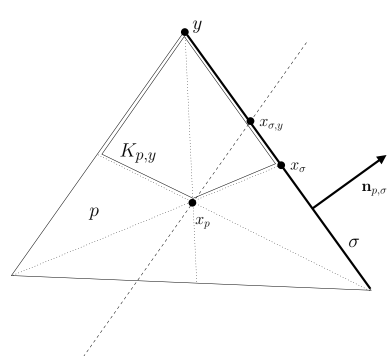

FIGURE 2. Notations for simplicial meshes of Section 3.3

Definition 3.5. A space discretisation in the sense of Definition 2.1 is now defined by the following method (see Figure 2).

1. We denote by $\mathcal{M}$ a set of disjoint open simplicial domains (triangles in 2D, tetrahedrons in 3D), such that $\overline{\Omega} = \bigcup_{p\in\mathcal{M}} \overline{p}$. The elements of $\mathcal{M}$ are denoted $p, q, \ldots$. We denote by x_p the centre of gravity of p. For any $p \in \mathcal{M}$, let $\partial p = \overline{p} \setminus p$ be the boundary of p; let $|p| > 0$ denote the measure of p and let h_p denote the diameter of p and $h_{\mathcal{D}}$ denote the maximum value of $(h_p)_{p\in\mathcal{M}}$.
2. We denote by $\mathcal{E}_p$ the set of all the faces of $p \in \mathcal{M}$, by $\mathcal{E}$ the union of all $\mathcal{E}_p$, and for all $\sigma \in \mathcal{E}$, we denote by $|\sigma|$ its $(d-1)$-dimensional measure. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{p \in \mathcal{M}, \sigma \in \mathcal{E}_p\}$. We assume that the mesh is conformal, in the sense that, if $\mathcal{M}_\sigma$ has one element, then $\sigma \subset \partial\Omega$ and if $\mathcal{M}_\sigma$ has two elements, then $\sigma \subset \Omega$. We then denote by $\mathcal{E}_p$ the faces of $p \in \mathcal{M}$ (it has $d+1$ elements) and by x_σ the centre of gravity of σ. We then denote $\mathbf{n}_{p,\sigma}$ the normal vector to σ, outward to p.
3. We denote by $\mathcal{V}_p$ the set of all the vertexes of $p \in \mathcal{M}$ (it has $d+1$ elements), by $\mathcal{V}$ the union of all $\mathcal{V}_p$, $p \in \mathcal{M}$. For $y \in \mathcal{V}_p$, we denote by $K_{p,y}$ the polyhedron, defined as the set of all $x \in p$ such that the barycentric coordinates $(s_{y'})_{y'\in\mathcal{V}_p}$ of x satisfy $s_y = \max_{y'\in\mathcal{V}_p} s_{y'}$ (recall that $(s_{y'})_{y'\in\mathcal{V}_p}$ is defined by $x - x_p = \sum_{y'\in\mathcal{V}_p} s_{y'}(y' - x_p)$, such that $s_{y'} \geq 0$ and $\sum_{y'\in\mathcal{V}_p} s_{y'} = 1$). We denote by $\mathcal{V}_\sigma$ the set of all vertexes of $\sigma \in \mathcal{E}$ (it has d elements), and by $\mathcal{E}_{p,y}$ the set of all $\sigma \in \mathcal{E}_p$ such that $y \in \mathcal{V}_\sigma$ (it has d elements). We then denote, for $\sigma \in \mathcal{E}$ and $y \in \mathcal{V}_\sigma$, by $x_{\sigma,y}$ the point of σ defined by the barycentric coordinates (in σ) $(s_{y'})_{y'\in\mathcal{V}_\sigma}$, such that $s_{y'} = 1/(d+1)$ for all $y' \in \mathcal{V}_\sigma \setminus \{y\}$ (therefore $s_y = 2/(d+1)$).
4. We define the set $H_{\mathcal{D}}$ of all $u = ((u_p)_{p\in\mathcal{M}}, (u_{\sigma,y})_{\sigma\in\mathcal{E}, y\in\mathcal{V}_\sigma})$, with $u_{\sigma,y} = 0$ for $\sigma \subset \partial\Omega$, $y \in \mathcal{V}_\sigma$ and $n = 1, \ldots, N_T$.
5. We denote, for all $u \in H_{\mathcal{D}}$, by $\Pi_{\mathcal{D}} u \in L^2(\Omega)$ the function defined by the constant value u_p a.e. in $p \in \mathcal{M}$.
6. We denote, for all $v \in H^1_0(\Omega)$, by $P_{\mathcal{D}} v \in H_{\mathcal{D}}$ the element defined by $(P_{\mathcal{D}} v)_p = \frac{1}{|p|}\int_p v(x)\mathrm{d}x$ for all $p \in \mathcal{M}$. We denote by σ_y the subset of all $x \in \sigma$ such that the barycentric coordinates $(s_{y'})_{y'\in\mathcal{V}_\sigma}$ of x in σ satisfy $s_y > 1/2$, and we set $(P_{\mathcal{D}} v)_{\sigma,y} = \frac{d-1}{(d+1)|\sigma|}\int_\sigma v(x)\mathrm{d}s(x) + \frac{2}{(d+1)|\sigma_y|}\int_{\sigma_y} v(x)\mathrm{d}s(x)$ for all $\sigma \in \mathcal{E}$ and $y \in \mathcal{V}_\sigma$ (hence computing a second order approximation at point $x_{\sigma,y}$).
7. For $u \in H_{\mathcal{D}}$, $p \in \mathcal{M}$ and $y \in \mathcal{V}_p$, we denote by
$$\nabla_{p,y} u = \frac{d+1}{|p|} \sum_{\sigma\in\mathcal{E}_{p,y}} \frac{|\sigma|}{d}(u_{\sigma,y} - u_p)\mathbf{n}_{p,\sigma}, \tag{21}$$
and by $\nabla_{\mathcal{D}} u$ the function defined a.e. on Ω by $\nabla_{p,y} u$ on $K_{p,y}$.

Remark 9. Note that the measure of $K_{p,y}$ is $|p|/(d+1)$ (this is easily shown, considering the affine transformation which sends p to a tetrahedron with all edges equal).

We then have the following result.

Lemma 3.6. *Let Ω be an open bounded polyhedron in $\mathbb{R}^d$. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$, with $\mathcal{D}_m = (H_{\mathcal{D}_m}, P_{\mathcal{D}_m}, \Pi_{\mathcal{D}_m}, \nabla_{\mathcal{D}_m})$ be defined by Definition 3.5, where the $h_m = \max_{p\in\mathcal{M}} \mathrm{diam}(p)$ tends to 0 as $m \to \infty$, with $\theta_m = \min_{p\in\mathcal{M}} \mathrm{diam}(p)/\rho_p$ (ρ_p being the supremum of the diameter of any ball included in p) remains bounded.*

Then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is an admissible sequence of space discretisations of Ω in the sense of Definition 2.3.

Proof. The proof of this lemma relies on the remarkable expression (21) (see [2, 13, 14]). Firstly, it allows similar compactness steps to that of the proof of Lemma 3.2. Secondly, this expression is consistent with regular functions, in the sense that, if there exists $G \in \mathbb{R}^d$ such that $u_p = G \cdot x_p$ and $u_{\sigma,y} = G \cdot x_{\sigma,y}$, then $\nabla_{p,y} u = G$. This consistency relation allows for the verification of the other properties required by Definition 2.3. $\square$

Let us write the schemes (10) and (11) in this case. For a given $p \in \mathcal{M}$ and $n = 1, \ldots, N_T$, we get

$$|p|\nu_p^m(u_p^n - u_p^{n-1}) - \tau \sum_{\sigma \in \mathcal{E}_p} \sum_{y \in \mathcal{V}_\sigma} \frac{|\sigma|}{d} \mu(|\nabla_{p,y} u^m|) \nabla_{p,y} u^n \cdot \mathbf{n}_{p,\sigma} = f_p^n, \tag{22}$$

where we set

$$(d+1)\nu_p^m = \sum_{y \in \mathcal{V}_p} \nu(u_p^m, \nabla_{p,y} u^m),$$

with $m = n$ for (10) and $m = n - 1$ for (11), and

$$f_p^n = \int_{(n-1)\tau}^{n\tau} \int_p f(x,t) \mathrm{d}x \mathrm{d}t.$$

For a given interior σ common to $p, q \in \mathcal{M}$, $y \in \mathcal{V}_\sigma$ and $n = 1, \ldots, N_T$, we have

$$\mu(|\nabla_{p,y} u^m|) \nabla_{p,y} u^n \cdot \mathbf{n}_{p,\sigma} = \mu(|\nabla_{q,y} u^m|) \nabla_{q,y} u^n \cdot \mathbf{n}_{p,\sigma}. \tag{23}$$

In the case of Scheme (11), the previous relations allow to eliminate the values $u_{\sigma,y}^n$ for all edges $y \in \mathcal{V}_\sigma$, as linear combinations of all u_q^n such that $y \in \mathcal{V}_q$. The resulting linear system only depends on the cell centred values, and the stencil connects all pairs of simplexes with a common vertex. In this linear system, the signs of the coefficients depend on the geometry, preventing from an easy derivation of an L^∞ estimate.

4. Properties of the schemes.

Before focusing on the estimates satisfied by the approximate solutions, we first present a few properties which are useful in the convergence study.

4.1. Estimates and existence of a solution to the fully implicit scheme.

Lemma 4.1. $L^2(\Omega\times]0,T[)$ estimate on $D_\tau u$ and $L^\infty(0,T;H_\mathcal{D})$ estimate, fully implicit scheme.

Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}, \tau)$ be a space-time discretisation of $\Omega\times]0,T[$ in the sense of Definition 2.2. Let $u \in H_{\mathcal{D},\tau}$ be a solution of (10). Then it holds:

$$\begin{aligned} &\nu_{\min} \int_0^{m\tau} \int_\Omega D_\tau u(x,t)^2 \mathrm{d}x\mathrm{d}t + \mu_{\min} \|\nabla_\mathcal{D} u^m\|_{L^2(\Omega)^d}^2 \\ &\le \mu_{\max} \|\nabla_\mathcal{D} P_\mathcal{D} u_0\|_{L^2(\Omega)^d}^2 + \frac{1}{\nu_{\min}} \|f\|_{L^2(\Omega\times]0,T[)}^2, \ \forall m = 1, \ldots, N_T. \end{aligned} \tag{24}$$

Proof. We set $v = D_\tau u$ in the scheme and in (10) we integrate in time on the interval $]0, m\tau[$, for $m = 1, \ldots, N_T$. Let us remark that, thanks to Hypothesis (H5) which implies the convexity of F, we have

$$\forall c_1, c_2 \in \mathbb{R}_+, F(c_2) - F(c_1) = \int_{c_1}^{c_2} z\mu(z) \mathrm{d}z \le c_2 \mu(c_2)(c_2 - c_1). \tag{25}$$

We can then write

$$F(|\nabla_{\mathcal{D}} u^n(x)|) - F(|\nabla_{\mathcal{D}} u^{n-1}(x)|) \leq \mu(|\nabla_{\mathcal{D}} u^n(x)|)|\nabla_{\mathcal{D}} u^n(x)|(|\nabla_{\mathcal{D}} u^n(x)| - |\nabla_{\mathcal{D}} u^{n-1}(x)|).$$

Note that the Cauchy-Schwarz inequality implies

$$|\nabla_{\mathcal{D}} u^n(x)|(|\nabla_{\mathcal{D}} u^n(x)| - |\nabla_{\mathcal{D}} u^{n-1}(x)|) \leq \nabla_{\mathcal{D}} u^n(x) \cdot (\nabla_{\mathcal{D}} u^n(x) - \nabla_{\mathcal{D}} u^{n-1}(x)).$$

Thanks to property (7), and to the Young inequality applied to the right hand side, we conclude (24). □

Lemma 4.2 (Existence of at least one solution to the fully implicit scheme)**.** *Under Hypotheses (H), let $(\mathcal{D}, \tau)$ be a space-time discretisation of $\Omega\times]0, T[$ in the sense of Definition 2.2. Then there exists at least one $u \in H_{\mathcal{D},\tau}$ such that (10) holds.*

Proof. We first define, for any $\lambda \in [0, 1]$, the functions μ_λ and ν_λ by $\mu_\lambda(s) = \mu_{\max}(1-\lambda) + \lambda\mu(s)$ and $\nu_\lambda(s, \xi) = \nu_{\min}(1-\lambda) + \lambda\nu(s, \xi)$. Since estimate (24) holds independently of λ, since the problem is linear for $\lambda = 0$, the topological degree argument [9], applied to the function $\Phi : H^\star_{\mathcal{D},\tau} \to H^\star_{\mathcal{D},\tau}$ defined by

$$\begin{aligned}\Phi(u)^n_i = \ & \int_\Omega \nu_\lambda(u^n(x), \nabla_{\mathcal{D}} u^n(x))(\Pi_{\mathcal{D}} u^n(x) - \Pi_{\mathcal{D}} u^{n-1}(x))\Pi_{\mathcal{D}} v_i(x)\mathrm{d}x \\ & +\tau \int_\Omega \mu_\lambda(|\nabla_{\mathcal{D}} u^n(x)|)\nabla_{\mathcal{D}} u^n(x) \cdot \nabla_{\mathcal{D}} v_i(x)\mathrm{d}x \\ & - \int_\Omega \int_{(n-1)\tau}^{n\tau} f(x, t)\mathrm{d}t\Pi_{\mathcal{D}} v_i(x)\mathrm{d}x,\end{aligned}$$

where $(v_i)_{i=1,\ldots,M}$ is a basis of $H_{\mathcal{D}}$, ensures the existence of at least one solution to Scheme (10) . □

Lemma 4.3. **$L^2(\Omega\times]0, T[)$ estimate on u_t and $L^\infty(0, T; H_{\mathcal{D}})$ estimate, semi-implicit scheme.** *Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}, \tau)$ be a space-time discretisation of $\Omega\times]0, T[$ in the sense of Definition 2.2. Let $u \in H_{\mathcal{D},\tau}$ be a solution of (11). Then it holds:*

$$\begin{aligned}& \nu_{\min} \int_0^{m\tau} D_\tau u(x, t)^2 \mathrm{d}x\mathrm{d}t + \mu_{\min}\|\nabla_{\mathcal{D}} u^m\|^2_{L^2(\Omega)^d} \\ & +\mu_{\min} \sum_{n=1}^{m} \int_\Omega (|\nabla_{\mathcal{D}} u^n(x)| - |\nabla_{\mathcal{D}} u^{n-1}(x)|)^2 \mathrm{d}x \\ & \leq \mu_{\max}\|\nabla_{\mathcal{D}} P_{\mathcal{D}} u_0\|^2_{L^2(\Omega)^d} + \frac{1}{\nu_{\min}}\|f\|^2_{L^2(\Omega\times]0,T[)},\ \forall m = 1, \ldots, N_T,\end{aligned} \tag{26}$$

hence proving the existence and uniqueness of the solution $u \in H_{\mathcal{D},\tau}$ to (11).

Proof. We proceed as in the proof of Lemma 4.1. We remark that, thanks to Hypothesis (H5),

$$\forall c_1, c_2 \in \mathbb{R}_+, \int_{c_1}^{c_2} z\mu(z)\mathrm{d}z + \frac{1}{2}(c_2 - c_1)^2\mu(c_1) \leq c_2\mu(c_1)(c_2 - c_1). \tag{27}$$

Indeed, we set, for $c_1, c_2 \in \mathbb{R}_+$, $\Phi_{c_1}(c_2) = c_2\mu(c_1)(c_2 - c_1) - \frac{1}{2}(c_2 - c_1)^2\mu(c_1) - \int_{c_1}^{c_2} z\mu(z)\mathrm{d}z$. We have $\Phi_{c_1}(c_1) = 0$, and $\Phi'_{c_1}(c_2) = c_2\mu(c_1) - c_2\mu(c_2)$, whose sign is that of $c_2 - c_1$ since μ is (non-strictly) decreasing. Hence $\Phi_{c_1}(c_2) \geq 0$ and we get

$$\begin{aligned}& F(|\nabla_{\mathcal{D}} u^n(x)|) - F(|\nabla_{\mathcal{D}} u^{n-1}(x)|) + \frac{\mu_{\min}}{2}(|\nabla_{\mathcal{D}} u^n(x)| - |\nabla_{\mathcal{D}} u^{n-1}(x)|)^2 \\ & \leq |\nabla_{\mathcal{D}} u^n(x)|\mu(|\nabla_{\mathcal{D}} u^{n-1}(x)|)(|\nabla_{\mathcal{D}} u^n(x)| - |\nabla_{\mathcal{D}} u^{n-1}(x)|).\end{aligned}$$

Then the conclusion follows, as in the proof of Lemma 4.1. □

5. **Convergence.** Thanks to the estimates proved in the above section, we are now in position for proving the convergence of the scheme, using the monotonicity properties of the operators.

5.1. **Convergence properties for the fully implicit scheme.** We consider the function $u_{\mathcal{D},\tau} \in H_{\mathcal{D},\tau}$ satisfying (10). We define

$$w_{\mathcal{D},\tau} = f - \nu(\Pi_{\mathcal{D},\tau} u_{\mathcal{D},\tau}, \nabla_{\mathcal{D},\tau} u_{\mathcal{D},\tau}) D_\tau u_{\mathcal{D},\tau}, \tag{28}$$

$$G_{\mathcal{D},\tau} = \mu(|\nabla_{\mathcal{D},\tau} u_{\mathcal{D},\tau}|)\nabla_{\mathcal{D},\tau} u_{\mathcal{D},\tau}, \tag{29}$$

Note that $u_{\mathcal{D},\tau}$ is the solution of

$$\begin{aligned} & u \in H_{\mathcal{D},\tau}, \\ & \int_0^T\!\!\int_\Omega G_{\mathcal{D},\tau}(x,t) \cdot \nabla_{\mathcal{D},\tau} v(x,t) \mathrm{d}x\mathrm{d}t = \int_0^T\!\!\int_\Omega w_{\mathcal{D},\tau}(x,t) \Pi_{\mathcal{D},\tau} v(x,t) \mathrm{d}x\mathrm{d}t, \ \forall v \in H^\star_{\mathcal{D},\tau}. \end{aligned} \tag{30}$$

We then have the following convergence lemma.

Lemma 5.1 (A convergence property of the fully implicit scheme)**.** *Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be an admissible sequence of space-time discretisations of $\Omega\times]0,T[$ in the sense of Definition 2.4. Let, for all $m \in \mathbb{N}$, $u_m \in H_{\mathcal{D}_m,\tau_m}$ be such that (10) hold.*

Then there exist a sub-sequence of $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, and functions

$\bar u \in L^\infty(0,T;H^1_0(\Omega)) \cap C^0(0,T;L^2(\Omega))$, with $\bar u_t \in L^2(\Omega\times]0,T[)$ and $u(.,0) = u_0$, $\bar G \in L^2(\Omega\times]0,T[)^d$ and $\bar w \in L^2(\Omega\times]0,T[)$ such that

1. *$\Pi_{\mathcal{D}_m,\tau_m} u_m$ converges in $L^\infty(0,T;L^2(\Omega))$ to $\bar u$ as $m \to \infty$,*
2. *$D_{\tau_m} u_m$ weakly converges in $L^2(\Omega\times]0,T[)$ to $\bar u_t$ as $m \to \infty$,*
3. *$G_{\mathcal{D}_m,\tau_m}$, defined by (29), weakly converges to $\bar G$ in $L^2(\Omega\times]0,T[)^d$ as $m \to \infty$,*
4. *$w_{\mathcal{D}_m,\tau_m}$, defined by (28), weakly converges to $\bar w$ in $L^2(\Omega\times]0,T[)$ as $m \to \infty$,*
5. *it holds*

$$\lim_{m\to\infty} \int_0^T \int_\Omega G_{\mathcal{D}_m,\tau_m}(x,t) \cdot \nabla_{\mathcal{D}_m,\tau_m} u_m(x,t) \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar G(x,t) \cdot \nabla \bar u(x,t) \mathrm{d}x\mathrm{d}t. \tag{31}$$

Proof. Thanks to (24), $G_{\mathcal{D}_m,\tau_m}$ remains bounded in $L^\infty(0,T;L^2(\Omega))$ and $w_{\mathcal{D}_m,\tau_m}$ remains bounded in $L^2(\Omega\times]0,T[)$. Hence, up to a sub-sequence, the existence of $\bar G \in L^2(\Omega\times]0,T[)^d$ and $\bar w \in L^2(\Omega\times]0,T[)$ such that $G_{\mathcal{D}_m,\tau_m}$ weakly converges to $\bar G$ in $L^2(\Omega\times]0,T[)^d$ and $w_{\mathcal{D}_m,\tau_m}$ weakly converges to $\bar w$ in $L^2(\Omega\times]0,T[)$.

We then remark that the sequence u_m is bounded in $L^\infty(0,T;H_{\mathcal{D}_m})$, which provides, thanks to compactness property assumed in Definition 2.3, to the $L^2(\Omega\times]0,T[)$ bound on $D_{\tau_m} u_m$ and to an adaptation of Ascoli's theorem similar to that done in [8], that there exists $\bar u \in L^\infty(0,T;H^1_0(\Omega)) \cap C^0(0,T;L^2(\Omega))$, with $\bar u_t \in L^2(\Omega\times]0,T[)$ such that, up to a sub-sequence, $\Pi_{\mathcal{D}_m,\tau_m} u_m$ converges in $L^\infty(0,T;L^2(\Omega))$ to $\bar u$ as $m \to \infty$. We then get that $D_{\tau_m} u_m$ weakly converges in $L^2(\Omega\times]0,T[)$ to $\bar u_t$ as $m \to \infty$. The proof that $u(.,0) = u_0$ results from the definition of u^0 and from Property (14). One of the difficulties is to respectively identify $\bar G$ and $\bar w$ with $\mu(|\nabla \bar u|)\nabla \bar u$ and $\nu(\bar u, \nabla \bar u)$. This will be done in further lemmas, thanks to the property (31) stated in the present lemma, that we have now to prove. Note that in the proof below, we drop some indexes m for the simplicity of notation.

Let $\varphi \in L^2(0,T;H_0^1(\Omega))$ be given. Letting $v = P_{\mathcal{D},\tau}\varphi$ in (30), and passing to the limit, we get

$$\int_0^T \int_\Omega \bar{G}(x,t) \cdot \nabla\varphi(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{w}(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \forall\varphi \in L^2(0,T;H_0^1(\Omega)). \tag{32}$$

Hence, setting $\varphi = \bar{u}$ in (32), we get

$$\int_0^T \int_\Omega \bar{G}(x,t) \cdot \nabla\bar{u}(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{w}(x,t)\bar{u}(x,t)\mathrm{d}x\mathrm{d}t.$$

Passing to the limit in (30) with $v = u_m$ (the right hand side converges thanks to weak/strong convergence), we then get (31). □

5.2. **Convergence properties for the semi-implicit scheme.** We consider $u_{\mathcal{D},\tau} \in H_{\mathcal{D},\tau}$, given by (11). We define

$$\widetilde{w}_{\mathcal{D},\tau} = f - \nu(\widetilde{\Pi}_{\mathcal{D},\tau}u_{\mathcal{D},\tau}, \widetilde{\nabla}_{\mathcal{D},\tau}u_{\mathcal{D},\tau})D_\tau u_{\mathcal{D},\tau}, \tag{33}$$

$$\widetilde{G}_{\mathcal{D},\tau} = \mu(|\widetilde{\nabla}_{\mathcal{D},\tau}u_{\mathcal{D},\tau}|)\nabla_{\mathcal{D},\tau}u_{\mathcal{D},\tau}, \tag{34}$$

and $G_{\mathcal{D},\tau}$ defined by (29).

Note that $u_{\mathcal{D},\tau}$ is the solution of

$$\begin{aligned}&u \in H_{\mathcal{D},\tau},\\ &\int_0^T \int_\Omega \widetilde{G}(x,t) \cdot \nabla_{\mathcal{D},\tau}v(x,t)\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \widetilde{w}(x,t)v_{\mathcal{D},\tau}(x,t)\mathrm{d}x\mathrm{d}t,\ \forall v \in H^\star_{\mathcal{D},\tau}.\end{aligned} \tag{35}$$

We then have the following convergence lemma.

Lemma 5.2 (A convergence property of the semi-implicit scheme)**.** *Let $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$ be an admissible sequence of space-time discretisations of $\Omega\times]0,T[$ in the sense of Definition 2.4. Let, for all $m \in \mathbb{N}$, $u_m \in H_{\mathcal{D}_m,\tau_m}$ be such that (11) holds.*

Then there exist a sub-sequence of $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$, and functions

$\bar{u} \in L^\infty(0,T;H_0^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$, with $\bar{u}_t \in L^2(\Omega\times]0,T[)$ and $u(.,0) = u_0$, $\bar{G} \in L^2(\Omega\times]0,T[)^d$ and $\bar{w} \in L^2(\Omega\times]0,T[)$ such that

1. *$\Pi_{\mathcal{D}_m,\tau_m}u_m$ converges in $L^\infty(0,T;L^2(\Omega))$ to $\bar{u}$ as $m \to \infty$,*
2. *$D_{\tau_m}u_m$ weakly converges in $L^2(\Omega\times]0,T[)$ to u_t as $m \to \infty$,*
3. *$\widetilde{G}_{\mathcal{D}_m,\tau_m}$, defined by (34), and $G_{\mathcal{D}_m,\tau_m}$, defined by (29), weakly converge to $\bar{G}$ in $L^2(\Omega\times]0,T[)^d$ as $m \to \infty$, and*
$$\lim_{m\to\infty} \int_0^T \int_\Omega (\widetilde{G}_{\mathcal{D}_m,\tau_m}(x,t) - G_{\mathcal{D}_m,\tau_m}(x,t)) \cdot \nabla_{\mathcal{D}_m,\tau_m}u_m(x,t)\mathrm{d}x\mathrm{d}t = 0, \tag{36}$$
4. *$\widetilde{w}_{\mathcal{D}_m,\tau_m}$, defined by (33), weakly converges to $\bar{w}$ in $L^2(\Omega\times]0,T[)$ as $m \to \infty$,*
5. *relation (31) holds.*

Proof. The proof mainly follows the same steps as that of Lemma 5.1. Let us focus on the points which are specific. Writing

$$\||\nabla_{\mathcal{D},\tau}u| - |\widetilde{\nabla}_{\mathcal{D},\tau}u|\|^2_{L^2(\Omega\times]0,T[)^d} = \tau\sum_{n=1}^{N_T}\int_\Omega(|\nabla_{\mathcal{D}}u^n(x)| - |\nabla_{\mathcal{D}}u^{n-1}(x)|)^2\mathrm{d}x, \tag{37}$$

we get, from (26), that $\||\nabla_{\mathcal{D}_m,\tau_m} u_m| - |\widetilde{\nabla}_{\mathcal{D}_m,\tau_m} u_m|\|_{L^2(\Omega\times]0,T[)^d}$ tends to 0 since $\tau_m \longrightarrow 0$ as $m \longrightarrow \infty$. This leads, for any $\psi \in L^2(\Omega\times]0,T[)$, that the quantity

$$\begin{aligned}&\int_0^T \int_\Omega (\widetilde{G}_{\mathcal{D}_m,\tau_m}(x,t) - G_{\mathcal{D}_m,\tau_m}(x,t)) \cdot \psi(x,t)\mathrm{d}x\mathrm{d}t \\ &\le \int_0^T \int_\Omega \left|\mu(|\widetilde{\nabla}_{\mathcal{D}_m,\tau_m} u_m(x,t)|) - \mu(|\nabla_{\mathcal{D}_m,\tau_m} u_m(x,t)|)\right| \\ &\quad\cdot|\nabla_{\mathcal{D}_m,\tau_m} u_m(x,t) \cdot \psi(x,t)|\mathrm{d}x\mathrm{d}t\end{aligned}$$

tends to 0 as $m \to \infty$ thanks to (37) and properties of function μ. The same holds for $\int_0^T \int_\Omega (\widetilde{G}_{\mathcal{D}_m,\tau_m}(x,t) - G_{\mathcal{D}_m,\tau_m}(x,t)) \cdot \nabla_{\mathcal{D}_m} u_m(x,t)\mathrm{d}x\mathrm{d}t$, which proves (36). □

5.3. **Strong convergence of $\nabla_{\mathcal{D}} u$.** The problem is now to show the convergence in $L^2(\Omega\times]0,T[)$ of $\nabla_{\mathcal{D}_m} u_m$ to $\nabla\bar{u}$. This will result from property (31) (which holds for both the fully implicit and the semi-implicit schemes), and from the properties of function μ. Indeed, this property is the key point of the proof of the following lemma which uses Minty's trick.

Lemma 5.3. *Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be an admissible sequence of space-time discretisations of $\Omega\times]0,T[$ in the sense of Definition 2.4.*

Let us assume that a sequence $(u_m)_{m\in\mathbb{N}}$ is such that $u_m \in H_{\mathcal{D}_m,\tau_m}$ for all $m \in \mathbb{N}$, and such that u_m converges in $L^2(\Omega\times]0,T[)$ to $\bar{u} \in L^\infty(0,T;H^1_0(\Omega))$, $\nabla_{\mathcal{D}_m} u_m$ weakly converges to $\nabla\bar{u}$ in $L^2(\Omega\times]0,T[)$, $G_{\mathcal{D}_m,\tau_m}$, defined by (29), weakly converges to $\bar{G}$ in $L^2(\Omega\times]0,T[)^d$ as $m \to \infty$ and we assume that (31) holds. For all $W \in L^2(\Omega\times]0,T[)^d$, we denote by

$$T_m(W) = \int_0^T \int_\Omega (G_{\mathcal{D}_m,\tau_m} - \mu(|W|)W) \cdot (\nabla_{\mathcal{D}_m,\tau_m} u_m - W)\mathrm{d}x\mathrm{d}t. \qquad (38)$$

Then the following holds

$$\lim_{m\to\infty} T_m(W) = \int_0^T \int_\Omega (\bar{G} - \mu(|W|)W) \cdot (\nabla\bar{u} - W)\mathrm{d}x\mathrm{d}t, \qquad (39)$$

and therefore

$$\bar{G}(x,t) = \mu(|\nabla\bar{u}(x,t)|)\nabla\bar{u}(x,t), \text{ for a.e. } (x,t) \in \Omega\times]0,T[. \qquad (40)$$

Proof. In order to pass to the limit in $T_m(W)$, we write $T_m(W) = T_m^{(1)}(W) - T_m^{(2)}(W) - T_m^{(3)}(W) + T^{(4)}(W)$ with

$$T_m^{(1)}(W) = \int_0^T \int_\Omega G_{\mathcal{D}_m,\tau_m}(x,t) \cdot \nabla_{\mathcal{D}_m,\tau_m} u_m\mathrm{d}x\mathrm{d}t,$$

$$T_m^{(2)}(W) = \int_0^T \int_\Omega G_{\mathcal{D}_m,\tau_m}(x,t) \cdot W\mathrm{d}x\mathrm{d}t,$$

$$T_m^{(3)}(W) = \int_0^T \int_\Omega \mu(|W|)W \cdot \nabla_{\mathcal{D}_m,\tau_m} u_m\mathrm{d}x\mathrm{d}t,$$

and

$$T^{(4)}(W) = \int_0^T \int_\Omega \mu(|W|)W \cdot W\mathrm{d}x\mathrm{d}t.$$

Thanks to properties of admissible sequences of discretisations, we get

$$\lim_{m\to\infty} T_m^{(2)}(W) = \int_0^T \int_\Omega \bar{G} \cdot W\mathrm{d}x\mathrm{d}t,$$

$$\lim_{m\to\infty} T_m^{(3)}(W) = \int_0^T \int_\Omega \mu(|W|)W(x,t)\cdot\nabla\bar{u}\mathrm{d}x\mathrm{d}t,$$

Relation (31) provides

$$\lim_{m\to\infty} T_m^{(1)}(W) = \int_0^T \int_\Omega \bar{G}\cdot\nabla\bar{u}\mathrm{d}x\mathrm{d}t.$$

Hence we get (39), which is sufficient to prove next Lemma 5.4. Nevertheless, let us apply Minty's trick (which remains available in the framework of non strictly monotonous operators): we set $W = \nabla\bar{u} - \lambda\psi$, with $\lambda > 0$ and $\psi \in C_c^\infty(\Omega\times]0,T[)^d$ in (39). We get, dividing by λ,

$$\int_0^T \int_\Omega \big(\bar{G} - \mu(|\nabla\bar{u} - \lambda\psi|)(\nabla\bar{u} - \lambda\psi)\big)\psi\mathrm{d}x\mathrm{d}t \geq 0.$$

We can let $\lambda \longrightarrow 0$ in the above inequality, using Lebesgue's dominated convergence theorem. We then get

$$\int_0^T \int_\Omega (\bar{G} - \mu(|\nabla\bar{u}|\nabla\bar{u})\psi\mathrm{d}x\mathrm{d}t \geq 0.$$

Since this also holds for $-\psi$, we get

$$\int_0^T \int_\Omega (\bar{G} - \mu(|\nabla\bar{u}|\nabla\bar{u})\psi\mathrm{d}x\mathrm{d}t = 0.$$

Hence $\bar{G} - \mu(|\nabla\bar{u}|\nabla\bar{u}) = 0$ a.e. in $\Omega\times]0,T[$, which achieves the proof of (40). □

We now have the following lemma.

Lemma 5.4. *Under the same hypotheses as Lemma 5.3, $\nabla_{\mathcal{D}_m,\tau_m} u_m$ converges in $L^2(\Omega\times]0,T[)$ to $\nabla\bar{u}$ as m tends to ∞.*

Proof. We first remark that, for all $V, W \in L^2(\Omega\times]0,T[)^d$, it holds

$$\forall V, W \in L^2(\Omega\times]0,T[)^d, \int_0^T \int_\Omega (\mu(|W|)W - \mu(|V|)V)\cdot(W-V)\,\mathrm{d}x\mathrm{d}t \\ \geq \alpha \| |W| - |V| \|^2_{L^2(\Omega\times]0,T[)}. \tag{41}$$

Indeed, thanks to the Cauchy-Schwarz inequality, we get

$$\int_0^T \int_\Omega \mu(|W|)W\cdot V\mathrm{d}x\mathrm{d}t \leq \int_0^T \int_\Omega \mu(|W|)|W|\ |V|\mathrm{d}x\mathrm{d}t,$$

and the same property holds exchanging the roles of W and V. Hence

$$\int_0^T \int_\Omega (\mu(|W|)W - \mu(|V|)V)\cdot(W-V)\,\mathrm{d}x\mathrm{d}t \\ \geq \int_0^T \int_\Omega (\mu(|W|)|W| - \mu(|V|)|V|)\,(|W| - |V|)\,\mathrm{d}x\mathrm{d}t.$$

Property (H5) on μ provides (41). Taking $W = \nabla_{\mathcal{D}_m} u_m$ and $V = \nabla\bar{u}$ in (41), we get

$$\| |\nabla_{\mathcal{D}_m} u_m| - |\nabla\bar{u}| \|^2_{L^2(\Omega\times]0,T[)} \leq \frac{1}{\alpha} T_m(\nabla\bar{u}),$$

and, thanks to (39), $\lim_{m\to\infty} T_m(\nabla\bar{u}) = 0$. Therefore

$$\lim_{m\to\infty} \| |\nabla_{\mathcal{D}_m} u_m| - |\nabla\bar{u}| \|_{L^2(\Omega\times]0,T[)} = 0,$$

which, in addition to the convergence of $\nabla_{\mathcal{D}_m} u_m$ to $\nabla \bar{u}$ for the weak topology of $L^2(\Omega\times]0,T[)$, provides the convergence in $L^2(\Omega\times]0,T[)$ of $\nabla_{\mathcal{D}_m} u_m$ to $\nabla \bar{u}$. □

We can now conclude the convergence of the scheme.

Theorem 5.5. *Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be an admissible sequence of space-time discretisations of $\Omega\times]0,T[$ in the sense of Definition 2.4.*

Let, for all $m \in \mathbb{N}$, u_m be such that (10) (fully implicit scheme) or (11) (semi-implicit scheme) hold.

Then there exists a sub-sequence of $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, and there exists a function $\bar{u} \in L^\infty(0,T;H_0^1(\Omega))$, weak solution of (2)-(3)-(4) in the sense of Definition 1.1, such that $u_{\mathcal{D}_m,\tau_m}$ tends to $\bar{u}$ in $L^\infty(0,T;L^2(\Omega))$, $\nabla_{\mathcal{D}_m} u_m$ tends to $\nabla\bar{u}$ in $L^2(\Omega\times]0,T[)^d$.

Proof. We first apply Lemmas 5.1 or 5.2. We get (30) or (35). We apply Lemma 5.4. We thus get that

$$\bar{w} = f - \nu(\bar{u}, \nabla\bar{u})\bar{u}_t \text{ a.e. in } \Omega\times]0,T[,$$

which, in addition to (40), concludes the proof. □

6. **Numerical experiments.** In this section we present several numerical examples to illustrate the properties of the proposed numerical schemes. They are devoted to the solution of regularised mean curvature flow and to the motion of 2D curves by curvature in level set formulation. In all examples we use both the semi-implicit and fully implicit schemes. We compute the errors and experimental order of convergence (EOC) for the whole level set function and also for the level set representing moving curve. Let us emphasise that for all the tests also proposed in [8], the order of the obtained errors and EOC are very close, using similar space and time steps. In the tables below n is number of finite volumes along each boundary side and n^2 is a total number of finite volumes. We consider the square domain $\Omega =]-1.25, 1.25[\times]-1.25, 1.25[$ and compute the errors of the solution in $L^2(\Omega\times]0,T[)$ norm denoted by E_2, $L^\infty(0,T;L^2(\Omega))$ denoted by E_∞ and for the gradient of the solution in $L^2(\Omega\times]0,T[)^d$ denoted by EG_2 and $L^\infty(0,T;L^2(\Omega)^d)$ norm denoted by EG_∞. We assume that μ and ν are defined by (5) with various values of a, taking for b a sufficiently large value (recall that the discrete gradient remains bounded if the discrete solution satisfies an L^∞ bound).

Example 1. In this example we compare a numerical solution with the exact solution

$$u(x,y,t) = \frac{x^2+y^2}{2} + t$$

to Equation (2), setting $a^2 = 1/2$ in (5) and defining f by

$$f(x,y,t) = -\frac{1/2}{(x^2+y^2+1/2)^{3/2}},$$

defining the initial condition and non-homogeneous Dirichlet boundary conditions according to the exact solution in the time interval $[0,T] = [0, 0.3125]$.

We consider two types of grids. The first type is a standard $n \times n$ square grid, on which we apply Schemes (15)-(17) and (18)-(20). Scheme (21)-(23) is applied on the second type, depicted in Figure 3, consisting in a triangular mesh obtained by $n \times n$ repetitions of a pattern build with 14 triangles. The advantage of this type of mesh is that the regularity factor is independent of n, and that no local higher order

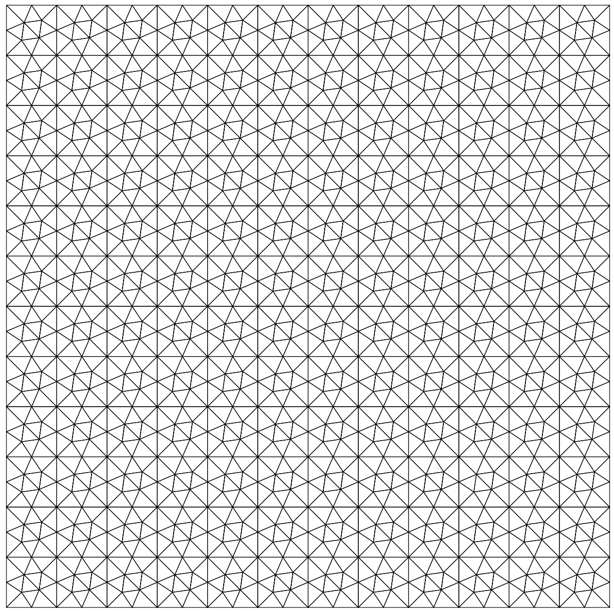

FIGURE 3. Triangular $n \times n \times 14$ grid with $n = 12$.

consistency is introduced, hence reproducing the properties of a general triangular mesh. In this example and in the other ones, the time step τ fulfils the relation $\tau = C/n^2$, where $1/n$ is proportional to the diameter of finite volumes (recall that the schemes are unconditionally stable, but this relation is used in order to ensure classical order of convergence in the case of parabolic PDEs). The linear systems are either solved by direct Gaussian elimination (which can be a very efficient method for solving such sparse linear systems, taking into account some algebraic properties of the stencil) or by iterative Gauss-Seidel linear solver. In the experiments on triangular meshes we use the direct solver, whereas on the rectangular grids we prefer the Gauss-Seidel method, which often provides smaller computing time on large cases. In order to prevent the choice of the linear solver from modifying the numerical results, the precision prescribed to the iterative solver is comparable to that of the direct solver.

All the computations are performed on a PC computer, using the double precision type for the real values. For the fully implicit scheme, we need about 30 nonlinear iterations when fixing the tolerance on the square of the L^2 norm of the residual to $TOL = 10^{-20}$. A sufficient precision is also obtained with $TOL = 10^{-10}$ in nonlinear iterations and then 10 iterations in one time step are needed.

The results with $m = n - 1$ (semi-implicit scheme) are presented in Tables 1, 3 and 5, and the results with $m = n$ (fully implicit scheme) are presented in Tables 2, 4 and 6.

They show, on this smooth example, that the the precision of the fully implicit scheme is comparable with that of the semi-implicit one in both cases of rectangular and triangular meshes. Let us mention that the fully implicit scheme demands more CPU time. All the schemes have $EOC = 2$ in solution error and, interestingly, the EOC in the L_2 norm is about 2 also for gradients for rectangular grids and it is about 1.5 for triangular meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	4.644e-03	-	1.140e-02	-	1.281e-02	-	2.387e-02	-
20	1.5625e-02	1.347e-03	1.786	3.145e-03	1.858	4.024e-03	1.669	8.480e-03	1.493
40	3.90625e-03	3.482e-04	1.952	7.729e-04	2.025	1.112e-03	1.845	2.996e-03	1.501
80	9.765625e-04	8.677e-05	2.005	1.897e-04	2.027	2.902e-04	1.949	1.066e-03	1.491
160	2.44141e-04	2.156e-05	2.009	4.693e-05	2.015	7.410e-05	1.970	3.780e-04	1.496
320	6.10352e-05	5.366e-06	2.007	1.167e-05	2.008	1.872e-05	1.985	1.340e-04	1.496

TABLE 1. Example 1, error reports and EOCs for the semi-implicit scheme (15)-(17) on $n \times n$ square meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	6.167e-03	-	1.394e-02	-	1.549e-02	-	3.461e-02	-
20	1.5625e-02	1.471e-03	2.068	3.260e-03	2.097	4.413e-03	1.811	1.305e-02	1.407
40	3.90625e-03	3.558e-04	2.048	7.782e-04	2.067	1.162e-03	1.925	4.772e-03	1.451
80	9.765625e-04	8.723e-05	2.028	1.900e-04	2.034	2.969e-04	1.969	1.716e-03	1.476
160	2.44141e-04	2.158e-05	2.015	4.694e-05	2.017	7.500e-05	1.986	6.117e-04	1.488
320	6.10352e-05	5.532e-06	1.964	1.206e-05	1.961	2.087e-03	1.845	2.078e-03	1.558

TABLE 2. Example 1, error reports and EOCs for the fully implicit scheme (15)-(17) on $n \times n$ square meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	3.750e-03	-	9.132e-03	-	1.339e-02	-	2.507e-02	-
20	1.5625e-02	1.218e-03	1.622	2.821e-03	1.694	4.064e-03	1.720	8.734e-03	1.521
40	3.90625e-03	3.318e-04	1.876	7.340e-04	1.942	1.108e-03	1.875	3.050e-03	1.518
80	9.765625e-04	8.479e-05	1.968	1.851e-04	1.988	2.892e-04	1.938	1.076e-03	1.503
160	2.44141e-04	2.131e-05	1.992	4.636e-05	1.997	7.395e-05	1.967	3.797e-04	1.503
320	6.10352e-05	5.336e-06	1.999	1.160e-05	1.999	1.870e-05	1.984	1.343e-04	1.499

TABLE 3. Example 1, error reports and EOCs for the semi-implicit scheme (18)-(20) on $n \times n$ square meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	4.894e-03	-	1.091e-02	-	1.593e-02	-	3.647e-02	-
20	1.5625e-02	1.324e-03	1.886	2.914e-03	1.905	4.402e-03	1.856	1.334e-02	1.451
40	3.90625e-03	3.388e-04	1.967	7.389e-04	1.980	1.155e-03	1.930	4.819e-03	1.469
80	9.765625e-04	8.523e-05	1.991	1.853e-04	1.996	2.958e-04	1.965	1.724e-03	1.483
160	2.44141e-04	2.134e-05	1.998	4.638e-05	1.998	7.483e-05	1.983	6.130e-04	1.492
320	6.10352e-05	5.969e-06	1.838	1.203e-05	1.947	1.937e-05	1.950	2.124e-04	1.529

TABLE 4. Example 1, error reports and EOCs for the fully implicit scheme (18) -(20) on $n \times n$ square meshes.

Example 2. Now we use the exact viscosity solution [15]

$$u(x,y,t) = \min\{\frac{x^2+y^2-1}{2} + t, 0\}$$

to the level set equation (1), which is (2) in the case $f = 0$ and

$$\mu(s) = 1/s,\ \forall s \in \mathbb{R}_+^\star,\ \text{ and } \nu(z,\xi) = \mu(|\xi|),\ \forall z \in \mathbb{R},\ \forall \xi \in \mathbb{R}^d \setminus 0, \tag{42}$$

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
3	6.25e-02	8.43e-03	-	1.96e-02	-	3.22e-02	-	6.53e-02	-
6	1.5625e-02	2.50e-03	1.75	5.65e-03	1.79	1.03e-02	1.64	2.78e-02	1.23
12	3.90625e-03	6.66e-04	1.90	1.45e-03	1.96	3.37e-03	1.61	1.26e-02	1.14
24	9.765625e-04	1.67e-04	2.000	3.62e-04	2.000	1.28e-03	1.40	5.95e-03	1.08

TABLE 5. Example 1, error reports and EOCs for the semi-implicit scheme (21)-(23) on $n \times n \times 14$ triangular meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
3	6.25e-02	1.00e-02	-	2.22e-02	-	3.19e-02	-	7.41e-02	-
6	1.5625e-02	2.65e-03	1.91	5.77e-03	1.94	9.85e-03	1.69	2.83e-02	1.39
12	3.90625e-03	6.75e-04	1.97	1.45e-03	2.00	3.26e-03	1.60	1.08e-02	1.39
24	9.765625e-04	1.68e-04	2.006	3.63e-04	2.000	1.26e-03	1.37	4.21e-03	1.36

TABLE 6. Example 1, error reports and EOCs for the fully implicit scheme (21)-(23) on $n \times n \times 14$ triangular meshes.

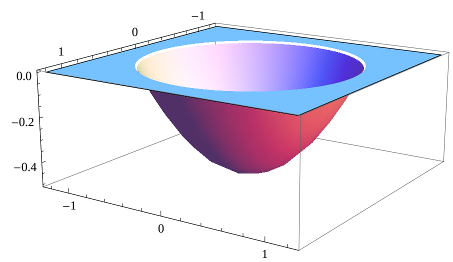

FIGURE 4. Example 2: Initial condition

with zero Dirichlet boundary conditions, in the time interval $[0, T] = [0, 0.3125]$. The initial condition and the exact and numerically computed solution (for a square 160×160 mesh) are plotted in Figures 4 and 5.

In this example, the solution contains flat regions, so we replace, for its numerical approximation, (42) by (5) with $a > 0$ (this is the so-called Evans-Spruck type regularisation [5]). Since the gradient of the solution jumps on a circular curve, we cannot expect a second order accurate approximation of the solution. However, as we see from Tables 7-10, the numerical schemes converge also in this singular case and naturally, EOC is equal (or close to) 1 for the solution error. In order to mimic convergence of numerical solution to (42) we use the coupling $a = h$ in (5) in order to fulfil the accuracy objectives. One can also observe that the errors obtained using the fully implicit scheme in this non-smooth example are slightly better than that provided by the semi-implicit one. However, since the CPU time needed by the fully-implicit scheme is greater, the semi-implicit scheme appears as a reasonable compromise (cf. [11, 4]) in practical applications.

Example 3. In this example we compute the displacement of the unit circle by its curvature, and we compare the numerical results with the exact solution. The exact radius $r(t)$ of a shrinking circle can be analytically expressed by

$$r(t) = \sqrt{r(0)^2 - 2t}, \quad t \in [0, T], \text{ where } T = \frac{r(0)^2}{2}. \tag{43}$$

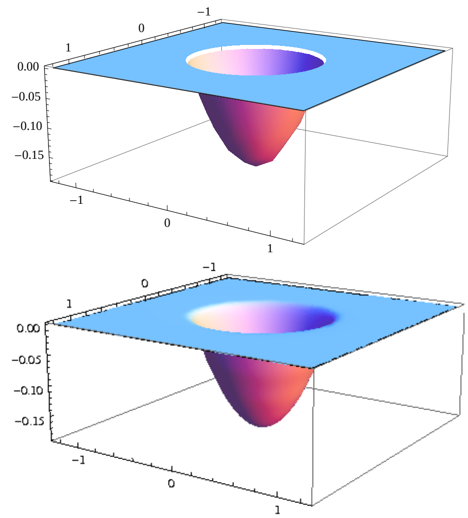

FIGURE 5. Example 2, the exact (left) and numerical solution (right) at time 0.3125.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	6.474e-02	-	1.413e-01	-	2.843e-01	-	5.659e-01	-
20	1.5625e-02	4.747e-02	0.448	1.138e-01	0.312	2.380e-01	0.256	4.655e-01	0.282
40	3.90625e-03	2.879e-02	0.721	7.273e-02	0.646	1.849e-01	0.364	3.617e-01	0.364
80	9.76563e-04	1.586e-02	0.860	4.072e-02	0.767	1.437e-01	0.364	2.807e-01	0.366
160	2.44141e-04	8.365e-03	0.923	2.159e-02	0.985	1.119e-01	0.361	2.171e-01	0.371
320	6.10352e-05	4.302e-03	0.960	1.115e-02	0.909	8.747e-02	0.355	1.695e-01	0.357

TABLE 7. Example 2, error reports and EOCs for semi-implicit scheme (15)-(17), $a = h$ on $n \times n$ square meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	6.530e-02	-	1.398e-01	-	2.898e-01	-	5.717e-01	-
20	1.5625e-02	4.279e-02	0.610	1.053e-01	0.410	2.375e-01	0.287	4.710e-01	0.078
40	3.90625e-03	2.421e-02	0.822	6.328e-02	0.735	1.908e-01	0.316	3.874e-01	0.511
80	9.76563e-04	1.299e-02	0.898	3.451e-02	0.875	1.537e-01	0.312	3.076e-01	0.445
160	2.44141e-04	6.819e-03	0.929	1.819e-02	0.924	1.227e-01	0.325	2.422e-01	0.358
320	6.10352e-05	3.517e-03	0.955	9.412e-03	0.950	9.603e-02	0.353	1.875e-01	0.369

TABLE 8. Example 2, error reports and EOCs for fully implicit scheme (15)-(17), $a = h$, on $n \times n$ square meshes.

The initial condition is given by

$$u_0(x, y) = -1 + \sqrt{x^2 + y^2}, \tag{44}$$

which represents the distance function to the initial unit circle. Since we use zero Neumann boundary conditions in this example, the initial level set function is deformed, see Figure 6, but the error on the interface decreases with respect to the space and time steps, indicating the convergence of the method, as it can be seen in Tables 11 and 12.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	6.478e-02	-	1.413e-01	-	2.892e-01	-	5.661e-01	-
20	1.5625e-02	4.754e-02	0.447	1.139e-01	0.312	2.398e-01	0.255	4.656e-01	0.282
40	3.90625e-03	2.881e-02	0.722	7.277e-02	0.646	1.852e-01	0.364	3.620e-01	0.363
80	9.76563e-04	1.587e-02	0.860	4.074e-02	0.837	1.439e-01	0.364	2.809e-01	0.366
160	2.44141e-04	8.368e-03	0.923	2.160e-02	0.915	1.120e-01	0.361	2.172e-01	0.371
320	6.10352e-05	4.303e-03	0.960	1.115e-02	0.954	8.752e-02	0.356	1.696e-01	0.357

TABLE 9. Example 2, error reports and EOCs for semi-implicit scheme (18)-(20), $a = h$, on $n \times n$ square meshes.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC
10	6.25e-02	6.536e-02	-	1.400e-01	-	2.899e-01	-	5.714e-01	-
20	1.5625e-02	4.301e-02	0.604	1.057e-01	0.405	2.374e-01	0.288	4.702e-01	0.281
40	3.90625e-03	2.439e-02	0.818	6.365e-02	0.732	1.888e-01	0.330	3.835e-01	0.294
80	9.76563e-04	1.307e-02	0.900	3.471e-02	0.875	1.506e-01	0.326	3.014e-01	0.348
160	2.44141e-04	6.842e-03	0.934	1.825e-02	0.927	1.196e-01	0.333	2.360e-01	0.353
320	6.10352e-05	3.522e-03	0.958	9.424e-03	0.953	9.361e-02	0.353	1.831e-01	0.366

TABLE 10. Example 2, error reports and EOCs for fully implicit scheme (18) -(20), $a = h$, on $n \times n$ square meshes.

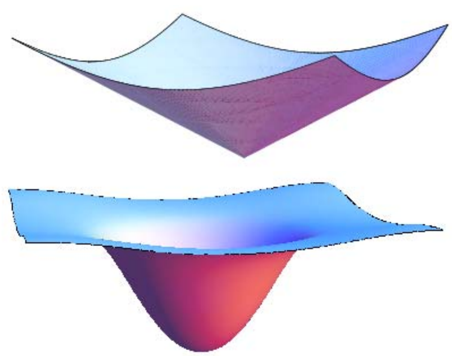

FIGURE 6. Example 3, the level set function at time $t = 0$ (left) and $t = 0.439$ (right).

The comparison of the numerical solution with the exact one (43) is performed within the time interval $[0, T]$, where $T = 0.375$, by a subsequent refinement of the grid. The measurement of the error is similar to that of [10]. For every discrete time step $k = 0, 1, \ldots, N$, we first compute all the points $x_i^k,\ i = 1, 2, \ldots, P$ where the piecewise linear representation of the numerical solution becomes equal to zero along the finite element grid lines. Then we compute the distances $r_i^k,\ i = 1, 2, \ldots, P$ between the origin and the points $x_i^k,\ i = 1, 2, \ldots, P$. Finally, these distances are compared to the radius $r(k\tau)$ of the exact evolving circle. Then the formula

$$E_2 = \sqrt{\sum_{k=0}^{N} \tau \frac{1}{P} \sum_{i=1}^{P} (r_i^k - r(k\tau))^2} \tag{45}$$

is used for assessing the error in the $L^2(0,T;L^2(S^1))$ norm, denoting by S^1 the unit circle and setting $T = N\tau$. The results for the semi-implicit and fully implicit schemes on rectangular grids are summarised in Tables 11 and 12. In all the cases, the convergence to the exact solution in the norm defined by (45) seems to be numerically observed. Since, in this definition, P is not constant during consecutive time steps, the $EOCs$ look less regular than in the previous two examples. Nevertheless, Table 12 seems to show second order convergence in the case of the fully-implicit scheme (18)-(20), using the coupling $\tau = h^2$.

n	τ	E_2 semi-implicit	EOC	E_2 fully-implicit	EOC
10	6.25e-02	9.035e-02	-	9.303e-02	-
20	1.5625e-02	4.831e-02	0.903	2.311e-02	2.009
40	3.90625e-03	1.200e-02	2.009	3.314e-04	6.124
80	9.76563e-04	9.864e-04	3.604	1.470e-04	1.172
160	2.44141e-04	5.122e-05	4.267	1.750e-05	3.070
320	6.10352e-05	1.867e-05	1.456	4.459e-06	1.973

TABLE 11. Example 3, error reports and EOCs for scheme (15)-(17) semi and fully implicit version, $a = h$, on $n \times n$ square meshes.

n	τ	E_2 semi-implicit	EOC	E_2 fully-implicit	EOC
10	6.25e-02	9.036e-02	-	9.309e-02	-
20	1.5625e-02	4.832e-02	0.903	2.339e-02	1.993
40	3.90625e-03	1.201e-02	2.008	2.880e-04	6.344
80	9.76563e-04	9.857e-04	3.607	6.922e-05	2.057
160	2.44141e-04	5.135e-05	4.263	1.408e-05	2.298
320	6.10352e-05	1.868e-05	1.458	2.510e-06	2.488

TABLE 12. Example 3, error reports and EOCs for scheme (18) -(20) semi and fully implicit version, $a = h$, on $n \times n$ square meshes.

In Figure 7 we represent the numerical evolution of a circle together with the exact solution, setting $n = 80$, $\tau = h^2$ and using 400 time steps. We hardly distinguish in this figure the numerical solution and the exact one.

Example 4. Finally, we consider the mean curvature flow of a quatrefoil, defined as the zero level set of the initial level set function constructed by the formula

$$u_0(x,y) = -1 + \frac{\sqrt{x^2+y^2}}{r_L}, \text{where } r_L = 0.6 + 0.4\sin\left(4\text{arctg}(\frac{y}{x})\right). \tag{46}$$

The evolution is computed in time interval $[0,T]$, $T = 0.22461$, $n = 80$, $\tau = h^2$ by both schemes (15)-(17) and (18) -(20) and the results are presented in Figure 8, showing very close results.

7. **Conclusions.** The family of discrete schemes presented in this paper shows very easy implementation properties, and satisfactory accuracy. The adaptation of the viscosity solution sense to this discrete framework remains an open problem.

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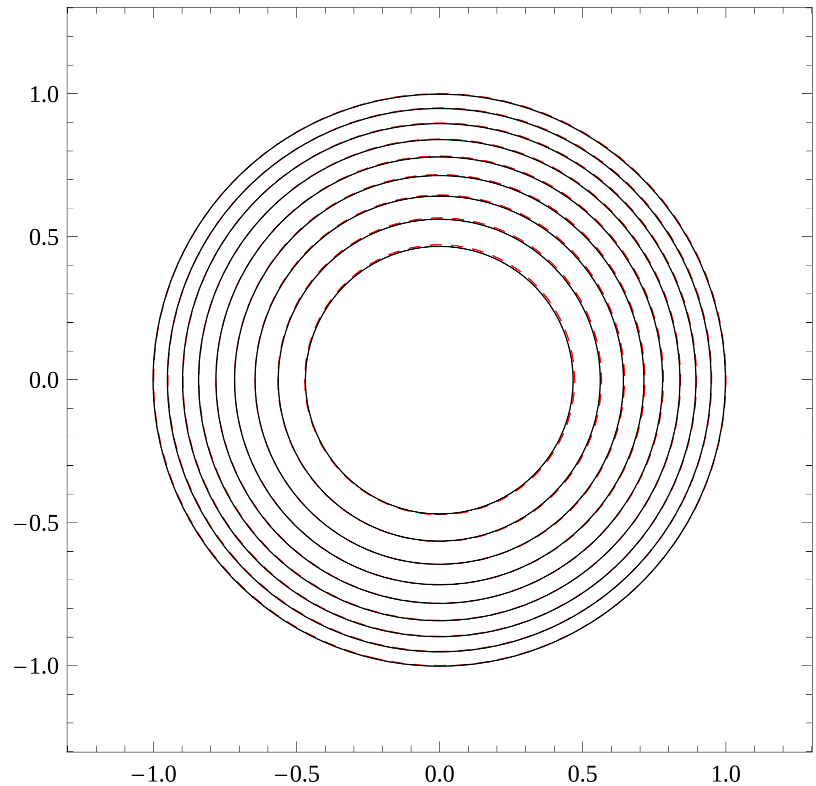

FIGURE 7. Example 3: Evolution of the unit circle, exact (black) and numerical(red dashed) solutions at time steps $t_N = N\tau$, $\tau = 9.76563e - 4$, $N = 0, 50, 100, 150, 200, 250, 300, 350, 400$.

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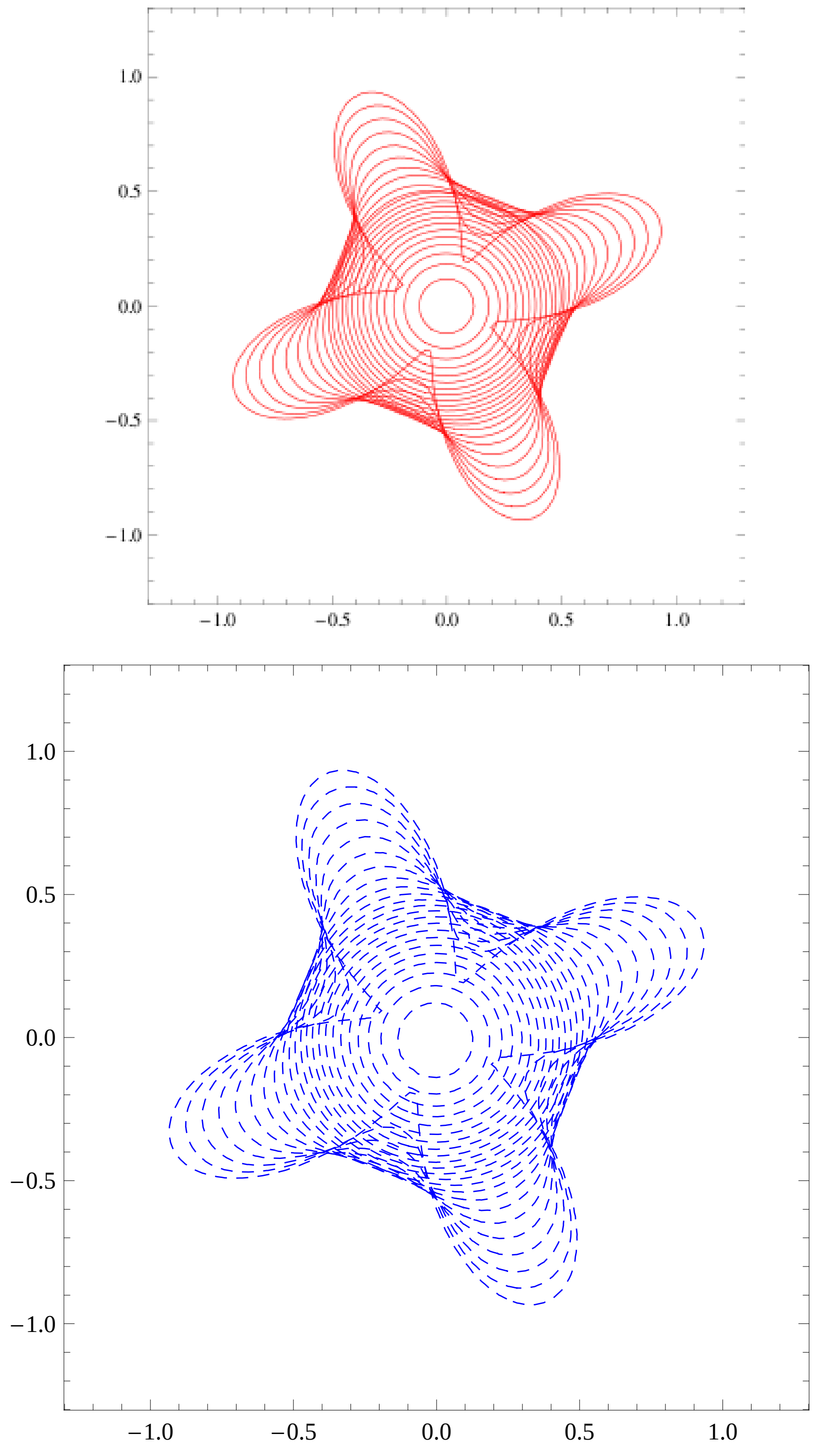

FIGURE 8. Example 3. Evolution of a quatrefoil, first scheme on rectangles (left, red, solid), second scheme on rectangles (right, blue, dashed) in time steps $t_N = N\tau$, $\tau = 9.76563e-4$, $N = 0, 10, 20, \dots, 220$.