A comparison of the variational solution to the Neumann geodetic boundary value problem with the geopotential model EGM-96

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Abstract: The Neumann geodetic boundary value problem (NGBVP) represents an exterior oblique derivative problem for the Laplace equation. The Neumann boundary conditions in the form of surface gravity disturbances correspond to derivatives of the unknown disturbing potential. The boundary element method (BEM) as a numerical method based on the variational formulation of the Laplace equation is applied to NGBVP. This approach gives a variational (approximate) solution directly on the Earth's surface, where the classical solution could be hardly found.

This paper discusses the 3D BEM application to NGBVP. It represents a new approach to the global quasigeoid modelling. The collocation technique with linear basis functions is applied for deriving the linear system from the boundary integral equations. With respect to a giant size of the Earth and in order to get accuracy as high as possible, computing on high-speed parallel computers is necessary. The Global Quasigeoid Models as the numerical results for two input data sets are compared with the geopotential model EGM-96.

Key words: the Neumann geodetic boundary value problem, boundary element method, variational solution, collocation method, linear basis functions, global quasigeoid modelling

1. Introduction

A determination of the external gravity field is usually formulated in terms of the geodetic boundary value problem for the Laplace equation.

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The present concepts use the fundamental gravimetric equation with input gravity anomalies as a boundary condition (BC), i.e. the Newton BC. A combination of gravimetric and satellite measurements allows to define the Neumann BC in the form of surface gravity disturbances that correspond to derivatives of the unknown disturbing potential. A formulation of the Neumann geodetic boundary value problem (NGBVP) is theoretically known for a long time but its practical solution for the real Earth's surface is still an open problem.

The boundary element method (BEM) as a numerical method based on the variational formulation of the partial differential equation (PDE) is suitable for solving exterior boundary value problems. The advantage of BEM arises from the fact that only the boundary of a solution domain requires a subdivision to elements. Thus the dimension of the problem is effectively reduced by one. The reformulation of PDE consists of surface integral equations defined on the boundary. They are transformed to the linear system of equations by an appropriate numerical technique, e.g. by the collocation method. Thanks to its simplicity this method is very popular in engineer applications. In great majority of applications constant basis functions are used for approximating boundary functions on each panel of the boundary surface.

The 3D BEM application to NGBVP gives a variational (approximate) solution directly on the Earth's surface, where the classical solution can be hardly found. It represents a new approach to the solution of the geodetic boundary value problem, i.e. to the global gravity field modelling. With respect to a giant size of the Earth and in order to get accuracy as high as possible, computing on high-speed parallel computers is necessary. In order to reduce requirements for the memory storage the collocation method with linear basis functions is applied in the presented numerical experiments.

2. The Neumann geodetic boundary value problem

The Earth as a spinning physical body generates the actual gravity potential. The disturbing potential T is defined as the difference between the actual gravity potential W and the normal gravity potential Uin a point $\boldsymbol{x}$

$$T(\boldsymbol{x}) = W(\boldsymbol{x}) - U(\boldsymbol{x}), \quad \boldsymbol{x} \in R^3. \tag{1}$$

The normal gravity potential is generated by a normal body. In order to keep its average difference from the actual gravity potential as small as possible, the normal body is defined as the "massive" biaxial geocentric and equipotential ellipsoid of revolution. This equipotential ellipsoid is completely determined by four parameters derived from the actual Earth; the semi-major axis a, the geopotential coefficient $J_{2,0}$, the geocentric gravitational constant GM and the spin angular velocity ω (Geodetic Reference System GRS-80 *(Moritz, 1992)*). Its minor axis coincides with the Earth's polar principal axis of inertia. As the spin angular velocity is the same for the Earth and the normal body, their centrifugal components are equal. Then the disturbing potential is a harmonic function outside the Earth (neglecting the atmosphere) and it satisfies the Laplace equation (geodetic boundary value problem).

A combination of gravimetric and satellite measurements allows to define the Neumann boundary conditions in the form of surface gravity disturbances that correspond to derivatives of the unknown disturbing potential. The surface gravity disturbance δg compares the actual gravity g and the normal gravity γ in the same point $\boldsymbol{x}$

$$\delta g(\boldsymbol{x}) = g(\boldsymbol{x}) - \gamma(\boldsymbol{x}), \quad \boldsymbol{x} \in R^3. \tag{2}$$

Let us apply the operator gradient to the definition of disturbing potential (1)

$$\nabla T(\boldsymbol{x}) = \nabla W(\boldsymbol{x}) - \nabla U(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{\gamma}(\boldsymbol{x}), \quad \boldsymbol{x} \in R^3. \tag{3}$$

Gradients of the actual and normal gravity potential have different directions. The spatial angle between them is negligibly small. In addition, directions of both vectors are very close to the opposite direction of the outer normal $\boldsymbol{n}_e$ to the geocentric equipotential ellipsoid. Neglecting these small angles (less than minute) we project the actual and normal gravity vectors to the normal $\boldsymbol{n}_e$

$$\langle \nabla T(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x}) \rangle = \langle \boldsymbol{g}(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x}) \rangle - \langle \boldsymbol{\gamma}(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x}) \rangle \approx \\ \approx -g(\boldsymbol{x}) + \gamma(\boldsymbol{x}) = -\delta g(\boldsymbol{x}), \quad \boldsymbol{x} \in R^3. \tag{4}$$

where $\langle \, , \rangle$ represents the scalar product of vectors.

The equation (4) defines Neumann boundary conditions for the Laplace equation. So we can formulate NGBVP

$$\Delta T(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in R^3 - \Omega, \tag{5a}$$

$$\langle \nabla T(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x})\rangle = -\delta g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \tag{5b}$$

$$T(\boldsymbol{x}) \to 0 \quad \text{for} \quad \boldsymbol{x} \to \infty. \tag{5c}$$

The domain Ω represents the body of the Earth. The boundary surface Γ is the Earth's surface. Equations (5) represent the exterior oblique derivative boundary value problem for the Laplace equation with the Neumann boundary conditions. The oblique derivative problem arises from the fact that the normal to the Earth's surface Γ doesn't coincide with the normal to ellipsoid $\boldsymbol{n}_e$.

The proposed NGBVP has an obvious advantage that input surface gravity disturbances do not require sea level heights. Ellipsoidal (geodetic) heights obtained from satellite observations are sufficient vertical information. Levelling is not needed in this case.

3. BEM applied to the Laplace equation

In the direct BEM formulation a boundary integral equation is derived from the weak (integral) formulation of the Laplace equation (5a) through the application of Green's second theorem *(Brebbia et al., 1984)*. In 3D it has the following form

$$4\pi T(\boldsymbol{x}) + \int_\Gamma T(\boldsymbol{y}) \frac{\partial G}{\partial n_\Gamma}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_\Gamma \frac{\partial T}{\partial n_\Gamma}(\boldsymbol{y}) G(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x} \in \Gamma. \tag{6}$$

where $\boldsymbol{n}_\Gamma$ is a normal to the boundary Γ. The kernel function G is known as the Green's function and it represents the fundamental solution of the Laplace equation.

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi\,|x-y|}, \quad \boldsymbol{x},\boldsymbol{y} \in R^3. \tag{7}$$

Since the directions of $\nabla T(\boldsymbol{x})$ and $\boldsymbol{n}_e(\boldsymbol{x})$ are almost identical (neglecting the deflection of vertical), we can approximate the term $\langle \nabla T(\boldsymbol{x}),\, \boldsymbol{n}_\Gamma(\boldsymbol{x})\rangle$ by a projection of input surface gravity disturbances $\delta g(\boldsymbol{x})$ to the vector $\boldsymbol{n}_\Gamma(\boldsymbol{x})$, i.e. by $\delta g(\boldsymbol{x})\,\cos\alpha(\boldsymbol{x})$, where $\alpha(\boldsymbol{x})$ is an angle $\angle(\boldsymbol{n}_\Gamma(\boldsymbol{x}), \boldsymbol{n}_e(\boldsymbol{x}))$. Thus we can replace the term $\partial T/\partial n_\Gamma$ in (6) by these corresponding quantities in any point $\boldsymbol{x} \in \Gamma$. In this way the oblique derivative BC (5b) is incorporated into the BEM formulation (6).

The collocation method with linear basis functions (the C^1 collocation) is used for deriving the linear system of equations from the boundary integral equation (6). The Earth's surface as a boundary surface is approximated by the triangulation of the topography – expressed as a set of panels $\Delta\Gamma_j$. Vertices $x_i, \ldots, x_N$ of triangles represent the nodes – collocation points. The C^1 collocation involves approximating the boundary functions by a linear function on each triangular panel using linear basis functions *(Brebbia et al., 1984)*, i.e.,

$$T(\boldsymbol{x}) \approx \sum_{k=1}^{3} T_k \psi_k(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Delta\Gamma_j, \tag{8a}$$

$$\delta g(\boldsymbol{x}) \approx \sum_{k=1}^{3} \delta g_k \psi_k(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Delta\Gamma_j, \tag{8b}$$

where T_k and δg_k for $k = 1, 2, 3$ represent values of the boundary functions at vertices of the triangular panel $\Delta\Gamma_j$. The linear basis functions $\{\psi_1, \psi_2, \ldots, \psi_N\}$ are given by

$$\begin{aligned} &\psi_j(\boldsymbol{x}_i) = 1, \quad \boldsymbol{x}_i = \boldsymbol{x}_j, \\ &\psi_j(\boldsymbol{x}_i) = 0, \quad \boldsymbol{x}_i \neq \boldsymbol{x}_j, \qquad i = 1, \ldots, N; j = 1, \ldots, N, \end{aligned} \tag{9}$$

where N is the number of collocation points. These approximations allow to reduce the boundary integral equation (6) to a discrete form for each collocation point i

$$c_i T_i \psi_i + \sum_{j=1}^{N} \int\limits_{\sup\psi_j} \frac{\partial G_{ij}}{\partial n_\Gamma} T_j \psi_j \, \mathrm{d}\Gamma_j = \sum_{j=1}^{N} \int\limits_{\sup\psi_j} G_{ij} \delta g_j \psi_j \, \mathrm{d}\Gamma_j, \quad i = 1, \ldots, N, \tag{10}$$

where supp ψ_j is the support of the j-th basis function. The function c_i represents "the spatial segment" bounded by panels joined in the node i. In case of linear basis functions it can be evaluated by the expression *(Balaš et al., 1985)*

$$c_i = \sum_{s=1}^{S} \frac{\varphi_{i_s}}{4\pi} \left(1 - \cos\phi_{i_s}\right), \tag{11}$$

where φ_{i_s} is the angle between two planes intersecting in $\boldsymbol{n}_e(\boldsymbol{x}_i)$ and creating two edges of the s-th triangle of the supp ψ_i and ϕ_{i_s} is the angle between $\boldsymbol{n}_e(\boldsymbol{x}_i)$ and the s-th triangle. S represents the number of triangles in the supp ψ_i. Equations (6.6) represent the system of approximations that can be rewritten in the matrix-vector form

$$\boldsymbol{M}\,\boldsymbol{T} = \boldsymbol{L}\ \boldsymbol{\delta g} \tag{12}$$

where $\boldsymbol{T} = (T_1, \ldots, T_N)$ and $\boldsymbol{\delta g} = (\delta g_1, \ldots, \delta g_N)$. Coefficients of the matrices $\boldsymbol{M}$ and $\boldsymbol{L}$ represent integrals that need to be computed using an appropriate disctretization of the integral operators in (10).

The discretization of the integral operators is influenced by a singularity of the kernel functions. The integrals with regular integrands, that represent non-diagonal coefficients, are approximated by the Gaussian quadrature rules defined on a triangle *(Laursen and Gellert, 1978).* Their discrete form is given by

$$L_{ij} = \frac{1}{4\pi} A_j \sum_{k=1}^{K} \frac{w_k}{r_{ik}} \qquad i \neq j, \tag{13a}$$

$$M_{ij} = \frac{1}{4\pi} A_j k_{ij} \sum_{k=1}^{K} \frac{w_k}{r_{ik}^3} \qquad i \neq j, \tag{13b}$$

where A_j – the area of the j-th planar triangular element,
k_{ij} – the perpendicular from the collocation point i to the j-th element,
K – the number of points used for the Gaussian quadrature,
w_k – corresponding weights,
r_{ik} – the distance from the i-th collocation point to the j-th quadrature point in the triangle Γ_j.

The non-regular integrals (singular elements) arise only for the diagonal components of the linear system. They require special evaluation techniques in order to handle the singularity of the kernel function. Thanks to the diagonal component c_i and the orthogonality of the normal to its triangle, the kernel function in integrals on the left hand side in (10) are regular *(Balaš et al., 1985)*. Then we obtain

$$M_{ii} = c_i. \tag{14}$$

Diagonal coefficients L_{ii} can be evaluated analytically using the software *Mathematica®* *(Wolfram, 1996)*.

In case of the pure Neumann BC, the right hand side of the system (12) can be replaced by a known vector. Solving this linear system of equations we obtain values of the unknown disturbing potential in collocation points. As the disturbing potential is known on the Earth's surface and we have at disposal only surface gravity data, it is more natural to use a strategy of the Molodenskij concept *(Molodenskij et al. 1962)*. Hence, the disturbing potential is transformed to the height anomalies (quasigeoidal heights) using the modified Bruns formula *(Moritz, 1980)*. However, there is a problem of unknown sea level heights. Thanks to a small value of the normal gravity gradient we can overcome this problem in an iterative way symbolically written by the expression

$$\zeta^{i+1}(B, L) = \frac{T(B, L, H)}{\gamma(B, H - \zeta^i)}, \tag{15}$$

where B, L, H – the ellipsoidal (geodetic) coordinates of the collocation point,
$\zeta^i(B, L)$ – the height anomaly of the i-th iteration,
$T(B, L, H)$ – the disturbing potential at the collocation point,
$\gamma(B, H - \zeta^i)$ – the normal gravity on the "iterative" telluroid.

Note: In order to obtain a convergence it is practically sufficient to use two or three iterations. In the $0^{-\text{th}}$ iteration the height anomalies equal to zero. Then the "$0^{-\text{th}}$ iterative" telluroid is identical with the Earth surface.

4. Numerical experiments

The numerical experiments deal with the global quasigeoid modelling. The Earth's surface is approximated by an approximately homogenous triangulation of the topography based on a subdivision of triangular faces of a "12-hedron". Each triangle is subdivided into 4 congruent sub-triangles by halving the sides until a required level (Fig.1). The vertices of triangles represent the collocation points. The developed algorithm generates their horizontal positions. Vertical components, i.e. ellipsoidal (geodetic) heights, are obtained approximately by adding an appropriate terrain model determined by sea level heights to the geopotential model EGM-96. Such approach involves three approximations:

- The approximate expression that the sea level height plus the geoidal height equals to the ellipsoidal (geodetic) height.
- Sea level heights are approximated by DTM.
- Geoidal heights are approximated by EGM-96, i.e. they provide only long-wavelength part and not true geoidal heights.

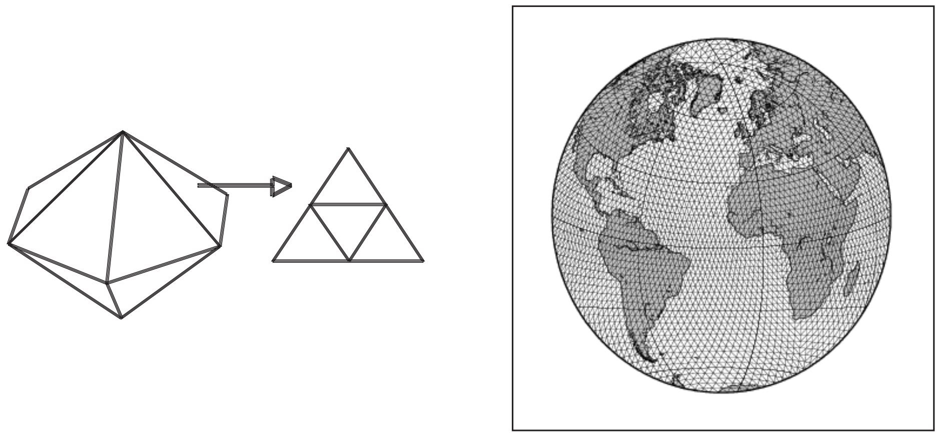

Fig. 1. The triangulation of the topography.

Input surface gravity disturbances need to be generated from available resources. We use two alternative data sets for preparing input data:

- **Set A: Gravity disturbances generated from EGM-96**

In the first alternative we use the available Fortran program $f477b$ written by prof. Rapp *(Rapp, 1994)*. This program is able to evaluate gravity disturbances at the collocation points inserting their ellipsoidal coordinates. Program $f477b$ is based on the strategy of geopotential models determined by spherical harmonics and geopotential coefficients. We use the geopotential coefficients of EGM-96 *(Lemoine et al., 1996)*, parameters of the reference ellipsoid defined by GRS-80 *(Moritz, 1992)* and the Global Digital Elevation Model GTOPO-30 *(EROS Data Centre)*.

The gravity disturbances generated by this alternative are dependent on the geopotential coefficients of EGM-96. Therefore numerical results in this case should converge to EGM-96. This fact will confirm a mathematical reliability of the variational solution.

- **Set B: Gravity disturbances generated from available gravity anomaly data**

In the second alternative we use available gravity anomaly data that were used in the development of EGM-96 *(Pavllis et al., 1996)*. This database provides a regular grid of points (grid size: $\Delta B \times \Delta L = 0.5° \times 0.5°$) all over the Earth's surface (except 2.3% that is an uncovered area) containing the Molodenskij free-air gravity anomalies defined on the Earth's surface. The sea level heights h used for the construction of this database were derived from the JGP95E global topographic database.

The surface gravity disturbances δg are generated from the Molodenskij free-air gravity anomalies Δg using an appropriate transformation (Fig.2). The ellipsoidal (geodetic) heights are obtained as a sum of the sea level heights from the available database and the geopotential model EGM-96.

With respect to a giant size of the Earth and in order to get accuracy as high as possible, computing on parallel computers is required. The final large scale computations were accomplished on the high-speed parallel computer TAJFUN: CRAY SV1-1/32 with 32 processors and 32 GB of the internal (shared) memory at ICM Warsaw (Acknowledgement). We approximated the Earth's surface by 44 378 collocation points and 88 752 triangles (latitude interval: Δ B = 1.0227°) using all 17 GB of the user disposable limit. The large nonsymetric linear system of equations was solved by nonstationary iterative method BiConjugate Gradient Stabilized (BiCGSTAB) *(Barrett et al., 1994)*. Only several iterations of Bi-CGSTAB were necessary to keep an error lower than the prescribed tolerance ε in absolute residual error, i.e. 16 iterations (Set A) and 17 iterations (Set B). There was no need for preconditioning thanks to the properties of the matrix $\boldsymbol{M}$, especially due

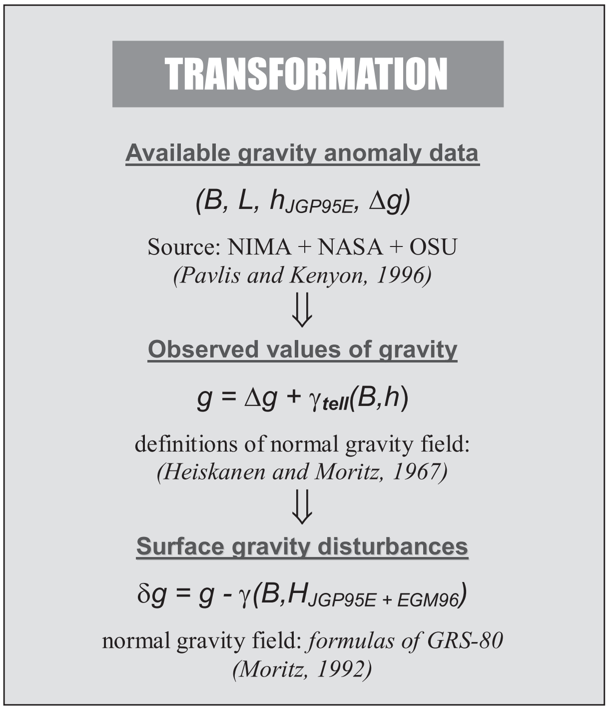

Fig. 2. Surface gravity disturbances generated from the available gravity anomalies.

to the strict diagonal dominance. Iterations took only several minutes while the matrices assembly several hours. Total CPU time took about 7 hours.

The Global Quasigeoid Models as the numerical results of the BEM application to NGBVP for both input data sets represent the variational (approximate) solutions of the geodetic boundary value problem. Instead of a complicated estimation of the theoretical accuracy for the achieved solutions, we compare our numerical results with EGM-96, i.e. with the geopotential model determined by a completely different mathematical approach based on the strategy of spherical harmonics and the Legendre polynomials *(Rapp, 1994)*.

- **Set A: Gravity disturbances generated from EGM-96**

While the gravity disturbances generated by this alternative depend on the geopotential coefficients of EGM-96, the numerical solution should agree with EGM-96. The comparison shows an evident correlation and agreement with EGM-96. This fact confirms a mathematical reliability of the proposed solution. Tab. 1 and Fig. 3 contains the basic statistical characteristics for residuals between EGM-96 and the Global Quasigeoid Model (set A). Their surface layout on the reference ellipsoid is depicted in Fig. 4.

Tab. 1.

STATISTICS					
Residuals: (EGM-96) - (BEM)					
Area	***Complete (Fig.3)****	***Oceans & Seas***	***Continents & Lands***	***Antarctica***	***Continents without Ant.***
Number of points	44 377	29 405	14 972	1 029	13 943
Mean	0.026 m	0.029 m	0.020 m	0.946 m	-0.048 m
St. deviation	0.706 m	0.570 m	0.918 m	1.723 m	0.786 m
Max. residual	6.161 m	3.613 m	6.161 m	5.010 m	6.161 m
Min. residual	-6.145 m	-5.682 m	-6.145 m	-5.509 m	-6.145 m
Note: The highest residual -15.147 m at the North Pole is omitted because EGM-96 is fixed to the reference ellipsoid at this point.					

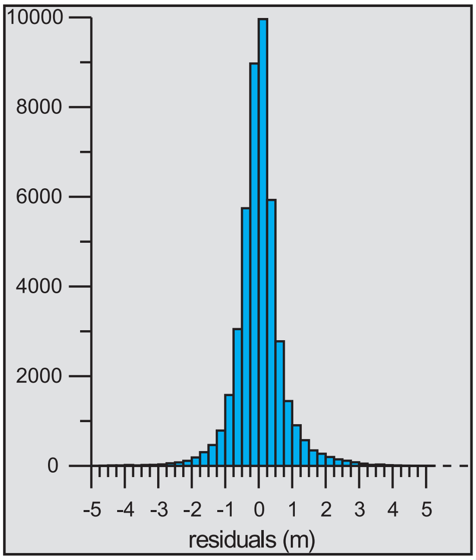

Fig. 3. Histogram of residuals (* Tab.1).

The comparison shows high accuracy of the achieved variational (approximate) solution. It demonstrates the obvious perspective of the proposed approach. Statistical parameters for partial regions (Tab. 1) show that the solution is more precise in regions of oceans and seas, while high residuals in Antarctica negatively affect accuracy on continents. Striking negative residuals in Himalayas correlate with the mountain range. They are probably due to a horizontal shift in the region of maximal deflections of vertical. Striking positive, but also negative, residuals in Antarctica may reflect a problem of ice sheet coverage that implies a problem of the Earth's surface determination as well as an influence of the ice sheet to gravity data.

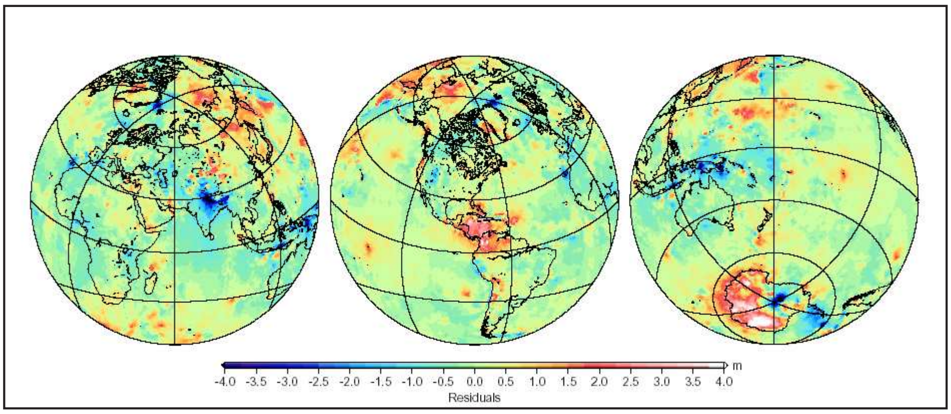

Fig. 4. Residuals between EGM-96 and the Global Quasigeoid Model (set A) (44 378 collocation points).

Further refining of the global triangulation could bring more precise numerical results but the problem of increasing requirements for the memory storage needs to be overcome. Tab. 2 shows how the accuracy of our solution increases with respect to the mesh size, i.e. to the number of collocation points. For a visual comparison, Fig. 5 depicts corresponding profiles along the parallel of latitude N 30°.

- **Set B: Gravity disturbances generated from available gravity anomaly data**

The gravity disturbances generated by this alternative are fully independent of the geopotential coefficients of EGM-96. Therefore the Global

Tab. 2.

STATISTICS ⇔ with respect to the mesh size				
Residuals: (EGM-96) - (BEM)				
Number of points	***1 946***	***5 402***	***38 402***	***44 378***
ΔB	5°	3°	1.125°	1.0227°
Mean value	0.580 m	-0.349 m	0.457 m	0.026 m
Standard deviation	7.525 m	3.951 m	0.869 m	0.706 m
Max. residual	47.875 m	32.778 m	7.987 m	6.161 m
Min. residual	-42.496 m	-33.064 m	-7.368 m	-6.145 m
Res.: the North Pole	-22.508 m	-20.939 m	-14.723 m	-15.147 m

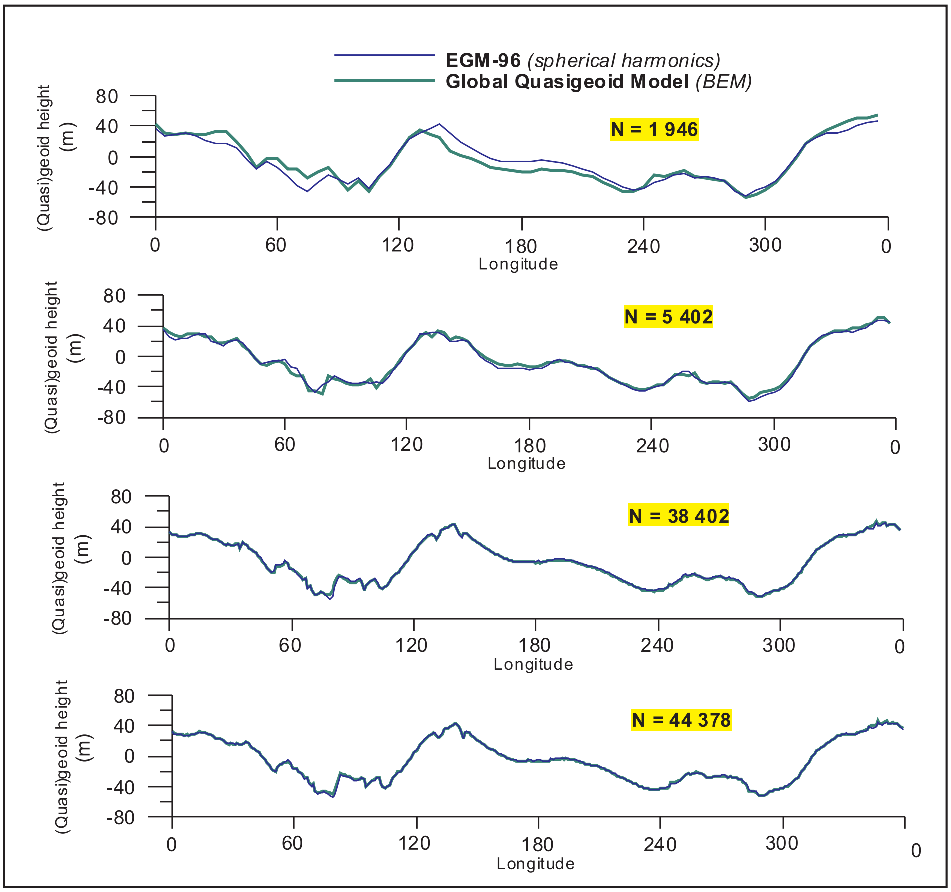

Fig. 5. Comparison with EGM-96 (profiles along the parallel of latitude N 30°).

Quasigeoid Model (set B) as the numerical result of the BEM application to NGBVP is also fully independent of EGM-96. The comparison with EGM-96 is depicted in Fig. 6. The basic statistical characteristics for arisen residuals are presented in Tab. 3 including the histogram of residuals (Fig. 7).

It is evident from depicted residuals (Fig. 6 + Tab. 3 + Fig. 7) that this solution (set B) is in less accordance with EGM-96 than the previous case (set A). It is due to the fact that both the input gravity data as well as the mathematical strategy are different from EGM-96 (spherical harmonics) and its geopotential coefficients. EGM-96 is involved only for generating ellipsoidal heights from available sea level heights.

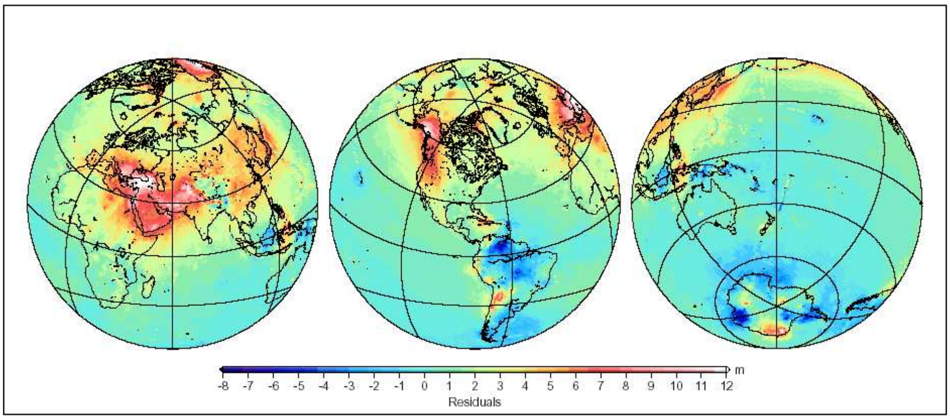

Fig. 6. Residuals between EGM-96 and the Global Quasigeoid Model (set B) (44 378 collocation points).

Tab. 3.

STATISTICS					
Residuals: (EGM-96) - (BEM)					
Area	***Complete (Fig. 7)****	***Oceans & Seas***	***Continents & Lands***	***Antarctica***	***Continents without Ant.***
Number of points	44 377	29 405	14 972	1 029	13 943
Mean	1.329 m	0.735 m	2.497 m	-0.187 m	2.695 m
St. deviation	2.122 m	1.365 m	2.763 m	2.645 m	2.666 m
Max. residual	15.367 m	11.079 m	15.367 m	12.213 m	15.367 m
Min. residual	-8.317 m	-7.581 m	-8.317 m	-8.317 m	-6.838 m

Note: The highest negative residual -12.139 m at the North Pole is omitted because EGM-96 is fixed to the reference ellipsoid at this point.

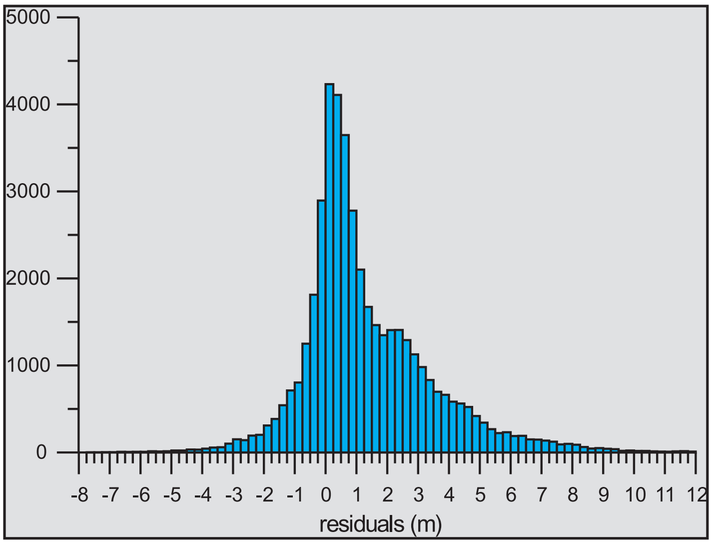

Fig. 7. Histogram of residuals (* Tab.3).

5. Conclusions

The boundary element method applied to the Neumann geodetic boundary value problem leads to the variational solution of the external Laplace problem on the Earth's surface. Although it is an approximate solution, the numerical results of the practical experiments and their comparison with the geopotential model EGM-96 show evident perspectives. The proposed approach represents an alternative method for the global gravity field modelling.

The use of surface gravity disturbances as the Neumann boundary conditions has obvious advantages with respect to gravimetric measurements. Ellipsoidal (geodetic) heights obtained from satellite observations represent sufficient vertical information. Economic and time demanding levelling is not needed in this case. These advantages are striking mainly in mountainous areas.

BEM using the collocation with linear basis functions appears to be very suitable and efficient numerical method for the solution of NGBVP. It provides solution on the Earth's surface where the classical solution can be hardly obtained. The variational solution by BEM involves several kinds of approximations. An error of approximations is theoretically known and can be reduced in order to get more precise numerical results, e.g. by using the higher-order interpolation functions or increasing an order of Gaussian quadrature.

The numerical results of the practical experiments (the Global Quasigeoid Models) show relatively high accuracy. They confirm the numerical reliability of the proposed approach. The developed computational algorithm is applicable for further investigation provided that the high-speed parallel computers will be at disposal.

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References

Balaš J., Sládek J., Sládek V., 1985: Analýza napätí metódou hraničných integrálnych rovníc (in Czech). VEDA, Bratislava.

Barrett R., Berry M., Chan T. F., Demmel J., Donato J., Dongarra J., Eijkhout V., Pozo R., Romine C., Van der Vorst H., 1994: Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. `http://www.netlib.org/templates/Templates.html`.

Brebbia C. A., Telles J. C. F., Wrobel L. C., 1984: Boundary Element Techniques, Theory and Applications in Engineering. Springer-Verlag, New York.

EROS Data Centre: GTOPO-30, `http://edcdaac.usgs.gov/gtopo30/gtopo30.html`.

Heiskanen W. A., Moritz H., 1967: Physical Geodesy. W. H. Freeman and Company, San Francisco.

Kirkup S., 2000: The BEM for Laplace Problems. Manual. `http://www.boundary-element-method.com/bemlap/manual.html`, Integrated Sound Software.

Laursen M. E., Gellert M., 1978: Some Criteria for Numerically Integrated Matrices and Quadrature Formulas for Triangles. International Journal for Numerical Methods in Engineering, **12**, 67–76.

Lemoine F. G., Kenyon S. C., Factor J. K., Trimmer R. G., Pavlis N. K., Chinn D. S., Cox C. M., Klosko S. M., Luthcke S. B., Torrence M. H., Wang Y. M., Williamson R. G., Pavlis E. C., Rapp R. H., Olson T. R., 1996: EGM-96 – The Development of the

NASA GSFC and NIMA Joint Geopotential Model. http://cddisa.gsfc.nasa.gov/926/egm96/.

Molodenskij M. S., Jeremejev B. F., Jurkina M. I., 1962: Methods for study of the external gravitational field and figure of the Earth. Israel program for scientific translations, Jerusalem (translated from Russian original, Moscow, 1960).

Moritz H., 1980: Advanced Physical Geodesy. Helbert Wichmann Verlag, Karlsruhe, Germany.

Moritz H., 1992: Geodetic reference system 1980. Bulletin Geodesique, **66**, (2), 187–192.

Pavllis N. K., Kenyon S. C., Manning D. M., 1996: Gravity anomaly data. http://www.ngdc.noaa.gov/seg/potfld/gravity/data/global/egm96/.

Rapp R. H., 1994: The Use of Potential Coefficient Models in Computing Geoid Undulation. In: International School for the Determination and Use of the Geoid. Lecture Notes. Milano. 71–99.

Wolfram S., 1996: The Mathematica book. Third edition. Wolfram Research Documentation Center.