Semi-implicit Diamond-cell Finite Volume Scheme for 3D Nonlinear Tensor Diffusion in Coherence Enhancing Image Filtering

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ABSTRACT. *This paper is devoted to a finite volume scheme for coherence enhancing diffusion filtering in 3D image processing. First, we derive the model, including a construction of its diffusion tensor. Then we design an original semi-implicit finite volume scheme for this 3D model with the help of the co-volume mesh. Our method is based on the choice of co-volumes as diamond-shaped polygons around each side of a 3D finite volume. Finally we discuss computational results in biomedical image processing illustrated in figures.*

KEYWORDS: *nonlinear tensor diffusion, 3D diamond-cell finite volume method, semi-implicit scheme, parabolic equation, coherence enhancing diffusion, image processing*

1. Introduction

Nonlinear diffusion is an interesting topic of study because of the amazing diversity of its applications, among which image processing has grown rapidly in recent decades, cf. [WEI 99, PRR 00, PRR 02, MNW 02, DRM 07]. In this paper we suggest an original 3D diamond-cell finite volume scheme for a 3D model of nonlinear tensor anisotropic filtering. The model is given in the following form

$$\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0, \qquad \text{in } Q_T \equiv I \times \Omega, \tag{1}$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega, \tag{2}$$

$$(D \nabla u) \cdot \mathbf{n} = 0, \qquad \text{on } I \times \partial\Omega, \tag{3}$$

where u denotes an intensity of greylevel 3D image, $u_0 \in L^2(\Omega)$, $I = [0, T]$ is a time interval, Ω is an image domain, D is a diffusion tensor depending on $u(x, t)$ and $\mathbf{n}$ is the outer normal unit vector to $\partial\Omega$. This model is useful in any situation, where strong smoothing is desirable in a particular direction, e.g. along 2D edge surfaces in 3D images, where a low smoothing is expected in the perpendicular direction. It has a capacity to improve the spatial coherence of structures, which can be deteriorated by a high level of noise.

2. Derivation of the diffusion tensor

First we build a gradient of the intensity function u given by

$$\nabla u_{\tilde{t}} = (u_{x_1}, u_{x_2}, u_{x_3})^T, \text{ where } \quad u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot, t))(x), \quad (\tilde{t} > 0) \qquad [4]$$

and $G_{\tilde{t}}$ is a smoothing kernel. We denote $||\nabla u_{\tilde{t}}||^2$ by μ. Provided $\mu > 0$ we choose the triplet of vectors (v_1, v_2, v_3) as follows

$$v_1 \parallel \nabla u_{\tilde{t}}, \qquad v_2 \perp \nabla u_{\tilde{t}}, \qquad v_3 \perp \nabla u_{\tilde{t}}, \qquad v_2 \perp v_3\,. \qquad [5]$$

The direction of vector v_1 represents in every point a direction of the largest change in image intensity. The other two vectors give a tangential plane to a level set of image intensity which may represent a 2D surface edge in 3D image, provided that μ is large, and we call it a coherence plane $\mathcal{P}$, cf. [MNW 02]. The coherence plane corresponds to an eigenspace corresponding to eigenvalue 0 of the outer product $\nabla u_{\tilde{t}} \otimes \nabla u_{\tilde{t}}$.

The idea of the nonlinear diffusion tensor filtering is as follows. We obtain a processed version $u(x,t)$ of an original image $u_0(x)$ with a scale parameter $t \geq 0$ as the solution of mathematical model [1]-[3], where D depends on solution u, satisfies smoothness and symmetry properties. In order to enhance coherence, the diffusion tensor D must steer a filtering process such that diffusion is strong and increasing with the level of μ along the coherence plane and is small in the direction of vector v_1. To that end, we choose the eigenvalues of the diffusion tensor D by

$$\begin{aligned} \kappa_1 &= \alpha, \quad \alpha \in (0,1),\ \alpha \ll 1, \qquad [6] \\ \kappa_2 &= \begin{cases} \alpha, & \text{if } \mu = 0, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{\mu}\right),\ C > 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and corresponding eigenspaces are given by v_1 and $\mathcal{P}$. In such a way, and, applying Gaussian smoothing with variance ρ we get the diffusion matrix D in the form

$$D = G_\rho * D_0, \text{ where } D_0 = \begin{cases} B, & \text{if } \mu = 0, \\ PBP^{-1} & \text{otherwise,} \end{cases} \quad B = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix} \quad [7]$$

and P is a transition matrix from the basis (v_1, v_2, v_3) to (e_1, e_2, e_3). The exponential function in [6] is used because it ensures that κ_2 does not exceed 1, and the positive parameter α guarantees that the process never stops; even if μ tends to zero, there still remains some small linear diffusion. C has the role of a threshold parameter. If $\mu \gg C$ then $\kappa_2 \approx 1$, and, conversely if $\mu \ll C$ then $\kappa_2 \approx \alpha$.

In fact, the matrix D_0 is uniformly positive definite, it does not depend on the concrete choice of v_2 and v_3 and, if $\mu > 0$, in standard basis it has the form

$$\frac{1}{\mu}\begin{pmatrix} u_{x_1}^2\kappa_1 + (u_{x_2}^2 + u_{x_3}^2)\kappa_2 & u_{x_1}u_{x_2}(\kappa_1 - \kappa_2) & u_{x_1}u_{x_3}(\kappa_1 - \kappa_2) \\ u_{x_1}u_{x_2}(\kappa_1 - \kappa_2) & u_{x_2}^2\kappa_1 + (u_{x_1}^2 + u_{x_3}^2)\kappa_2 & u_{x_2}u_{x_3}(\kappa_1 - \kappa_2) \\ u_{x_1}u_{x_3}(\kappa_1 - \kappa_2) & u_{x_2}u_{x_3}(\kappa_1 - \kappa_2) & u_{x_3}^2\kappa_1 + (u_{x_1}^2 + u_{x_2}^2)\kappa_2 \end{pmatrix}$$

It depends explicitly on $\nabla u_{\tilde{t}}$ and can thus be evaluated in a direct and fast way using the diamond-cell finite volume method (see also next section). Then the matrices, and correspondingly the coherence and gradient directions, are spatially averaged using convolution applied to the matrix elements. Since

$$z^T D(x) z = z^T (G_\rho * D_0(x)) z = \sum_{i=1}^{3}\sum_{j=1}^{3} z_i z_j \int_{R^3} G_\rho(x-\xi) d_{ij}(\xi) d\xi =$$

$$\int_{R^3} G_\rho(x-\xi) \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j d_{ij}(\xi) d\xi \geq \alpha \int_{R^3} G_\rho(x-\xi) d\xi = \alpha > 0,$$

where $x \in \Omega$, $z = (z_1, z_2, z_3)$ is any non-zero vector in R^3 and d_{ij} are elements of the matrix D_0, we see that the diffusion tensor D is positive definite. At this point we differ from other possible approaches, see e.g. [WEI 99], where the so-called 3D structure tensor is first built using convolution applied to the outer product of the intensity gradient, and its eigenvalues and eigenvectors are constructed and used for building diffusion tensor. In the general 3D case this procedure is more complicated than our method (with explicitly given matrix D_0) because it has to deal with eigenvector and eigenvalue analysis of general 3×3 matrices in every image pixel. However, it can also be made fast using, e.g., AOS schemes [WEI 99,WES 02]. A similar approach to ours (without explicitly stating the convolution by G_ρ) is also given in [MNW 02].

3. Finite volume scheme for 3D nonlinear tensor anisotropic diffusion

The goal of this section is to derive our method of calculation. Let the image be represented by $n_1 \times n_2 \times n_3$ voxels (finite volumes) such that it looks like a mesh with n_1 rows, n_2 columns and n_3 layers. Let $\Omega = (0, n_1 h) \times (0, n_2 h) \times (0, n_3 h)$, h be a voxel size and let the image $u(x)$ be given by a bounded mapping $u : \Omega \to R$. We consider the smoothing process in a time interval $I = [0, T]$. Let $0 = t_0 \leq t_1 \leq \cdots \leq t_{N_{max}} = T$ denote the time discretization with $t_n = t_{n-1} + k$, where k is a length of discrete time step. In our scheme we will look for u^n an approximation of solution at time t_n, for every $n = 1, ..., N_{max}$. As usual in finite volume methods, we integrate equation [1] over finite volume K, then provide a semi-implicit time discretization and use a divergence theorem to get

$$\frac{u_K^n - u_K^{n-1}}{k} m(K) - \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds \quad = \quad 0, \qquad [8]$$

where u_K^n, $K \in \mathcal{T}_h$, represents the mean value of u^n on K and $\mathcal{T}_h$ is a cubic finite volume mesh. Further quantities and notations are described as follows: $m(K)$ is a 3D measure of the finite volume K with boundary ∂K, $\sigma_{KL} = K \cap L$ is a side of the finite volume K, where $L \in \mathcal{T}_h$ is an adjacent finite volume to K such that a 2D measure $m(K \cap L) \neq 0$. Due to simplifying notations, we use σ instead of σ_{KL} at several places if no confusion can appear. $\mathcal{E}_K$ is set of sides such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \sigma$ and $\mathcal{E} = \bigcup_{K \in \mathcal{T}_h} \mathcal{E}_K$. The set of boundary sides is denoted by $\mathcal{E}_{ext}$, that is $\mathcal{E}_{ext} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$ and let $\mathcal{E}_{int} = \mathcal{E} \setminus \mathcal{E}_{ext}$. Υ is the set of pairs of adjacent finite volumes, defined by $\Upsilon = \{(K,L) \in \mathcal{T}_h^2,\ K \neq L,\ m(K \cap L) \neq 0\}$ and $\mathbf{n}_{K,\sigma}$ is the normal unit vector to σ outward to K.

Let our numerical solution be $u_{h,k}(x,t) = \sum\limits_{n=0}^{N_{max}} \sum\limits_{K \in \mathcal{T}_h} u_K^n \chi_{\{x \in K\}} \chi_{\{t_{n-1} < t \leq t_n\}}$, where the function $\chi_{\{A\}}$ is defined as $\chi_{\{A\}} = \begin{cases} 1, & \text{if A is true,} \\ 0, & \text{elsewhere.} \end{cases}$ In our scheme we start the computation by defining initial values $u_K^0 = \frac{1}{m(K)} \int_K u_0(x) dx$, $K \in \mathcal{T}_h$ and let $u_{h,k}^n(x) = \sum\limits_{K \in \mathcal{T}_h} u_K^n \chi\{x \in K\}$ denote a finite volume approximation at the n-th time step. In order to obtain the scheme we write [8] in the form

$$\frac{u_K^n - u_K^{n-1}}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \phi_\sigma^n(u_{h,k}^n) m(\sigma) = 0, \qquad [9]$$

where $m(\sigma)$ is the measure of side σ and $\phi_\sigma^n(u_{h,k}^n)$ denotes an approximation of the exact averaged flux $\frac{1}{m(\sigma)} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{K,\sigma} ds$ for any K and $\sigma \in \mathcal{E}_K$.

We construct $\phi_\sigma^n(u_{h,k}^n)$ with the help of a co-volume mesh, cf. e.g. [CVV 99, DRM 07], for the 2D case. The co-volume χ_σ associated with σ is constructed around each finite volume side by joining four vertices of this side and midpoints of finite volumes which are common to this side, cf. Fig.1. The co-volume boundary is given by triangles $\bar\sigma \subset \partial\chi_\sigma$ (we denote their vertices by $N_1(\bar\sigma)$, $N_2(\bar\sigma)$ and $N_3(\bar\sigma)$) and let $\mathbf{n}_{\chi_\sigma,\bar\sigma}$ be the normal unit vector to $\bar\sigma$ outward to χ_σ. In order to approximate diffusion flux, using divergence theorem, we first derive an approximation of the averaged gradient on χ_σ, namely $\frac{1}{m(\chi_\sigma)} \int_{\chi_\sigma} \nabla u^n dx = \frac{1}{m(\chi_\sigma)} \int_{\partial\chi_\sigma} u^n \mathbf{n}_{\chi_\sigma,\bar\sigma} ds$ and then we approximate it by $p_\sigma^n(u) = \frac{1}{m(\chi_\sigma)} \sum\limits_{\bar\sigma \in \partial\chi_\sigma} \frac{1}{3} \left(u_{N_1(\bar\sigma)}^n + u_{N_2(\bar\sigma)}^n + u_{N_3(\bar\sigma)}^n \right) m(\bar\sigma) \mathbf{n}_{\chi_\sigma,\bar\sigma}$. For each side σ, let the values at x_E and x_W be denoted as u_E and u_W, and let the values u_{TN}, u_{TS}, u_{BN}, and u_{BS} at the vertices x_{TN}, x_{TS}, x_{BN}, and x_{BS}, cf. Fig. 1, be computed as the arithmetic mean of u_K, where K are finite volumes which are common to the vertex. Since our mesh is uniform and squared, we can use the following relations: $m(\chi_\sigma) = \frac{h^3}{3}$, $m(\bar\sigma) = \frac{\sqrt{2}}{4} h^2$ and after a short calculation we are ready to state

$$\begin{aligned} p_\sigma^n(u) = \frac{u_E^n - u_W^n}{h} \mathbf{n}_{K,\sigma} \quad &+ \quad \frac{u_{TN}^n + u_{BN}^n - u_{TS}^n - u_{BS}^n}{2h} \mathbf{t1}_{K,\sigma} \\ &+ \quad \frac{u_{TN}^n + u_{TS}^n - u_{BN}^n - u_{BS}^n}{2h} \mathbf{t2}_{K,\sigma}, \qquad [10] \end{aligned}$$

where $\mathbf{t1}_{K,\sigma}$ is a unit vector parallel to $x_{TN} - x_{TS}$ such that $(x_{TN} - x_{TS}) \cdot \mathbf{t1}_{K,\sigma} > 0$ and $\mathbf{t2}_{K,\sigma}$ is a unit vector parallel to $x_{TN} - x_{BN}$ such that $(x_{TN} - x_{BN}) \cdot \mathbf{t2}_{K,\sigma} > 0$. Replacing the exact gradient ∇u^n by the numerical gradient $p^n_\sigma(u)$ in approximation

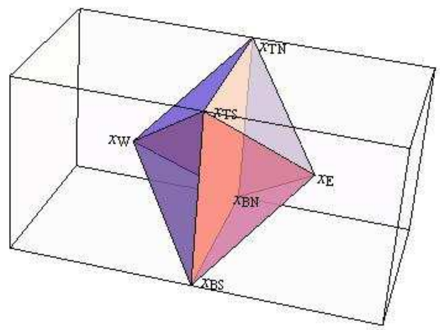

Figure 1. *The vertices of co-volume χ_σ associated with side σ*

of $\phi^n_\sigma(u^n_{h,k})$ we get the numerical flux in the form

$$\phi^n_\sigma(u^n_{h,k}) = (D_\sigma p^n_\sigma(u)) \cdot \mathbf{n}_{K,\sigma}, \qquad [11]$$

where $D_\sigma = D^{n-1}_\sigma = \begin{pmatrix} \bar{D}^\sigma_{11} & \bar{D}^\sigma_{12} & \bar{D}^\sigma_{13} \\ \bar{D}^\sigma_{12} & \bar{D}^\sigma_{22} & \bar{D}^\sigma_{23} \\ \bar{D}^\sigma_{13} & \bar{D}^\sigma_{23} & \bar{D}^\sigma_{33} \end{pmatrix}$ is an approximation of the mean value of matrix D along σ evaluated at the previous time step. To that end we take $u^{n-1}_{h,k}$ for the construction of the diffusion tensor. Because of the convolutions in (4) and (7), the elements of matrix D_σ are C^∞ functions.

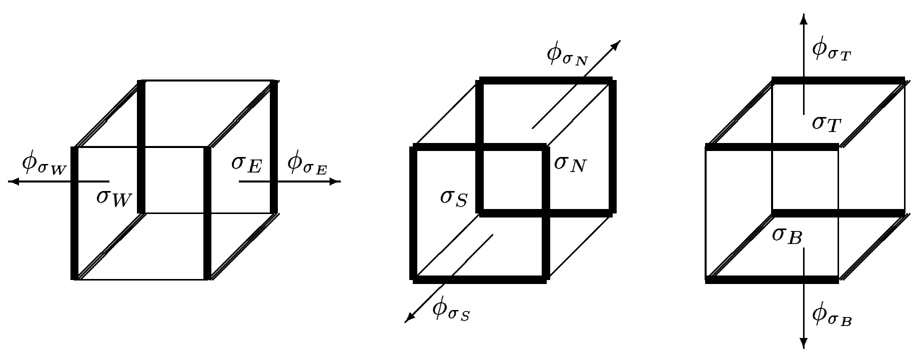

Figure 2. *A finite volume K, its boundaries $\sigma_i,\ i = E, W, N, S, T, B$ and the fluxes outward to the finite volume K*

It is important to note that in [11] we always consider the matrix D_σ written in the basis $(\mathbf{n}_{K,\sigma}, \mathbf{t1}_{K,\sigma}, \mathbf{t2}_{K,\sigma})$, cf. [CVV 99, DRM 07] for an analogy with the

2D model. Although it may look artificial, it will simplify further considerations. In practice it means that, cf. Fig. 2, if the matrix D is given in standard basis on side σ by $\begin{pmatrix} D^\sigma_{11} & D^\sigma_{12} & D^\sigma_{13} \\ D^\sigma_{12} & D^\sigma_{22} & D^\sigma_{23} \\ D^\sigma_{13} & D^\sigma_{23} & D^\sigma_{33} \end{pmatrix}$ then it does not change in new basis on two sides σ_W and σ_E. For two other sides σ_S and σ_N, in new basis it has the form $\begin{pmatrix} D^\sigma_{22} & D^\sigma_{12} & D^\sigma_{23} \\ D^\sigma_{12} & D^\sigma_{11} & D^\sigma_{13} \\ D^\sigma_{23} & D^\sigma_{13} & D^\sigma_{33} \end{pmatrix}$, and for the last two sides σ_B and σ_T it becomes $\begin{pmatrix} D^\sigma_{33} & D^\sigma_{23} & D^\sigma_{13} \\ D^\sigma_{23} & D^\sigma_{22} & D^\sigma_{12} \\ D^\sigma_{13} & D^\sigma_{12} & D^\sigma_{11} \end{pmatrix}$. Using such matrix representations, definition [11] can be written in this compact form

$$\phi^n_\sigma(u^n_{h,k}) = \left[\begin{pmatrix} \bar{D}^\sigma_{11} & \bar{D}^\sigma_{12} & \bar{D}^\sigma_{13} \\ \bar{D}^\sigma_{12} & \bar{D}^\sigma_{22} & \bar{D}^\sigma_{23} \\ \bar{D}^\sigma_{13} & \bar{D}^\sigma_{23} & \bar{D}^\sigma_{33} \end{pmatrix} \begin{pmatrix} \frac{u^n_E - u^n_W}{h} \\ \frac{u^n_{TN} + u^n_{BN} - u^n_{TS} - u^n_{BS}}{2h} \\ \frac{u^n_{TN} + u^n_{TS} - u^n_{BN} - u^n_{BS}}{2h} \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= \bar{D}^\sigma_{11} \frac{u^n_E - u^n_W}{h} + \bar{D}^\sigma_{12} \frac{u^n_{TN} + u^n_{BN} - u^n_{TS} - u^n_{BS}}{2h} + \bar{D}^\sigma_{13} \frac{u^n_{TN} + u^n_{TS} - u^n_{BN} - u^n_{BS}}{2h} .$$

Finally, let us summarize our **semi-implicit finite volume scheme**:

$$\frac{u^n_K - u^{n-1}_K}{k} - \frac{1}{m(K)} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{int}} \phi^n_\sigma(u^n_{h,k}) m(\sigma) = 0, \qquad [12]$$

$$\text{where} \quad \phi^n_\sigma(u^n_{h,k}) = \bar{D}^\sigma_{11} \frac{u^n_E - u^n_W}{h} + \bar{D}^\sigma_{12} \frac{u^n_{TN} + u^n_{BN} - u^n_{TS} - u^n_{BS}}{2h} + \bar{D}^\sigma_{13} \frac{u^n_{TN} + u^n_{TS} - u^n_{BN} - u^n_{BS}}{2h}. \qquad [13]$$

Let us note that when we consider the arithmetic mean of the voxel values for the values of u_{TN}, u_{TS}, u_{BN} and u_{BS} in [13], we end up with 27 point finite volume scheme. We solve the resulting linear system by the Gauss-Seidel iterative method.

4. Numerical experiments

In this section we present computational results using real 3D images coming from multiphoton laser scanning microscopy. It represents the membranes of sea urchin cells in the early stages of embryogenesis and its size is $200 \times 200 \times 94$ voxels.

The images of the membranes are well suited for processing by this type of diffusion, which is documented by comparing the edge detection results before and after filtering in Figures 3-4. The edge detection is a well suited measure of the filtering quality, because filtering usually serves as a preliminary step of segmentation, which strongly depends on proper edge detection result. In the experiments we use $h = 0.01$, $k = 0.00001$, $C = 1$, $\alpha = 0.001$, $\tilde{t} = 0.00001$, $\rho = 0.002$. The satisfactory results were obtained after a few filtering steps and in the presented experiments we did not observe any instability problem, which is a usual drawback of explicit schemes, cf. [WES 02].

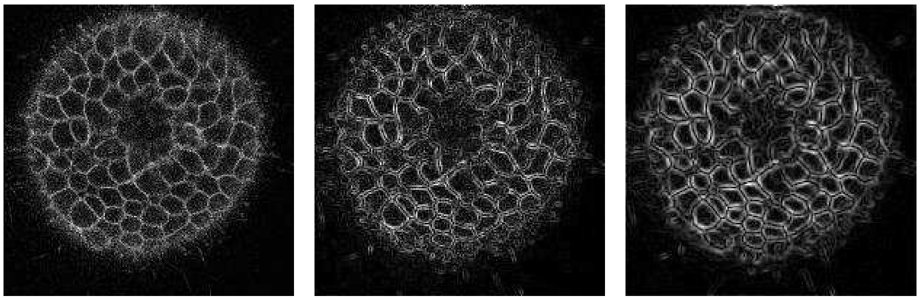

Figure 3. *2D slice of 3D membranes image. Original image (left), edge detection of the original image (in the middle), edge detection of the filtered image after* 3 *steps (right)*

In Figure 3 and in its zoom in Figure 4, we can clearly see the enhancement of the structure connectivity and improvement of the quality of the edge detection using 3 filtering steps. It is not possible to correctly recognize noisy membranes in the central part of the original image (Figure 3 left). Comparing the edge detection of original (middle) and after 3D filtering (right), we can see that membranes become visible after diffusion and thus can also be segmented. A more detailed result is presented in Figure 4. In the upper left part is a zoom of a noisy original, in the upper right is a result of filtering. At the bottom there are two edge detections, on the left using original, on the right using the filtered image. The connectivity and denoising of edges given by the black pixels in the edge detection images is highly improved, especially for the cells in the bottom part.

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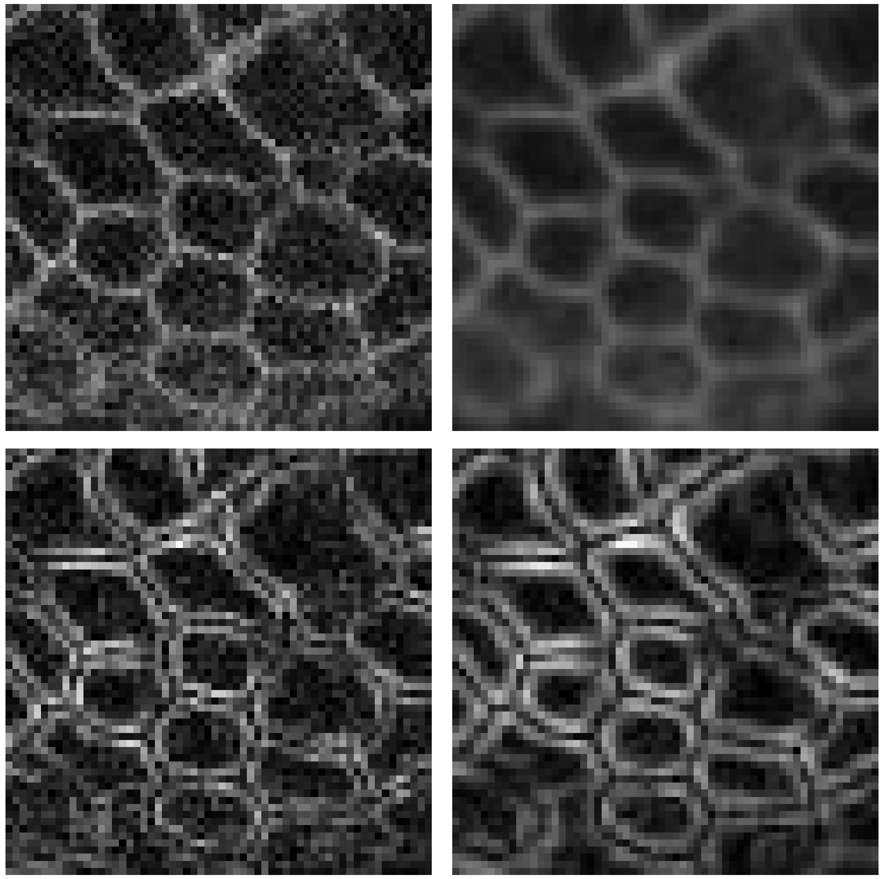

Figure 4. *Detail (*70×70 *pixels) of image from Fig. 3. Original image (top left), image filtered by tensor diffusion after* 3 *steps (top right), edge detection of the original image (bottom left), edge detection of the filtered image (bottom right)*

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