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An inequality between the edge-Wiener index and the Wiener index of a graph

Martin Knor\textsuperscript{a,b}, Riste Škrekovski\textsuperscript{b,c,d},
Aleksandra Tepeh\textsuperscript{b,e}

\textsuperscript{a}Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68, Bratislava, Slovakia,
knor@math.sk

\textsuperscript{b}Faculty of Information Studies, 8000 Novo Mesto, Slovenia,
\textsuperscript{c}FAMNIT, University of Primorska, 6000 Koper, Slovenia,
\textsuperscript{d}Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana,
skrekovski@gmail.com

\textsuperscript{e}Faculty of Electrical Engineering and Computer Science, University of Maribor, Smetanova ulica 17, 2000 Maribor, Slovenia,
aleksandra.tepeh@gmail.com

Abstract

The Wiener index $W(G)$ of a connected graph $G$ is defined to be the sum $\sum_{u,v} d(u,v)$ of distances between all unordered pairs of vertices in $G$. Similarly, the edge-Wiener index $W_e(G)$ of $G$ is defined to be the sum $\sum_{e,f} d(e,f)$ of distances between all unordered pairs of edges in $G$, or equivalently, the Wiener index of the line graph $L(G)$. Wu [37] showed that $W_e(G) \geq W(G)$ for graphs of minimum degree 2, where equality holds only when $G$ is a cycle. Similarly, in [24] it was shown that $W_e(G) \geq \frac{\delta^2}{4} W(G)$ where $\delta$ denotes the minimum degree in $G$. In this paper, we extend/improve these two results by showing that $W_e(G) \geq \frac{\delta^2}{4} W(G)$ with equality satisfied only if $G$ is a path on 3 vertices or a cycle. Besides this, we also consider the upper bound for $W_e(G)$ as well as the ratio $\frac{W_e(G)}{W(G)}$. We show that among graphs $G$ on $n$ vertices $\frac{W_e(G)}{W(G)}$ attains its minimum for the star.

Keywords: Wiener index, Gutman Index, Line graph

1 Introduction

For a graph $G$, let $\text{deg}(u)$ and $d(u,v)$ denote the degree of a vertex $u \in V(G)$ and the distance between vertices $u, v \in V(G)$, respectively. Let $L(G)$ denote the line graph of
$G$, that is, the graph with vertex set $E(G)$ and two distinct edges $e, f \in E(G)$ adjacent in $L(G)$ whenever they share an end-vertex in $G$. Furthermore, for $e, f \in E(G)$, we let $d(e, f)$ denote the distance between $e$ and $f$ in the line graph $L(G)$.

In this paper we consider three important graph invariants, called Wiener index (denoted by $W(G)$ and introduced in [36]), edge-Wiener index (denoted by $W_e(G)$ and introduced in [21]) and Gutman index (denoted by $\text{Gut}(G)$ and introduced in [12]), which are defined as follows:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v),$$
$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e, f),$$
$$\text{Gut}(G) = \sum_{\{u,v\} \subseteq V(G)} \deg(u) \deg(v) \cdot d(u, v).$$

Observe that the edge-Wiener index of $G$ is nothing but the Wiener index of the line graph $L(G)$ of $G$. Note also that in the literature a slightly different definition of the edge-Wiener index is sometimes used; for example, in [20] edge-Wiener index is defined to be $W_e(G) + \binom{n}{2}$ where $W_e(G)$ is defined as above and $n$ is the order of $G$.

The Wiener index and related distance-based graph invariants have found extensive application in chemistry, see for example [14, 15, 34], and [2, 8, 16, 17, 18, 30, 31] for some recent studies. The Wiener index of a graph was investigated also from a purely graph-theoretical point of view (for early results, see for example [9, 33], and [4, 25, 26, 38] for some surveys). Generalizations of Wiener index and relationships between these were studied in a number of papers (see for example [3, 5, 6, 20]), and relationships between generalized graph entropies and the Wiener index (among other related topological indices) were established in [28]. New results on the Wiener index are constantly being reported, see for instance [10, 19, 23, 29, 35] for recent research trends.

Wu [37] showed that $W_e(G) \geq W(G)$ for graphs of minimum degree 2 where equality holds only when $G$ is a cycle. Similarly, in [24] it was shown that $W_e(G) \geq \frac{2\delta - 1}{\delta} W(G)$ where $\delta$ denotes the minimum degree in $G$. In this paper, we improve these two results by showing that $W_e(G) \geq \frac{4}{\delta} W(G)$ with equality satisfied only if $G$ is a path on 3 vertices or a cycle. One of the closely related distance-based graph invariant is the Szeged index [11], and a relation between the Szeged index and its edge version was recently established in [27].

In [3] it was proved that $W_e(G) \leq \frac{2^2}{3\pi} + O(n^{3/2})$ for graphs of order $n$. Using the result of [32] we improve this bound to $W_e(G) \leq \frac{2^2}{3\pi} + O(n^4)$. We also consider the ratio $\frac{W_e(G)}{W(G)}$ and show that this ratio is minimum if $G$ is the star $S_n$ on $n$ vertices. Consequently, if $G$ is a graph on $n$ vertices, then $\frac{W_e(G)}{W(G)} \geq \frac{n^2-2}{2(n-1)}$.

### 2 Distances, average distance and $D_\alpha$ relations

Note that for any two distinct edges $e = u_1u_2$ and $f = v_1v_2$ in $E(G)$, the distance between $e$ and $f$ equals

$$d(e, f) = \min\{d(u_i, v_j) : i, j \in \{1, 2\}\} + 1. \quad (1)$$

In the case when $e$ and $f$ coincide, we have $d(e, f) = 0$. In addition to the distance between two edges we will also consider the average distance between the endpoints of
two edges, defined by
\[
s(u_1u_2, v_1v_2) = \frac{1}{4} \left( d(u_1, v_1) + d(u_1, v_2) + d(u_2, v_1) + d(u_2, v_2) \right).
\]

Notice that \( s(e, f) = \frac{1}{2} \) when \( e \) and \( f \) coincide. The average distance of endpoints is in an interesting relationship with the Gutman index of a graph. Namely, if one likes to consider the version of edge-Wiener index where the distances between edges are replaced by the average distances of their endpoints, then what one gets is essentially the Gutman index, see Lemma 1.

A variation to the following result was mentioned in [24, 37], where the sum in (2) is taken over all ordered pairs of edges. In our case the sum runs over all 2-element subsets of \( E(G) \).

**Lemma 1.** Let \( G \) be a connected graph. Then

\[
\sum_{\{e,f\} \subseteq E(G)} s(e, f) = \frac{1}{4} \left( \operatorname{Gut}(G) - |E(G)| \right).
\]

**Proof.** Consider the sum on the left-hand side of (2). We can rewrite it as

\[
\frac{1}{4} \sum_{uw, vz \subseteq E(G)} \left( d(u, v) + d(u, z) + d(w, v) + d(w, z) \right).
\]

Now, for any two non-adjacent vertices of \( G \), say \( u \) and \( v \), the distance \( d(u, v) \) appears in the above sum precisely once for each pair of edges, where one of these edges is incident with \( u \) and the other is incident with \( v \). Thus, \( d(u, v) \) appears in total precisely \( \deg(u) \cdot \deg(v) \) times. And, if \( u \) and \( v \) are two adjacent vertices of \( G \), then the distance \( d(u, v) = 1 \) appears in that sum precisely \( \deg(u) \cdot \deg(v) - 1 \) times. Thus, the above sum equals

\[
\frac{1}{4} \left[ \sum_{uw \in E(G)} \deg(u)\deg(v)d(u, v) + \sum_{uv \in E(G)} \left( \deg(u)\deg(v) - 1 \right)d(u, v) \right],
\]

which is the right-hand side of (2).

Now we define the following notions. Let \( G \) be a graph. For a pair of edges \( e \) and \( f \) of \( G \) we define the *difference*

\[
D(e, f) = d(e, f) - s(e, f).
\]

Moreover, if \( D(e, f) = \alpha \), we say that \( e, f \) form a pair of type \( D_\alpha \) or that the pair \( e, f \) belongs to the set \( D_\alpha \). Note that if \( e = f \), then \( D(e, f) = -\frac{1}{2} \). Denote by \( \mathcal{I} \) the set \( \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \). Note that \( \sum_{\alpha \in \mathcal{I}} |D_\alpha| = \binom{|E(G)|}{2} \). Next easy lemma shows that \( D(e, f) \in \mathcal{I} \) whenever \( e \neq f \).

**Lemma 2.** In a connected graph, every pair of distinct edges belongs to \( D_\alpha \) for some \( \alpha \in \mathcal{I} \).
Proof. Let \( e = u_1u_2 \) and \( f = v_1v_2 \) be two distinct edges. We may assume that
\[
d(u_1, v_1) = \min_{i,j \in \{1,2\}} \{d(u_i, v_j)\}.
\]
Let \( k = d(u_1, v_1) \). Notice that
\[
d(u_1, v_2), d(u_2, v_1) \in \{k, k + 1\} \quad \text{and} \quad d(u_2, v_2) \in \{k, k + 1, k + 2\}.
\]
If \( d(u_2, v_2) = k + 2 \), then it must hold \( d(u_1, v_2) = d(u_2, v_1) = k + 1 \), and hence \( D(e, f) = 0 \), which means that the pair \( e, f \) belongs to \( D_0 \). So, in the sequel, we assume that \( d(u_2, v_2) = k \) or \( k + 1 \). Suppose \( d(u_1, v_2) = d(u_2, v_1) = k \). If \( d(u_2, v_2) = k \), then the pair \( e, f \) belongs to \( D_1 \). And, if \( d(u_2, v_2) = k + 1 \) then the pair \( e, f \) belongs to \( D_2 \).
Suppose now that \( d(u_1, v_2) = k + 1 \) and \( d(u_2, v_1) = k \). If \( d(u_2, v_2) = k \), then again the pair \( e, f \) belongs to \( D_2 \). On the other hand, if \( d(u_2, v_2) = k + 1 \), then the pair \( e, f \) belongs to \( D_2 \). We argue similarly if \( d(u_1, v_2) = k \) and \( d(u_2, v_1) = k + 1 \). Finally, suppose that \( d(u_1, v_2) = d(u_2, v_1) = k + 1 \). If \( d(u_2, v_2) = k \), the pair \( e, f \) belongs to \( D_2 \). If \( d(u_2, v_2) = k + 1 \), the pair \( e, f \) belongs to \( D_2 \).

To prove our main result we will have to distinguish two possibilities for \( \alpha = \frac{1}{2} \).
If (according to the notation in the proof of Lemma 2) \( d(u_1, v_1) = d(u_2, v_2) = k \) and \( d(u_1, v_2) = d(u_2, v_1) = k + 1 \), then we say that the pair belongs to \( D_1 \), and if \( d(u_1, v_1) = d(u_2, v_1) = k \) and \( d(u_1, v_2) = d(u_2, v_2) = k + 1 \), we say that the pair belongs to \( D_2 \).

In Figure 1, where all different configurations of pairs of edges are presented, full lines represent the edges \( u_1u_2 \) and \( v_1v_2 \).

**Proposition 3.** Let \( G \) be a connected graph. Then
\[
W_e(G) = \frac{\text{Gut}(G)}{4} - \frac{|E(G)|}{4} + |D_1| + \frac{1}{4}|D_\frac{1}{2}| + \frac{1}{2}|D_\frac{3}{2}| + \frac{3}{4}|D_2|.
\]

**Proof.** By Lemma 1, we have
\[
W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e,f)
= \sum_{\{e,f\} \subseteq E(G)} s(e,f) + \sum_{\{e,f\} \subseteq E(G)} D(e,f)
= \frac{\text{Gut}(G)}{4} - \frac{|E(G)|}{4} + \sum_{\{e,f\} \subseteq E(G)} D(e,f).
\]
Now, as every pair \( e, f \) belongs to precisely one of \( D_\alpha \) for some \( \alpha \in \mathcal{I} \), we have
\[
\sum_{\{e,f\} \subseteq E(G)} D(e,f) = 0 \cdot |D_0| + \frac{1}{4} \cdot |D_\frac{1}{2}| + \frac{1}{2} \cdot |D_\frac{3}{2}| + \frac{3}{4} \cdot |D_2| + 1 \cdot |D_1|,
\]
and the proof follows. \( \square \)
3 Bounds for $W_e(G)$

Klavžar and Lipovec [22] proved the following result.

**Lemma 4.** Let $G$ be a 2-connected graph that is not a cycle. Then $G$ contains two isometric cycles $C_1 = u_1u_2 \ldots u_ku_{k+1} \ldots u_ru_1$ and $C_2 = u_1u_2 \ldots u_kv_{k+1} \ldots v_rv_1$, where $r \geq s > k \geq 2$ and $v_i \neq u_j$ for $i, j \geq k + 1$.

Moreover, equality holds if and only if $G$ is a cycle.

**Proof.** Let $G$ be a 2-connected graph. It is well-known that if an edge $e = xy$ belongs to a cycle, then it must belong to an isometric cycle. (In order to obtain such a cycle just take the edge $e$ and a shortest path distinct from the path $e$ connecting $x$ and $y$, which must exist since $xy$ lies on a cycle.) Let $E_0$ (resp. $E_1$) be the set of edges that belong to some isometric cycle in $G$ of even (resp. odd) length. Since $|E(G)| = |E_0| + |E_1| - |E_0 \cap E_1|$, we have $|E(G)| \leq |E_0| + |E_1|$.

Notice that if $e$ is an edge of an even isometric cycle $C$, and $e'$ is its antipodal edge on $C$, then the pair $e, e'$ belongs to $D_2'$. Let $G_0$ be a graph such that $V(G_0) = E_0$ and two vertices are adjacent in $G_0$ if the corresponding edges in $G$ belong to a pair in $D_2'$. This gives us

$$|E_0| = |V(G_0)| \leq \sum_{v \in V(G_0)} \deg(v) = 2|E(G_0)| = 2|D_2'|,$$

as every vertex in $V(G_0)$ is of degree at least 1, since every edge of $E_0$ is at least in some pair of $D_2'$.

Similarly, if $e$ is an edge of an odd isometric cycle $C$, and $e_1, e_2$ are antipodal edges of $e$, then the pairs $e, e_1$ and $e, e_2$ belong to $D_4'$. Defining a graph $G_1$ with $V(G_1) = E_1$ and two vertices being adjacent in $G_1$ if the corresponding edges in $G$ belong to a pair in $D_4'$, we get

$$|E_1| = 2|V(G_1)| \leq \sum_{v \in V(G_1)} \deg(v) = 2|E(G_1)| = 2|D_4'|,$$

since every vertex in $V(G_1)$ is of degree at least 2, as every edge of $E_1$ is at least in two pairs of $D_4'$. Thus $|E_1| \leq |D_4'|$ and $2|D_2'| + |D_4'| \geq |E_0| + |E_1| \geq |E(G)|$.

If $G$ is an even cycle, we clearly have $|E_0| = 2|D_2'|$ and $|E_1| = 0$, and if $G$ is an odd cycle, then $|E_1| = |D_4'|$ and $|E_0| = 0$. Thus, if $G$ is a cycle, we have equality in (4). Now, we show that as soon as $G$ is not a cycle, strict inequality holds in (4). By Lemma 4, there exist two different isometric cycles $C$ and $C'$ such that $C \cap C'$ is a path of length at least one. Denote this path by $S$ and let $u_1u_2$ be the first edge on this path.
If one of the cycles $C$ and $C'$ is even and the other is odd, we have $u_1 u_2 \in E_0 \cap E_1$, thus $|E(G)| < |E_0| + |E_1|$, which readily implies $|E(G)| < 2|D_{1/2}^+| + |D_{1/4}|$.

Now assume that both $C$ and $C'$ are even. Observe that every pair of edges that lie on an isometric path belongs to $D_0$. Thus, since $S$ is isometric, the edge that is antipodal to the edge $u_1 u_2$ on $C$ ($C'$, respectively) belongs to $C \setminus S$ ($C' \setminus S$, respectively). This means that the degree of the vertex in $G_0$ that corresponds to $u_1 u_2$ is at least 2, which implies strict inequality in (5), i.e $|E_0| < 2|D_{1/2}^+|$ and thus $|E(G)| \leq |E_0| + |E_1| < 2|D_{1/2}^+| + |D_{1/4}|$.

Similarly, if both $C$ and $C'$ are odd, we observe that the two antipodal edges of $u_1 u_2$ in $C$ are different from the antipodal edges of $u_1 u_2$ in $C'$. This yields a strict inequality in (6) (since the vertex corresponding to $u_1 u_2$ is of degree at least 4 in $G_1$) and the result follows.

To prove the main theorem in case of regular graphs the following observation will be needed.

**Lemma 6.** Suppose that $G \neq K_2$ is a regular graph containing bridges. Then every end-block of $G$ contains an edge $e$ such that for every bridge $b$ the pair $e, b$ is in $D_{1/2}^+$. 

**Proof.** Let $G$ be a regular graph of degree $k$. Since $G \neq K_2$, we have $k \geq 2$. Let $B$ be an end-block, and let $v$ be the cut-vertex incident with $B$. Since $k \geq 2$, $B$ contains at least 3 vertices. Moreover, all vertices of $B$ are of degrees $k$ in $B$ except $v$.

We claim that $B$ is a non-bipartite graph. Suppose to the contrary that $B$ is bipartite with bipartition $L, R$ of $V(B)$. Assume that $v \in R$. Then $k|L| = |E(B)| = k(|R| - 1) + \deg_B(v)$, which implies that $k$ divides $\deg_B(v)$, a contradiction.

For each $i \geq 0$, denote by $L_i$ the vertices of $B$ at the distance $i$ from $v$. As $B$ is non-bipartite, some $L_i$ will contain adjacent vertices. Hence, there is an edge $e = u_1 u_2$ of $B$ with $d(u_1, v) = d(u_2, v)$.

Now we will show that $e$ is the required edge. For any bridge $b = v_1 v_2$ notice that $d(v_1, v) \neq d(v_2, v)$, otherwise we obtain that $b$ lies on a cycle. So, we may assume that $d(v_1, v) = d(v_2, v) + 1$. As $B$ is an end-block attached to the rest of the graph at $v$, every shortest path from a vertex of $B$ to a vertex in $G - B$ must contain the vertex $v$. Hence

$$d(u_1, v_2) = d(u_1, v) + d(v, v_2) = d(u_2, v) + d(v, v_2) = d(u_2, v_2),$$

and similarly, $d(u_1, v_1) = d(u_2, v_1)$. Thus,

$$d(u_1, v_2) = d(u_2, v_2) = d(u_1, v_1) - 1 = d(u_2, v_1) - 1,$$

and hence the pair $e, b$ is in $D_{1/2}^+$. \qed

Now we are ready to prove the main result.

**Theorem 7.** Let $G$ be a connected graph of minimum degree $\delta$. Then,

$$W_e(G) \geq \frac{\delta^2}{4} W(G)$$

with equality holding if and only if $G$ is isomorphic to a path on three vertices or a cycle.
Proof. We distinguish two cases.

Case 1: $G$ is non-regular.

Then $G$ has a vertex $w \in V(G)$ of degree at least $\delta + 1$. By Proposition 3, we have

$$4W_e(G) = \text{Gut}(G) - |E(G)| + 4|D_1| + |D_{\frac{3}{2}}| + 2|D_{\frac{1}{2}}| + 3|D_{\frac{4}{2}}|$$

$$\geq \text{Gut}(G) - |E(G)|$$

$$\geq \delta^2 \sum_{\{u,v\} \in V(G) \setminus \{w\}} d(u,v) + (\delta + 1) \sum_{u \in V(G) \setminus \{w\}} \deg(u)d(u,w) - |E(G)|$$

$$\geq \delta^2 W(G) + \sum_{u \in V(G) \setminus \{w\}} \deg(u) - |E(G)|$$

$$\geq \delta^2 W(G).$$

Note that in order to obtain equality in (7), no edge lies on a cycle by Lemma 5, otherwise we have $|D_{\frac{3}{2}}| > 0$ or $|D_{\frac{1}{2}}| > 0$. This implies that $G$ is a tree, and so $\delta = 1$. Moreover, each edge is incident with $w$, as we need that

$$\sum_{u \in V(G) \setminus \{w\}} \deg(u) = |E(G)|,$$

which implies that $G$ is a star. And finally, we need $\deg(w) = \delta + 1 = 2$, which implies that $G$ is isomorphic to $P_3$. This establishes the case.

Case 2: $G$ is regular.

Let $B$ be the set of bridges of $G$ and let $E_c$ be the set of edges of $G$ that lie on at least one cycle. Then $E(G) = B \cup E_c$ and $B \cap E_c = \emptyset$. One can check that if a pair of edges belongs to $D_{\frac{1}{2}}$ or $D_{\frac{3}{2}}$, then this pair belongs to the same block. Now, applying Lemma 5 to every nontrivial block of $G$, i.e. to every block containing a cycle, we obtain cumulatively that

$$2|D_{\frac{1}{2}}| + |D_{\frac{3}{2}}| \geq |E_c|.$$

If $G$ has bridges, i.e. if $B \neq \emptyset$, then $G$ has at least two end-blocks. Now, Lemma 6 assures the existence of two distinct edges $e'$ and $e''$ such that for every bridge $b$ each of the pairs $b, e'$ and $b, e''$ belongs to $D_{\frac{3}{2}}$. So we have

$$|D_{\frac{3}{2}}| \geq 2|B|.$$

Now, starting with the equality (3) and using the fact that $\text{Gut}(G) = \delta^2 W(G)$ for regular graphs, we obtain

$$4W_e(G) = \text{Gut}(G) - |E(G)| + 4|D_1| + |D_{\frac{3}{2}}| + 2|D_{\frac{1}{2}}| + 3|D_{\frac{4}{2}}|$$

$$= \delta^2 W(G) - |E_c| - |B| + 4|D_1| + |D_{\frac{3}{2}}| + 2|D_{\frac{1}{2}}| + 2|D_{\frac{3}{2}}| + 3|D_{\frac{4}{2}}|$$

$$\geq \delta^2 W(G) - |E_c| - |B| + 4|D_1| + |E_c| + 4|B| + 3|D_{\frac{4}{2}}|$$

$$= \delta^2 W(G) + 4|D_1| + 3|B| + 3|D_{\frac{4}{2}}|$$

$$\geq \delta^2 W(G).$$

Note that in order to obtain equality in (7), $B = \emptyset$, i.e., $G$ must have no bridges. So there are no trivial blocks in $G$. Next, in order to have the equality, by Lemma 5 every nontrivial block must be a cycle. This means that all blocks of $G$ are cycles. Consequently, since $G$ is regular, we conclude that $G$ is a cycle. \qed

Now we consider the upper bound for $W_e(G)$. In [32] we have the following theorem:
Theorem 8. Let $G$ be a connected graph on $n$ vertices. Then

$$\text{Gut}(G) \leq \frac{2^4}{5^5} n^5 + O(n^4).$$

Using this theorem we prove the following statement.

Theorem 9. Let $G$ be a connected graph on $n$ vertices. Then

$$\text{We}(G) \leq \frac{2^2}{5^5} n^5 + O(n^4).$$

Proof. For all pairs of edges $e, f \in E(G)$ and for all $u_i, v_j$, where $e = u_iu_2$, $f = v_1v_2$ and $i, j \in \{1, 2\}$, we sum the distances $d_{L(G)}(e, f)$. In this way we get $4\text{We}(G)$. Now we group these distances according to the pairs $u_i, v_j$. That is, for all pairs of vertices $u, v \in V(G)$ (including the pairs of identical vertices) we take all edges $e$ incident to $u$ and all edges $f$ incident to $v$, and we sum $d_{L(G)}(e, f)$. Let $e$ be an edge incident to $u$ and let $f$ be an edge incident to $v$. Then

$$d_{L(G)}(e, f) \leq d_G(u, v) + 1.$$

By $c(u, v)$ we denote the sum $\sum_{e, f} d_{L(G)}(e, f)$ taken over all edges $e, f$ such that $e$ is incident with $u$ and $f$ is incident with $v$. Then

$$c(u, v) = \sum_{e, f} d_{L(G)}(e, f) \leq \deg(u)\deg(v)\left(d_G(u, v) + 1\right).$$

By Theorem 8, we have

$$4\text{We}(G) = \sum_{u \neq v} c(u, v) + \sum_u c(u, u) \leq \sum_{u \neq v} \deg(u)\deg(v)\left(d_G(u, v) + 1\right) + \sum_u (\deg(u))^2 \cdot 1 \leq \text{Gut}(G) + \sum_{u \neq v} \deg(u)\deg(v) + \sum_u (\deg(u))^2 \leq \frac{2^4}{5^5} n^5 + O(n^4) + O(n^4) + O(n^3) = \frac{2^4}{5^5} n^5 + O(n^4).$$

4 A lower bound for $\text{We}(G)/\text{W}(G)$

The problem of finding the graphs on $n$ vertices, whose line graph has maximal Wiener index (i.e. whose edge-Wiener index is maximal) was given by Gutman [13] (see also [7]). Moreover, Dobrynin and Mel’nikov [7] proposed to estimate the ratio $\text{W}(L^i(G))/\text{W}(G)$, where $L^i(G)$ stands for an iterated line graph, defined inductively as

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In this section we consider the case $i = 1$ and give a tight lower bound for $\frac{\text{We}(G)}{\text{W}(G)}$.

We need two well-known results. While the first one is already a folklore (and follows from a result in [9]), the second was proved by Buckley in [1].
Theorem 10. Among all trees on \( n \) vertices, the star \( S_n \) has the smallest Wiener index.

Theorem 11. If \( T \) is a tree on \( n \) vertices, \( n \geq 2 \), then \( W_e(T) = W(T) - \binom{n}{2} \).

Now we are able to prove our lower bound.

Theorem 12. Among all connected graphs on \( n \) vertices, the fraction \( \frac{W_e(G)}{W(G)} \) is minimum for the star \( S_n \), in which case \( \frac{W_e(G)}{W(G)} = \frac{n-2}{2(n-1)} \).

Proof. First we prove that if \( G \) is not a tree, then \( \frac{W_e(G)}{W(G)} \geq \frac{1}{2} \). Thus, assume that \( G \) is not a tree. We start with the following claim.

Claim 1. There is \( f : V(G) \to E(G) \) such that for every \( v \in V(G) \) the edge \( f(v) \) is incident with \( v \) and \( f(u) \neq f(v) \) whenever \( u \neq v \).

By Claim 1, \( G \) has a collection of \( n \) edges that can be considered as a system of distinct representatives for the vertices in such a way, that a vertex and an edge representing the vertex must be incident.

Proof of Claim 1. We start with trees. Since trees have only \( n-1 \) edges, they cannot satisfy Claim 1. However, for every tree \( T \) and for every vertex \( v_0 \in V(T) \), one can find \( f : V(T) \setminus \{v_0\} \to E(T) \) satisfying Claim 1. To see this, it suffices to set \( f(v) \) to be the first edge of the unique \( v, v_0 \)-path in \( T \).

Now let \( e_0 \) be an edge of \( G \) such that deleting \( e_0 \) results in a connected graph. Further, let \( T \) be a spanning tree of \( G \) which does not contain \( e_0 \). Denote by \( v_0 \) a vertex incident with \( e_0 \) in \( G \) and construct \( f : V(T) \setminus \{v_0\} \to E(T) \) as described above. Then the extension of \( f \) to \( V(T) = V(G) \) by setting \( f(v_0) = e_0 \) satisfies Claim 1.

Now we proceed with the proof of Theorem 12. Consider a function \( f \) satisfying Claim 1. We have

\[
W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d_{L(G)}(e,f) \geq \sum_{\{u,v\} \subseteq V(G)} d_{L(G)}(f(u),f(v)),
\]

where the sums are taken over all pairs of distinct elements of \( E(G) \) and \( V(G) \), respectively. Hence,

\[
\frac{W_e(G)}{W(G)} \geq \frac{\sum_{\{u,v\} \subseteq V(G)} d_{L(G)}(f(u),f(v))}{\sum_{\{u,v\} \subseteq V(G)} d_G(u,v)}.
\]

The fraction on the right-hand side is the smallest when the denominator is as big as possible compared with the numerator. Since \( d_{L(G)}(f(u),f(v)) \geq d_G(u,v) - 1 \), that is \( d_G(u,v) \leq d_{L(G)}(f(u),f(v)) + 1 \), we get

\[
\frac{W_e(G)}{W(G)} \geq \frac{\sum_{\{u,v\} \subseteq V(G)} d_{L(G)}(f(u),f(v))}{\sum_{\{u,v\} \subseteq V(G)} (d_{L(G)}(f(u),f(v)) + 1)}.
\]

Since \( f(u) \neq f(v) \) whenever \( u \neq v \), we have \( d_{L(G)}(f(u),f(v)) \geq 1 \), which gives \( \frac{W_e(G)}{W(G)} \geq \frac{1}{2} \).

Thus, assume that \( G \) is a tree. By Theorem 11, we have

\[
\frac{W_e(G)}{W(G)} = \frac{W(G) - \binom{n}{2}}{W(G)}.
\]

Hence, \( \frac{W_e(G)}{W(G)} \) achieves its minimum for a tree with the minimum Wiener index. Since \( W(S_n) = 2^{\binom{n}{2}} + (n-1) = (n-1)^2 \), Theorem 10 completes the proof.
Acknowledgment. The first author acknowledges partial support by Slovak research grants VEGA 1/0781/11, VEGA 1/0065/13 and APVV 9223-10. All authors are partially supported by Slovenian research agency ARRS, program no. P1–00383, project no. L1–4292, and Creative Core–FISNM–3330-13-500033.

References


Figures

Figure 1: Different configurations of pairs of edges.