Smallest regular graphs of given degree and diameter

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Abstract

In this note we present a sharp lower bound on the number of vertices in a regular graph of given degree and diameter.

1 Introduction

The *degree/diameter* problem consists in determination of the largest order $N(d,k)$ of a graph with (maximum) degree d and diameter k. An upper bound for $N(d,k)$ is the *Moore bound* $M(d,k) = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}$ and graphs achieving this bound are called *Moore graphs*. As shown in [1, 3, 5], Moore graphs exist only when $d = 2$ or $k = 1$ or when $k = 2$ and the degree is either 3 or 7 or possibly 57. For all other pairs (d,k) we have $N(d,k) \leq M(d,k) - 2$, see [2, 4]. Recently, there are plenty of papers dealing with the degree/diameter problem, some of them constructing "large" graphs of given degree and diameter, which increases the lower bound for $N(d,k)$ for special pairs (d,k), other decreasing $N(d,k)$ for special classes of graphs. For a nice survey see [7].

In this note we consider the inverse of degree/diameter problem. Since usually the degree/diameter problem is formulated for regular graphs (although some authors require only that d is the maximum degree), we ask what is the minimum order $n(d,k)$ of a regular graph of degree d and diameter k. In this note we answer this question completely.

We start with some notation. Let G be a graph, $G = (V(G), E(G))$. For two of its vertices, say x and y, by $\mathrm{dist}_G(x,y)$ we denote their distance in G. By $N_i(x)$ we denote the set of vertices that are at distance i from x. As usual, $N_1(x)$ is often abbreviated to $N(x)$. The longest distance in G is the *diameter* $\mathrm{diam}(G)$. The

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complete graph on n vertices is denoted by K_n and the discrete graph on n vertices (the complement of K_n) is denoted by D_n. If G is a graph, then by $G^{(-1)}$ (and $G^{(-2)}$) we denote a graph obtained from G by removing all the edges of one 1-factor (one 2-factor).

If G and H are graphs, then $G + H$ denotes the *join* of G and H, that is, a graph obtained from the disjoint union of G and H by adding all edges xy, where $x \in V(G)$ and $y \in V(H)$. The *sequential join* of graphs G_1, G_2, ..., G_r is denoted by $G_1 + G_2 + \ldots + G_r$ and is defined by

$$G_1 + G_2 + \ldots + G_r = (G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{r-1} + G_r).$$

Thus, one can obtain $G_1 + G_2 + \ldots + G_r$ from the disjoint union $G_1 \cup G_2 \cup \ldots \cup G_r$ by adding all edges xy where $x \in V(G_i)$ and $y \in V(G_{i+1})$ for $i = 1, 2, \ldots, r-1$. To simplify the expressions, instead of

$$\ldots + \underbrace{G + G + \ldots + G}_{k \text{ times}} + \ldots \qquad \text{we write} \qquad \ldots + (G)_k + \ldots.$$

Finally, denote by $G \div H$ a graph obtained from the disjoint union of G and H by adding all edges of one 1-factor, every edge of which joins a vertex of G with a vertex of H. Obviously, $G \div H$ is defined only if $|V(G)| = |V(H)|$. Analogously as in the case of join, by $G_1 \div G_2 \div \ldots \div G_r$ we denote the graph $(G_1 \div G_2) \cup (G_2 \div G_3) \cup \ldots \cup (G_{r-1} \div G_r)$. We can form also more complicated expressions using both $+$ and $\div$. In such a way, $\mathrm{K}_1 + \mathrm{D}_2 \div \mathrm{D}_2 \div \mathrm{K}_2$ is a cycle of length 7; see Figure 1.

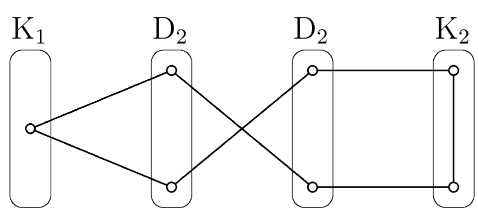

Figure 1: The graph $\mathrm{K}_1 + \mathrm{D}_2 \div \mathrm{D}_2 \div \mathrm{K}_2$.

2 Results

For small diameters we have the following statement.

Proposition 2.1. *Let $d \geq 2$. We have:*

(i) $n(d, 1) = d + 1$*;*

(ii) *if d is even then $n(d, 2) = d + 2$;*

(iii) *if d is odd then $n(d,2) = d+3$;*

(iv) $n(d,3) = 2d+2$.

Proof. The case $k=1$ is obvious since K_{d+1} is the unique graph of diameter 1 and degree d.

Let $k=2$. Let G be a d-regular graph of diameter 2, and let $x, y \in V(G)$ such that $\mathrm{dist}_G(x,y) = 2$. Then $\{x\} \cup N(x) = N_0(x) \cup N_1(x)$, which gives $|N_0(x)| + |N_1(x)| = d+1$. Since $y \in N_2(x)$, we have $|V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| \geq d+2$, which gives $n(d,2) \geq d+2$. However, if d is odd then $|V(G)|$ cannot be odd and so $n(d,2) \geq d+3$ in this case. If d is even then $\mathrm{K}_{d+2}^{(-1)}$ is a d-regular graph of diameter 2 on $d+2$ vertices, which shows $n(d,2) \leq d+2$; while if d is odd then $\mathrm{K}_{d+3}^{(-2)}$ is a d-regular graph of diameter 2 on $d+3$ vertices, which shows $n(d,2) \leq d+3$.

Finally, let $k=3$. Analogously as above, let G be a d-regular graph of diameter 3, and let $x, y \in V(G)$ such that $\mathrm{dist}_G(x,y) = 3$. Then $\{x\} \cup N(x) = N_0(x) \cup N_1(x)$, which gives $|N_0(x)| + |N_1(x)| = d+1$, and $\{y\} \cup N(y) \subseteq N_2(x) \cup N_3(x)$, which gives $|N_2(x)| + |N_3(x)| \geq d+1$. Thus, $|V(G)| = |N_0(x)| + |N_1(x)| + |N_2(x)| + |N_3(x)| \geq 2d+2$, and so $n(d,3) \geq 2d+2$. On the other hand, denote by $\mathrm{K}_{n,n}$ a complete bipartite graph on $2n$ vertices in which the two partite sets have n vertices each. Then $\mathrm{K}_{d+1,d+1}^{(-1)}$ is a d-regular graph of diameter 3 on $2d+2$ vertices, which shows $n(d,3) \leq 2d+2$. $\square$

Now we turn our attention to larger diameters. Since there are only two 2-regular graphs of diameter k, namely the cycle on $2k$ vertices and the cycle on $2k+1$ vertices, we have the following trivial observation.

Proposition 2.2. *If $k \geq 4$ then $n(2,k) = 2k$.*

For larger degrees we have a slightly different bound.

Theorem 2.3. *Let $k = 3j+t$, where $k \geq 4$ and $0 \leq t \leq 2$, and let $d \geq 3$. Then $n(d,k) = (d+1)(j+1)+t+\delta$, where $\delta = 1$ if either d is odd and $t=1$ or d is even and $t=2$. Otherwise $\delta = 0$.*

Proof. First we prove a lower bound for $n(d,k)$. Let G be a regular graph of degree d and diameter k and let $x, y \in V(G)$ such that $\mathrm{dist}_G(x,y) = k$. Denote $n_i = |N_i(x)|$. Since $x \in N_0(x)$, we have $\{x\} \cup N(x) \subseteq N_0(x) \cup N_1(x)$. Thus, $n_0 + n_1 \geq d+1$. Analogously $n_{k-1} + n_k \geq d+1$ since $y \in N_k(x)$. Further, for every i, $1 \leq i \leq j-1$, we have $n_{3i-1} + n_{3i} + n_{3i+1} \geq d+1$ since for $z_i \in N_{3i}(x)$ it holds $\{z_i\} \cup N(z_i) \subseteq N_{3i-1}(x) \cup N_{3i}(x) \cup N_{3i+1}(x)$. Finally, if $t \geq 1$ then $n_{k-1-\ell} \geq 1$ where $1 \leq \ell \leq t$. Summing up all these inequalities we get

$$|V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1)+t.$$

If $t = 2$ then we used $n_{k-3} \geq 1$ and $n_{k-2} \geq 1$. But if d is even then G cannot have a bridge, and so $n_{k-3}+n_{k-2} \geq 3$. Thus, we get $|V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1)+t+1$ in this case.

Similarly, if $t = 1$ and d is odd then $(d+1)(j+1)+t$ is an odd number. But a regular graph of odd degree cannot have an odd number of vertices, and so $|V(G)| = \sum_{i=0}^{k} n_i \geq (d+1)(j+1)+t+1$ also in this case.

To prove the upper bound we construct extremal graphs, that is, regular graphs of degree d and diameter k on $n(d,k)$ vertices. First we define an extremal graph G for odd d. The case $k = 4$ is treated separately. If $d = 3$ then one extremal graph G is on Figure 2. For $d \geq 5$ we set:

$G = \mathrm{K}_2 + \mathrm{K}_{d-1}^{(-2)} + \mathrm{D}_2 \div \mathrm{D}_2 + \mathrm{K}_{d-1}$.

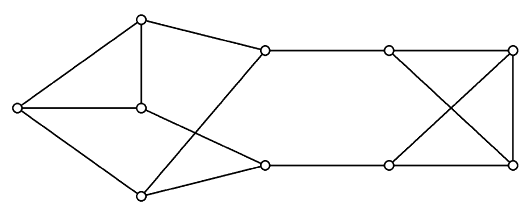

Figure 2: An extremal graph for $d = 3$ and $k = 4$.

Recall that $k = 3j + t$. To cover the remaining diameters, that is, 5, 6, 7, ..., in the next we assume $j \geq 1$ if $t = 2$, and $j \geq 2$ if $t = 0$ or $t = 1$:

$G = \mathrm{K}_2 + \mathrm{K}_{d-1}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1})_{j-1} + \mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1}^{(-1)} + \mathrm{K}_2$ if $t = 2$;
$G = \mathrm{K}_2 + \mathrm{K}_{d-1}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1})_{j-2} + \mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1} \div \mathrm{K}_{d-1}^{(-1)} + \mathrm{K}_2$ if $t = 0$;
$G = \mathrm{K}_2 + \mathrm{K}_{d-1}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1})_{j-2} + \mathrm{K}_1 + \mathrm{K}_1 + \mathrm{K}_{d-1}^{(-1)} + \mathrm{D}_2 \div \mathrm{D}_2 + \mathrm{K}_{d-1}$ if $t = 1$.

Now we define an extremal graph G for even d. To cover all possible diameters, that is, 4, 5, 6, ..., in the next we assume $j \geq 1$ if $t = 1$ or $t = 2$, and $j \geq 2$ if $t = 0$:

$G = \mathrm{K}_3 + \mathrm{K}_{d-2}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_{d-2})_{j-1} + \mathrm{K}_1 + \mathrm{D}_2 + \mathrm{K}_{d-1}$ if $t = 1$;
$G = \mathrm{K}_3 + \mathrm{K}_{d-2}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_{d-2})_{j-1} + \mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_{d-2}^{(-2)} + \mathrm{K}_3$ $t = 2$;
$G = \mathrm{K}_3 + \mathrm{K}_{d-2}^{(-1)} + (\mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_{d-2})_{j-2} + \mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_{d-2} \div \mathrm{K}_{d-2}^{(-1)} + \mathrm{K}_3$ if $t = 0$.

Observe that in all these graphs, whenever we removed a 1-factor out of K_q, then the number of vertices q was even. Obviously, in each case G has diameter k and it is a matter of routine to check that G is a regular graph of degree d. (For example, a vertex in the last copy of $K_{d-2}^{(-1)}$ in the last graph is joined to 1 vertex of K_{d-2}, $d-4$ vertices of $\mathrm{K}_{d-2}^{(-1)}$ and to 3 vertices of K_3, so its degree is $1 + d - 4 + 3 = d$.) Also, in each of these cases the number of vertices of G attains the bound of the theorem. To verify this statement it suffices to check the number of vertices for the smallest admissible values of j since in each case in the brackets we have exactly $d+1$ vertices. $\square$

By Proposition 2.2, if $d = 2$ then $n(d,k) = dk$. However, for higher degrees we get $n(d,k) \sim \frac{1}{3}dk$. Denote by $n_{\mathrm{VT}}(d,k)$ the minimum number of vertices in a vertex-transitive d-regular graph with diameter k. As shown in [6], for $k \geq 4$ and "large" d we have $n_{\mathrm{VT}}(d,k) \sim \frac{2}{3}dk$, and so $n_{\mathrm{VT}}(d,k) \doteq 2n(d,k)$ in this case. On the other hand, since the extremal graphs constructed in the proof of Proposition 2.1 are vertex-transitive, we have $n_{\mathrm{VT}}(d,k) = n(d,k)$ when $k \leq 3$.

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