Biembedding Abelian groups with mates having transversals

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Abstract

A certain recursive construction for biembeddings of Latin squares has played a substantial role in generating large numbers of nonisomorphic triangular embeddings of complete graphs. In this paper we prove that, except for the groups C_2, C_2^2 and C_4, each Latin square formed from the Cayley table of an Abelian group appears in a biembedding in which the second Latin square has a transversal. Such biembeddings may then be freely used as ingredients in the recursive construction.

Running head: Abelian groups and biembeddings

AMS classifications: 05B15, 05C10.

Keywords: Abelian group, Cayley table, Complete tripartite graph, Latin square, Topological embedding, Transversal.

1 Background

A graph embedding is *face 2-colourable* if the faces may be coloured by two colours in such a way that any two faces with a common boundary edge receive different colours. A face 2-colourable triangular embedding of $K_{n,n,n}$ determines two Latin squares of order n, one for each colour class, by assigning the vertices of the three partite sets as labels for the rows, columns and entries (in any one of the six possible orders) of the two Latin squares. Two Latin squares L and L' of the same order are said to be *paratopic* if they lie in the same main class, which is to say that there are three bijections from the rows, columns and symbols of L to those of some conjugate of L'. We say that two Latin squares of order n are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of $K_{n,n,n}$ in which the face sets forming the two colour classes give paratopic copies of the two squares. In [3] a recursive construction was presented for such biembeddings. This construction requires that in one of the ingredient biembeddings, one of the two squares should possess a transversal. In applications, it has proved advantageous for the other square to possess a nice structure, such as being a Cayley table of some group. The construction can be used to produce large numbers of nonisomorphic triangular embeddings of both $K_{n,n,n}$ and of K_n for suitable values of n. By this method, the best lower bounds on these numbers have been established, each of which is of the form n^{an^2} and apply to infinite sets of values of n for suitable positive constants a [1].

In [4] we proved that, except for the group C_2^2, every Latin square formed from a Cayley table of an Abelian group appears in a biembedding. It would be useful for future applications of the recursive construction if it could be determined which Cayley tables of Abelian groups appear in a biembedding in which the second square (the *mate*) has a transversal. In the current paper we show that such biembeddings exist except in three cases, namely when the Abelian group is C_2, C_2^2 or C_4.

The reasons for the three exceptions are easy to see. The Cayley table of C_2^2 appears in no biembedding, while the unique biembeddings in which the Cayley tables of C_2 and C_4 appear are with copies of themselves, and both these squares lack transversals, see [2].

For general background material on topological embeddings, we refer the reader to [5] and [6]. Our embeddings will always be 2-cell embeddings in closed connected 2-manifolds without a boundary. It was shown in [2] that a triangular embedding of $K_{n,n,n}$ is face 2-colourable if and only if the supporting surface is orientable, and the surface is therefore a sphere with an appropriate number of handles.

Given a Latin square L of order n, we may use the notation $k = L(i, j)$ to denote that entry k appears in row i column j of L; alternatively we may write $(i, j, k) \in L$. In this latter form, the triples of any Latin square will always be specified in (row, column, entry) order. A *transversal* in a Latin square L is a set of triples of L covering every row, every column and every entry, each precisely once. A *partial transversal* is defined similarly with rows, columns and entries covered at most once. If L and L' are Latin squares of the same order, with common sets of row labels, of column labels, and of entries, then a *shared transversal* is a set of triples of $L \cup L'$ covering every row, every column and every entry, each precisely once. We will write $L \bowtie L'$ (to be read as L *biembeds with* L' *without relabelling*), if the particular realizations of L and L' form an embedding in a surface; that is to say that the triangles formed by the (row, column, entry) triples of L and L' may be sewn together along their common edges to form the surface. With a slight abuse of notation we also use $L \bowtie L'$ to denote the actual embedding itself. We will also identify a group G with its Cayley table, so that we may write $G \bowtie H$, meaning that the Latin square formed by a Cayley table of G biembeds with the Latin square H.

2 The theorem

Our aim is to prove the following result.

Theorem 2.1 *Suppose that G is an Abelian group and that $G \neq C_2, C_2^2$ or C_4. Then $G \bowtie H$ for some Latin square H that has a transversal.*

Some of the steps in the proof are simplified by aiming instead for a stronger result.

Theorem 2.2 *Suppose that G is an Abelian group and that $G \neq C_2, C_2^2$ or C_4. Then $G \bowtie H$ for some Latin square H that has at least two disjoint transversals.*

As indicated above, a major role is played by the following recursive construction given first in [3].

Theorem 2.3 [3] *Suppose that $L \bowtie L'$, where L and L' are of order n and have row, column and entry labels $\{0, 1, \ldots, n-1\}$. Suppose also that $Q \bowtie Q'$, where Q and Q' are of order m and have row, column and entry labels $\{0, 1, \ldots, m-1\}$, and that the square Q' has a transversal $\mathcal{T}$. Define squares $Q(L)$ and $Q'(L, \mathcal{T}, L')$ by*

$$\begin{aligned} Q(L)(nu+i, nv+j) &= nQ(u,v) + L(i,j), \\ Q'(L,\mathcal{T},L')(nu+i, nv+j) &= nQ'(u,v) + k, \end{aligned}$$

for $0 \leq u, v \leq m-1$ and $0 \leq i, j \leq n-1$, where

$$k = \begin{cases} L(i,j) & \text{if } (u,v,w) \notin \mathcal{T} \text{ for any } w, \\ L'(i,j) & \text{if there exists } w \text{ such that } (u,v,w) \in \mathcal{T}, \end{cases}$$

Then $Q(L)$ and $Q'(L, \mathcal{T}, L')$ are Latin squares of order mn with row, column and entry labels $\{0, 1, \ldots, mn-1\}$, and $Q(L) \bowtie Q'(L, \mathcal{T}, L')$.

The square $Q(L)$ is partitioned into $n \times n$ subsquares which are just relabelled copies of L. Note also that if Q and L are groups then $Q(L)$ is a Cayley table for the group $Q \times L$. The square $Q'(L, \mathcal{T}, L')$ has a similar structure to $Q(L)$ but the subsquares corresponding to the transversal $\mathcal{T}$ are relabelled copies of L'. Note that if L' has a transversal, then among the relabelled copies of L' one can find a transversal in $Q'(L, \mathcal{T}, L')$. This feature facilitates re-application of the construction. The following Lemma guarantees the existence of such a transversal under different conditions.

Lemma 2.1 *Suppose that, in addition to the conditions of Theorem 2.3, L and L' have a shared transversal and Q' has a second transversal disjoint from $\mathcal{T}$. Then $Q'(L, \mathcal{T}, L')$ has a transversal.*

Proof. Suppose that L and L' have a shared transversal $\mathcal{P} \cup \mathcal{P}'$, where $\mathcal{P} \subseteq L$ and $\mathcal{P}' \subseteq L'$, and that Q' has a second transversal $\mathcal{S}$ disjoint from $\mathcal{T}$.

If $\mathcal{P} = \emptyset$, so that $\mathcal{P}'$ is a (full) transversal of L', then by the argument following the statement of Theorem 2.3, $Q'(L, \mathcal{T}, L')$ has a transversal. Similarly if $\mathcal{P}' = \emptyset$, so that $\mathcal{P}$ is a (full) transversal of L, then the subsquares of $Q'(L, \mathcal{T}, L')$ corresponding to the transversal $\mathcal{S}$ are relabelled copies of L, and the resulting relabelled copies of $\mathcal{P}$ form a transversal in $Q'(L, \mathcal{T}, L')$. We may therefore assume that neither $\mathcal{P}$ nor $\mathcal{P}'$ is empty, and if $p = |\mathcal{P}|$, then $1 \leq p \leq n-1$.

Permuting the rows or columns of a Latin square does not affect the existence of transversals. We may therefore considerably simplify notation and argument, without loss of generality, by making the following assumptions.
(i) The partial transversals $\mathcal{P}$ and $\mathcal{P}'$ are located respectively on the leading diagonals of L and L', with $\mathcal{P}$ corresponding to the cells (t, t) for $0 \leq t \leq p-1$, and $\mathcal{P}'$ corresponding to the cells (t, t) for $p \leq t \leq n-1$.
(ii) The transversal $\mathcal{T}$ is located on the leading diagonal of Q'.

With these assumptions $Q'(L, \mathcal{T}, L')$ has the form

$$\left(\begin{array}{c|c|c|c}
nQ'(0,0)+L' & nQ'(0,1)+L & nQ'(0,2)+L & \cdots \\ \hline
nQ'(1,0)+L & nQ'(1,1)+L' & nQ'(1,2)+L & \cdots \\ \hline
nQ'(2,0)+L & nQ'(2,1)+L & nQ'(2,2)+L' & \cdots \\ \hline
\vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

where the (i, j) entry is an $n \times n$ subsquare formed from a copy of L or L' shifted by a constant amount $nQ'(i, j)$. Clearly this $nm \times nm$ Latin square has a partial transversal $\mathcal{R}'$ formed from copies of $\mathcal{P}' \subseteq L'$ in the cells $(ni+t, ni+t)$ for $p \leq t \leq n-1$ and $0 \leq i \leq m-1$. In fact,

$$\mathcal{R}' = \{(ni+t, ni+t, nQ'(i,i) + L'(t,t)) : p \leq t \leq n-1, 0 \leq i \leq m-1\}.$$

Now suppose that the transversal $\mathcal{S}$ of Q' is given by

$$\mathcal{S} = \{(i, j_i, Q'(i, j_i)) : 0 \leq i \leq m-1\}.$$

Then $j_i \neq i$ for $0 \leq i \leq m-1$ since $\mathcal{S}$ is disjoint from $\mathcal{T}$. Once again examining the form of $Q'(L, \mathcal{T}, L')$ it will be seen that it contains a partial transversal $\mathcal{R}$ formed from copies of $\mathcal{P} \subseteq L$ in the cells $(ni+t, nj_i+t)$ for $0 \leq t \leq p-1$ and $0 \leq i \leq m-1$. In fact,

$$\mathcal{R} = \{(ni+t, nj_i+t, nQ'(i,j_i) + L(t,t)) : 0 \leq t \leq p-1, 0 \leq i \leq m-1\}.$$

It is now routine to check that $\mathcal{R} \cup \mathcal{R}'$ forms a (full) transversal in $Q'(L, \mathcal{T}, L')$.

$\square$

Observe that if L and L' have a second shared transversal $\bar{\mathcal{P}} \cup \bar{\mathcal{P}}'$ disjoint from $\mathcal{P} \cup \mathcal{P}'$, then the resulting transversal of $Q'(L, \mathcal{T}, L')$ (say, $\bar{\mathcal{R}} \cup \bar{\mathcal{R}}'$) will be disjoint from $\mathcal{R} \cup \mathcal{R}'$. We state this result in the form of a corollary. This is the main tool we will use to prove Theorem 2.2.

Corollary 2.1 *Suppose that, in addition to the conditions of Theorem 2.3, L and L' have two disjoint shared transversals and that Q' has a second transversal disjoint from $\mathcal{T}$. Then $Q'(L, \mathcal{T}, L')$ has two disjoint transversals.*

We next recall the existence of the *regular* biembedding of each cyclic square C_n. For the meaning of this term, see the discussion in [4].

Theorem 2.4 [2] *If the Latin squares C_n and C'_n are defined by $C_n(i,j) = i + j \mod n$ and $C'_n(i,j) = i + j + 1 \mod n$, then $C_n \bowtie C'_n$.*

In applications of Theorem 2.4 we will use the facts that

(i) if $n > 1$ is odd then C'_n has at least two disjoint transversals $\mathcal{T} = \{(i, i, 2i + 1) : i \in \mathbb{Z}_n\}$ and $\mathcal{S} = \{(i, i+1, 2i+2) : i \in \mathbb{Z}_n\}$, and

(ii) if n is even then C_n and C'_n have at least two disjoint shared transversals $\mathcal{P} \cup \mathcal{P}'$ and $\bar{\mathcal{P}} \cup \bar{\mathcal{P}}'$ where $\mathcal{P} = \{(i, i, 2i) : 0 \leq i \leq \frac{n}{2} - 1\}$, $\mathcal{P}' = \{(i, i, 2i+1) : \frac{n}{2} \leq i \leq n-1\}$, $\bar{\mathcal{P}} = \{(i, i, 2i) : \frac{n}{2} \leq i \leq n-1\}$ and $\bar{\mathcal{P}}' = \{(i, i, 2i+1) : 0 \leq i \leq \frac{n}{2} - 1\}$.

We will make particular use of the disjoint shared transversals in the cases $n = 2$ and $n = 4$.

As an example, take the regular biembeddings $C_4 \bowtie C'_4$ and $C_k \bowtie C'_k$ when $k \geq 3$ is odd, and apply the recursive construction of Theorem 2.3 to obtain a biembedding of $C_4 \times C_k = C_{4k}$, say $C_{4k} \bowtie H$. By Corollary 2.1, the mate H has at least two disjoint transversals. These are shown in Figure 1, one highlighted and the other boxed.

$$
H = \begin{array}{c|cccc|cccc|cccc|c}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \\
\hline
0 & \boxed{5} & 6 & 7 & 4 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \cdots \\
1 & 6 & \boxed{7} & 4 & 5 & 9 & 10 & 11 & 8 & 13 & 14 & 15 & 12 & \cdots \\
2 & 7 & 4 & 5 & 6 & 10 & 11 & \boxed{8} & 9 & 14 & 15 & 12 & 13 & \cdots \\
3 & 4 & 5 & 6 & 7 & 11 & 8 & 9 & \boxed{10} & 15 & 12 & 13 & 14 & \cdots \\
4 & 8 & 9 & 10 & 11 & \boxed{13} & 14 & 15 & 12 & 16 & 17 & 18 & 19 & \cdots \\
5 & 9 & 10 & 11 & 8 & 14 & \boxed{15} & 12 & 13 & 17 & 18 & 19 & 16 & \cdots \\
6 & 10 & 11 & 8 & 9 & 15 & 12 & 13 & 14 & 18 & 19 & \boxed{16} & 17 & \cdots \\
7 & 11 & 8 & 9 & 10 & 12 & 13 & 14 & 15 & 19 & 16 & 17 & \boxed{18} & \cdots \\
8 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & \boxed{21} & 22 & 23 & 20 & \cdots \\
9 & 13 & 14 & 15 & 12 & 17 & 18 & 19 & 16 & 22 & \boxed{23} & 20 & 21 & \cdots \\
10 & 14 & 15 & 12 & 13 & 18 & 19 & 16 & 17 & 23 & 20 & 21 & 22 & \cdots \\
11 & 15 & 12 & 13 & 14 & 19 & 16 & 17 & 18 & 20 & 21 & 22 & 23 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Figure 1. A mate for C_{4k}.

Lemma 2.2 *If $i \geq 3$ then $C_{2^i} \bowtie H$ for some H having at least two disjoint transversals.*

Proof. In [4, Lemma 2.1] it was shown that for each $i \geq 3$ there is a mate H such that $C_{2^i} \bowtie H$. In the case when $i = 3$ the mate H is given by

$$
H = \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 7 & 2 & 3 & 0 & 1 & 4 & \boxed{5} & 6 \\
1 & 3 & 4 & 5 & 1 & 6 & 7 & 0 & \boxed{2} \\
2 & \boxed{1} & 5 & 6 & 4 & 7 & 0 & 2 & 3 \\
3 & 6 & \boxed{3} & 7 & 5 & 0 & 2 & 4 & 1 \\
4 & 5 & 7 & \boxed{0} & 6 & 2 & 3 & 1 & 4 \\
5 & 2 & 0 & 4 & \boxed{7} & 3 & 1 & 6 & 5 \\
6 & 0 & 6 & 1 & 2 & \boxed{4} & 5 & 3 & 7 \\
7 & 4 & 1 & 2 & 3 & 5 & \boxed{6} & 7 & 0
\end{array}
$$

As one can see, H has two disjoint transversals, one highlighted and one boxed.

In the case when $i \geq 4$, the square H was obtained from the square C'_{2^i} mentioned in Theorem 2.4 by replacing the 16 triples $(\frac{rn}{4}, \frac{sn}{4}, \frac{n(r+s)}{4} + 1)$ for $0 \leq r, s \leq 3$ by the following triples.

$$
\begin{array}{cccc}
(0, 0, \frac{n}{2} + 1), & (0, \frac{n}{4}, \frac{3n}{4} + 1), & (0, \frac{n}{2}, 1), & (0, \frac{3n}{4}, \frac{n}{4} + 1), \\
(\frac{n}{4}, 0, \frac{3n}{4} + 1), & (\frac{n}{4}, \frac{n}{4}, 1), & (\frac{n}{4}, \frac{n}{2}, \frac{n}{4} + 1), & (\frac{n}{4}, \frac{3n}{4}, \frac{n}{2} + 1), \\
(\frac{n}{2}, 0, 1), & (\frac{n}{2}, \frac{n}{4}, \frac{n}{4} + 1), & (\frac{n}{2}, \frac{n}{2}, \frac{n}{2} + 1), & (\frac{n}{2}, \frac{3n}{4}, \frac{3n}{4} + 1), \\
(\frac{3n}{4}, 0, \frac{n}{4} + 1), & (\frac{3n}{4}, \frac{n}{4}, \frac{n}{2} + 1), & (\frac{3n}{4}, \frac{n}{2}, \frac{3n}{4} + 1), & (\frac{3n}{4}, \frac{3n}{4}, 1).
\end{array}
$$

As noted in [4], the resulting mate H contains the transversal $\mathcal{T}$ formed by triples

$$\begin{gathered}(x, x+2, 2x+3) \text{ for } x = 0, 1, \dots, \tfrac{n}{2} - 3, \text{ except for } x = \tfrac{n}{4} - 1,\\ (x, x-1, 2x) \text{ for } x = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n-1, \text{ except for } x = \tfrac{3n}{4},\\ (\tfrac{n}{4} - 1, \tfrac{3n}{4} - 1, n-1), \quad (\tfrac{n}{2} - 2, 1, \tfrac{n}{2}), \quad (\tfrac{n}{2} - 1, \tfrac{n}{2}, 0),\\ (\tfrac{n}{2}, 0, 1), \quad (\tfrac{n}{2} + 1, n-1, \tfrac{n}{2} + 1), \quad (\tfrac{3n}{4}, \tfrac{n}{4} + 1, 2).\end{gathered}$$

Now consider the following 2^i triples of H

$$\begin{gathered}(x, x+2+\tfrac{n}{2}, 2x+3+\tfrac{n}{2}) \text{ for } x = 0, 1, \dots, \tfrac{n}{2} - 3, \text{ except for } x = \tfrac{n}{4} - 1,\\ (x, x-1+\tfrac{n}{2}, 2x+\tfrac{n}{2}) \text{ for } x = \tfrac{n}{2} + 2, \tfrac{n}{2} + 3, \dots, n-1, \text{ except for } x = \tfrac{3n}{4},\\ (\tfrac{n}{4} - 1, \tfrac{n}{4} - 1, \tfrac{n}{2} - 1), \quad (\tfrac{n}{2} - 2, \tfrac{n}{2} + 1, 0), \quad (\tfrac{n}{2} - 1, 0, \tfrac{n}{2}),\\ (\tfrac{n}{2}, \tfrac{n}{2}, \tfrac{n}{2} + 1), \quad (\tfrac{n}{2} + 1, \tfrac{n}{2} - 1, 1), \quad (\tfrac{3n}{4}, \tfrac{3n}{4} + 1, \tfrac{n}{2} + 2)\end{gathered}$$

Since these triples cover every row, every column and every entry, they form a transversal $\mathcal{S}$ in H which is clearly disjoint from $\mathcal{T}$. □

In the proof of Lemma 2.2 of [4], it was shown that $C_4^2 \bowtie K$ for some K having two disjoint transversals. Take the regular biembedding $C_4 \bowtie C_4'$ (Theorem 2.4) with $C_4^2 \bowtie K$ and apply the recursive construction of Theorem 2.3 to obtain a biembedding of $C_4 \times C_4^2 = C_4^3$, say $C_4^3 \bowtie L$. By Corollary 2.1, the mate L has at least two disjoint transversals. Repeating this process gives the following result.

Lemma 2.3 *If $i \geq 2$ then $C_4^i \bowtie H$ for some H having at least two disjoint transversals.*

In the proof of Lemma 2.3 of [4], we have $(C_2 \times C_4) \bowtie H$, where H is a copy of the Cayley table of the dihedral group D_4 of order 8. It is very easy to see that H has two disjoint transversals. Consequently, we have the following result.

Lemma 2.4 *$(C_2 \times C_4) \bowtie H$ for some H having at least two disjoint transversals.*

In [3] it was shown that $C_2^3 \bowtie K$, where

$K =$

	0	1	2	3	4	5	6	7
0	2	4	6	7	0	1	$\boxed{3}$	5
1	4	2	7	6	1	$\boxed{0}$	5	3
2	$\boxed{7}$	1	4	5	2	3	6	0
3	1	7	5	$\boxed{4}$	3	2	0	6
4	6	3	0	2	$\boxed{5}$	7	4	1
5	3	$\boxed{6}$	2	0	7	5	1	4
6	0	5	$\boxed{1}$	3	4	6	2	7
7	5	0	3	1	6	4	7	$\boxed{2}$

The square K has two disjoint transversals, one highlighted and one boxed. Take the regular biembedding $C_2 \bowtie C_2'$ (Theorem 2.4) with $C_2^3 \bowtie K$ and apply the recursive construction of Theorem 2.3 to obtain a biembedding of $C_2 \times C_2^3 = C_2^4$, say $C_2^4 \bowtie L$. By Corollary 2.1, the mate L has at least two disjoint transversals. Repeating this process gives the following result.

Lemma 2.5 *If $i \geq 3$ then $C_2^i \bowtie H$ for some H having at least two disjoint transversals.*

Proof of Theorem 2.2. Suppose that G is an Abelian group. In general, we may write G as a direct product of cyclic groups in the form

$$G = C_{2^{i_1}}^{j_1} \times C_{2^{i_2}}^{j_2} \times \cdots \times C_{2^{i_m}}^{j_m} \times C_{k_1}^{l_1} \times C_{k_2}^{l_2} \times \cdots \times C_{k_n}^{l_n},$$

where each i_s, j_s and l_s is a positive integer, and each k_s is an odd positive integer. Without loss of generality we may assume that $i_1 < i_2 < \ldots < i_m$ and $k_1 < k_2 < \ldots < k_n$. If G has no factor C_{2^i}, that is if $m = 0$, then starting with the regular biembedding of each C_{k_s} and applying Corollary 2.1 repeatedly, we have $G \bowtie H$ for some H having at least two disjoint transversals. In view of Lemma 2.2, the same is true if G has factors C_{2^i} for $i \geq 3$ but no factors C_2 or C_4. It remains to deal with the cases when G has factors C_2 and/or C_4.

Consider first the case when G has no factors apart from C_2 and C_4, that is $G = C_2^{j_1} \times C_4^{j_2}$. If $(j_1, j_2) = (0, 0)$ there is nothing to prove. Other cases are dealt with in Table 1, where R denotes use of regular biembeddings (Theorem 2.4), L a lemma, C the Corollary 2.1, and $t(H)$ denotes the number of transversals in H.

j_1	j_2	G	$G \bowtie H$ (?)	$t(H)$
0	1	C_4	R	0
	≥ 2	$C_4^{j_2}$	L2.3	≥ 2
1	0	C_2	R	0
	1	$C_2 \times C_4$	L2.4	≥ 2
	≥ 2	$C_4^{j_2-1} \times (C_2 \times C_4)$	C, R, L2.4	≥ 2
2	0	C_2^2	No biembedding	–
	1	$C_2 \times (C_2 \times C_4)$	C, R, L2.4	≥ 2
	2	$(C_2 \times C_4) \times (C_2 \times C_4)$	C, L2.4	≥ 2
	≥ 3	$C_4^{j_2-2} \times (C_2 \times C_4) \times (C_2 \times C_4)$	C, R, L2.4	≥ 2
≥ 3	0	$C_2^{j_1}$	L2.5	≥ 2
	≥ 1	$C_4^{j_2} \times C_2^{j_1}$	C, R, L2.5	≥ 2

Table 1. $G = C_2^{j_1} \times C_4^{j_2}$.

Next consider the case when $G = C_2^{j_1} \times C_4^{j_2} \times G^*$ where G^* is non-trivial but has no factors C_2 or C_4. We already have $G^* \bowtie H^*$ for some H^* having at least two disjoint transversals. For $(j_1, j_2) = (1, 0)$ or $(0, 1)$, recall the existence of disjoint shared transversals in the regular biembeddings of C_2 and C_4. Then if $(j_1, j_2) \neq (2, 0)$, by using Corollary 2.1 and the appropriate case from Table 1, we obtain $G \bowtie H$ for some H having at least two disjoint transversals.

All that remains to consider is the case $C_2^2 \times G^*$. But this may be written as $C_2 \times (C_2 \times G^*)$, and dealt with using the regular biembedding of C_2 and two applications of Corollary 2.1. We have therefore proved that if G is an Abelian group, $G \neq C_2, C_2^2$ or C_4, then $G \bowtie H$ for some Latin square H that has two disjoint transversals. This completes the proof of Theorem 2.2. $\square$

Acknowledgement The first author acknowledges partial support by Slovak research grants VEGA 1/0489/08, APVT-20-000704 and APVV-0040-06.

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