Biembeddings of Latin squares obtained from a voltage construction

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Abstract

We investigate a voltage construction for face 2-colourable triangulations by $K_{n,n,n}$ from the viewpoint of the underlying Latin squares. We prove that if the vertices are relabelled so that one of the Latin squares is exactly the Cayley table C_n of the group $\mathbb{Z}_n$, then the other square can be obtained from C_n by a cyclic permutation of row, column or entry identifiers, and we identify these cyclic permutations. As an application, we improve the previously known lower bound for the number of nonisomorphic triangulations by $K_{n,n,n}$ obtained from the voltage construction in the case when n is a prime number.

Running head: Voltage construction.

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1 Introduction

Research into biembeddings of Latin squares, or equivalently, face 2-colourable triangulations by complete tripartite graphs $K_{n,n,n}$, is motivated by their connection with the Heawood map colouring theorem. To prove this result, Ringel and Youngs constructed one minimum genus embedding of each complete graph K_n [15, 16]. For n lying in certain residue classes modulo 12, such embeddings have all their faces triangular. By varying the rotations at the vertices of current graphs similar to those used by Ringel and Youngs, a lower bound 2^{an} for the number of nonisomorphic minimum genus embeddings of K_n was established for all sufficiently large n, where a is a positive constant [10, 11, 12, 13, 14]. For some residue classes this lower bound was improved to 2^{bn^2}, where b is a positive constant [1, 6]. Recently, a lower bound of the form n^{cn^2}, where c is a positive constant, was established for a very sparse class of n in both the orientable and nonorientable cases [3, 7]. A trivial upper bound is $n^{n^2/3}$ [7]. The lower bounds 2^{bn^2} and n^{cn^2} were obtained using constructions first given in [1, 6] and later generalized in [3, 7]. These use face 2-colourable triangulations by complete tripartite graphs to generate many nonisomorphic face 2-colourable triangulations by K_n. The key step in [3, 7] was the construction of n^{dn^2} face 2-colourable triangulations by $K_{n,n,n}$, where d is a positive constant.

When a triangular embedding of $K_{n,n,n}$ is face 2-colourable, the triangular faces in each colour class determine a Latin square of order n by taking these faces as the (row, column, entry) triples, where the row labels, the column labels and the entries form the three sets of the partition. Hence, a face 2-colourable embedding of $K_{n,n,n}$ can be regarded as a biembedding of two Latin squares; if these squares are L and L', we write $L \bowtie L'$ to denote the fact that the L biembeds with L', and we also use this notation to denote the biembedding itself, taking the faces of L to be white and those of L' to be black.

In [5] we generalized a well-known voltage construction for a face 2-

colourable triangulation by $K_{n,n,n}$ and we proved that the generalized construction generates exponentially many nonisomorphic face 2-colourable triangulations by $K_{n,n,n}$ when n is prime. Since both the squares in each of these embeddings are isotopic to the cyclic Latin square C_n defined by $C_n(i,j) = i + j$, with arithmetic in $\mathbb{Z}_n$, the vertices can be relabelled so that the embedding is exactly $C_n \bowtie D_n$ for some D_n. In this paper we identify all the possibilities for D_n and we illustrate our results with two examples. We also improve, by a factor of 3, the lower bound for the number of these embeddings when n is a prime number.

In the next section we recall the voltage construction of [5], we state our results and give the examples. The necessary proofs are postponed to the last section. For background material and for notation and terminology not defined here we refer the reader to [2, 9].

2 Results

We start with the description of the voltage construction used in [5]. Let M be an embedding in a sphere of a graph with two vertices u and v having n parallel edges so that each face of the embedding is a 2-gon. Further, let $a_0, a_1, \ldots, a_{n-1}$ be voltages in the clockwise rotation on the arcs emanating from u, see Figure 1, such that $\{a_0, a_1, \ldots, a_{n-1}\} = \{0, 1, \ldots, n-1\}$. Then the voltages around v in the clockwise rotation are $-a_{n-1}, -a_{n-2}, \ldots, -a_0$. Suppose that for each i, $0 \le i \le n-1$, the differences $a_i - a_{i-1}$ are coprime with n (the indices are always taken modulo n).

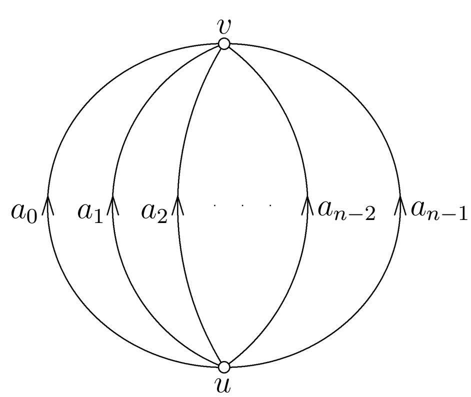

Figure 1: The embedding M.

Now consider the lift of M with voltages in $\mathbb{Z}_n$. In the lift there are vertex sets $U = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V = \{v_0, v_1, \ldots, v_{n-1}\}$, and as all $a_i - a_{i-1}$ are coprime with n, each face (2-gon) of M is lifted to a $2n$-gon. Hence, the lift gives an embedding of the complete bipartite graph $K_{n,n}$ in an orientable surface in which every face is bounded by a Hamiltonian cycle. We denote this embedding by $B(u, v; \alpha)$, where α is the cyclic permutation $(a_0, a_1, \ldots, a_{n-1})$. Now place into each of the n faces of $B(u, v; \alpha)$ a vertex and join it to all the vertices lying on the boundary of that face, to create a triangular embedding of the complete tripartite graph $K_{n,n,n}$ in an orientable surface. More exactly, let $W = \{w_0, w_1, \ldots, w_{n-1}\}$ be a set of n vertices disjoint from U and V. The vertex w_i is placed into that face of $B(u, v; \alpha)$ which is obtained by lifting the 2-gon with voltages a_i and $-a_{i-1}$, $0 \leq i \leq n-1$. In what follows, we denote this embedding by $T(u, v; \alpha)$, and a permutation $\alpha = (a_0, a_1, \ldots, a_{n-1})$ of $\{0, 1, \ldots, n-1\}$ such that $(a_i - a_{i-1}, n) = 1$, $0 \leq i \leq n-1$, will be called *admissible*.

In [4] it was shown that a triangulation by $K_{n,n,n}$ is face 2-colourable if and only if the underlying surface is orientable. Hence, $T(u, v; \alpha)$ is an orientable face 2-colourable triangulation by $K_{n,n,n}$. Colour white the triangles of $T(u, v; \alpha)$ with anti-clockwise rotation of vertices (u_i, v_j, w_k), and colour black the triangles with clockwise rotation of vertices (u_i, v_j, w_k). The two colour classes form a pair of Latin squares, one for each colour, by taking each triangle $u_i v_j w_k$ of the specified colour and placing the entry k into the i-th row and j-th column of an $n \times n$ array. Of course there are potentially five more pairs of Latin squares that can be formed in this way by permuting the roles of row labels, column labels and entries; for example we might place entry j into row k, column i. In what follows, we denote by $r_0, r_1, \ldots, r_{n-1}$ (respectively, $c_0, c_1, \ldots, c_{n-1}$ and $e_0, e_1, \ldots, e_{n-1}$), the row labels (column labels and entries) of a Latin square of order n. If the entry k occurs in row i and column j of a Latin square L, then we write $L(i, j) = k$ or equivalently $(r_i, c_j, e_k) \in L$.

There are four operations on permutations which were used in [5] to solve the isomorphism problem for $B(u, v; \alpha)$ and which we now recall. All the arithmetic is taken in $\mathbb{Z}_n$. So suppose that $\alpha = (a_0, a_1, \ldots, a_{n-1})$ is a permutation of $\{0, 1, \ldots, n-1\}$.

For $e \in \mathbb{Z}_n$ put $\beta = (a_{0+e}, a_{1+e}, \ldots, a_{n-1+e})$. Then, although α and β are identical permutations, they are differently labelled. We say that β is a *rotation* of α. We do not need to use rotations to state our results, although rotations can relabel the triangles of $T(u, v; \alpha)$ as noted below.

For $f \in \mathbb{Z}_n$ denote by $f + \alpha$ the permutation defined by

$$f + \alpha = (f + a_0, f + a_1, \dots, f + a_{n-1}).$$

For $g \in \mathbb{Z}_n$ and coprime with n, denote by $g\alpha$ the permutation defined by

$$g\alpha = (ga_0, ga_1, \dots, ga_{n-1}).$$

Finally, denote by α^{-1} the permutation

$$\alpha^{-1} = (a_{n-1}, a_{n-2}, \dots, a_0),$$

and define $\alpha^1 = \alpha$. If β is a permutation obtained from α by applying the above operations arbitrarily many times, then we say that α and β are *equivalent* permutations.

The following result comes from [5]; see Lemma 4 of that paper and the discussions immediately preceding and following it. Here $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{n-1}\}$.

Theorem 2.1 *(a) If the admissible permutations α and β are equivalent then $T(u, v; \alpha)$ and $T(x, y; \beta)$ are isomorphic triangulations with an isomorphism mapping $\{U, V\}$ to $\{X, Y\}$.*
(b) If α and β are admissible permutations and $T(u, v; \alpha)$ and $T(x, y; \beta)$ are isomorphic triangulations with an isomorphism mapping $\{U, V\}$ to $\{X, Y\}$ then α and β are equivalent permutations.

¿From the proof of [5, Lemma 4], it is easily seen that if α and β are equivalent admissible permutations then there exist $f, g \in \mathbb{Z}_n$ with $(g, n) = 1$, and $h \in \{-1, 1\}$, such that $\beta = f + g\alpha^h$.

Now we define three Latin squares of order n, namely F_n^β, G_n^β and H_n^β, depending on a cyclic permutation $\beta = (b_0, b_1, \dots, b_{n-1})$ of $\{0, 1, \dots, n-1\}$. Again, all the arithmetic is taken in $\mathbb{Z}_n$.

For every k and i, $0 \le k, i \le n-1$, denote by j that element of $\mathbb{Z}_n$ for which $i = b_{j-1} - k$. The square F_n^β is defined by

$$F_n^\beta(k, i) = F_n^\beta(k, b_{j-1} - k) = b_j.$$

For every k and i, $0 \le k, i \le n-1$, denote by ℓ that element of $\mathbb{Z}_n$ for which $i = b_\ell$. The squares G_n^β and H_n^β are defined by

$$\begin{aligned} G_n^\beta(k, i) &= G_n^\beta(k, b_\ell) = b_{\ell-1} + k, \\ H_n^\beta(i, k) &= H_n^\beta(b_\ell, k) = b_{\ell-1} + k. \end{aligned}$$

Observe that if γ is obtained from β by rotation, then $F_n^\beta = F_n^\gamma$, $G_n^\beta = G_n^\gamma$ and $H_n^\beta = H_n^\gamma$, so that the squares are well-defined. Moreover, F_n^β is obtained from the cyclic Latin square C_n by the cyclic permutation of the entries $b_t \to b_{t+1}$, $0 \le t \le n-1$. Likewise, G_n^β and H_n^β are obtained from C_n by the cyclic permutation $b_t \to b_{t+1}$, $0 \le t \le n-1$, of the columns and rows, respectively. Hence, each of F_n^β, G_n^β and H_n^β is isotopic to C_n.

In the following results we replace the identifiers u, v and w in $T(u, v; \alpha)$ by r, c and e in any one of the six possible orders and we describe the Latin squares involved in $T(r, c; \alpha)$, $T(c, r; \alpha)$, $T(r, e; \alpha)$, $T(e, r; \alpha)$, $T(c, e; \alpha)$ and $T(e, c; \alpha)$. Although all six of these embeddings are isomorphic, they may yield different (but isotopic) Latin squares. Theorem 2.2 describes relabellings in which either the white or the black square is taken to be C_n.

Theorem 2.2 *Consider the embedding $T(u, v; \alpha)$ where α is an admissible permutation. Then there are relabellings of rows, columns and entries such that*

- *(a) both $T(r, c; \alpha)$ and $T(c, r; \alpha)$ are represented by $C_n \bowtie F_n^\alpha$ and also by $F_n^{-\alpha^{-1}} \bowtie C_n$,*
- *(b) both $T(r, e; \alpha)$ and $T(e, r; \alpha)$ are represented by $C_n \bowtie G_n^\alpha$ and also by $G_n^{-\alpha^{-1}} \bowtie C_n$,*
- *(c) both $T(c, e; \alpha)$ and $T(e, c; \alpha)$ are represented by $C_n \bowtie H_n^\alpha$ and also by $H_n^{-\alpha^{-1}} \bowtie C_n$.*

The proof of Theorem 2.2 is given in Section 3. We illustrate the result with two examples.

Example 2.1 Denote by C_n^- and C_n^+ squares isotopic to C_n, defined by $C_n^-(i, j) = i + j - 1$ and $C_n^+(i, j) = i + j + 1$, the arithmetic being in $\mathbb{Z}_n$ and $0 \le i, j \le n-1$. In [5] we proved that $C_n \bowtie C_n^-$ is, up to isomorphism, the unique regular triangular embedding of $K_{n,n,n}$ in an orientable surface. Observe that for $\alpha = (0, 1, \ldots, n-1)$ we have $F_n^\alpha = C_n^+$ and $G_n^\alpha = H_n^\alpha = C_n^-$. Thus $C_n \bowtie C_n^-$ and $C_n \bowtie C_n^+$ are isomorphic embeddings since they are both obtained by relabellings of the same $T(u, v; \alpha)$. □

Example 2.2 Consider $\alpha = (a_0, a_1, \ldots, a_{n-1}) = (0, 1, n-1, n-2, \ldots, 2)$. Then the differences are given by

$$(a_0 - a_{n-1}, a_1 - a_0, \ldots, a_{n-1} - a_{n-2}) = (n-2, 1, n-2, -1, -1, \ldots, -1).$$

Hence, if n is odd then α is an admissible permutation, so that $T(u,v;\alpha)$ is an orientable face 2-colourable triangulation by $K_{n,n,n}$. By Theorem 2.2 we have $C_n \bowtie F_n^\alpha$, $C_n \bowtie G_n^\alpha$ and $C_n \bowtie H_n^\alpha$. Figure 2 shows these squares in the case $n = 5$. Observe that F_n^α (G_n^α and H_n^α) is obtained from C_n by applying the cyclic permutation $\alpha : a_t \to a_{t+1}$, $0 \le t \le n-1$, to entry (column and row) labels. The embedding $C_n \bowtie G_n^\alpha$ (n odd) is employed in [8] to demonstrate a new generalized product construction using shared transversals. $\square$

$$C_5 = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 & 0 \\ 2 & 2 & 3 & 4 & 0 & 1 \\ 3 & 3 & 4 & 0 & 1 & 2 \\ 4 & 4 & 0 & 1 & 2 & 3 \\ \hline \end{array} \qquad F_5^\alpha = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 4 & 0 & 2 & 3 \\ 1 & 4 & 0 & 2 & 3 & 1 \\ 2 & 0 & 2 & 3 & 1 & 4 \\ 3 & 2 & 3 & 1 & 4 & 0 \\ 4 & 3 & 1 & 4 & 0 & 2 \\ \hline \end{array}$$

$$G_5^\alpha = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2 & 0 & 3 & 4 & 1 \\ 1 & 3 & 1 & 4 & 0 & 2 \\ 2 & 4 & 2 & 0 & 1 & 3 \\ 3 & 0 & 3 & 1 & 2 & 4 \\ 4 & 1 & 4 & 2 & 3 & 0 \\ \hline \end{array} \qquad H_5^\alpha = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 2 & 3 & 4 & 0 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 & 2 \\ 3 & 4 & 0 & 1 & 2 & 3 \\ 4 & 1 & 2 & 3 & 4 & 0 \\ \hline \end{array}$$

Figure 2: The Latin squares C_5, F_5^α, G_5^α and H_5^α.

Theorem 2.2 gives a description of some relabellings of $T(u,v;\alpha)$ in which either the first or second square is C_n. However, C_n has many automorphisms, and so in both cases there are many possibilities for the other square. Theorem 2.3 describes *all* these possibilities.

Theorem 2.3 *Consider the embedding $T(u,v;\alpha)$ where α is an admissible permutation. Whenever the vertices in one of the colour classes are relabelled so that the corresponding triangles form C_n, then the triangles in the other colour class form F_n^β, G_n^β or H_n^β, where $\beta = f + g\alpha^h$ for some $f, g \in \mathbb{Z}_n$, $(g,n) = 1$, and $h \in \{-1, 1\}$.*

The proof of Theorem 2.3 is given in Section 3. This result establishes that whenever we have an embedding $C_n \bowtie D_n$ (or $D_n \bowtie C_n$) which is an isomorphic copy of $T(u,v;\alpha)$ for some cyclic permutation α, then D_n is F_n^β, G_n^β or H_n^β for some cyclic permutation β equivalent to α. Furthermore, it will be shown in the proof of Theorem 2.3 that for $T(r,c;\alpha)$ and $T(c,r;\alpha)$ we have $D_n = F_n^\beta$, for $T(r,e;\alpha)$ and $T(e,r;\alpha)$ we have $D_n = G_n^\beta$, and for $T(c,e;\alpha)$ and $T(e,c;\alpha)$ we have $D_n = H_n^\beta$.

It is also possible to improve Theorem 2.1 to deal with isomorphisms that do *not* map $\{U,V\}$ to $\{X,Y\}$. The improvement is summarized in the following theorem where α and β are admissible permutations. The proof is given in Section 3.

Theorem 2.4 *(a) Suppose that $T(u,v;\alpha)$ and $T(x,y;\beta)$ are isomorphic triangulations with an isomorphism that does* not *map $\{U,V\}$ to $\{X,Y\}$. Then these triangulations are isomorphic copies of the unique regular triangulation $T(r,c;\delta)$ given by $\delta = (0,1,\ldots,n-1)$.*
(b) The triangulations $T(u,v;\alpha)$ and $T(x,y;\beta)$ are isomorphic if and only if α and β are equivalent permutations.

Our last result is an application of Theorem 2.4. Again, the proof is given in Section 3.

Theorem 2.5 *If n is a prime number then there are at least $(n-2)!/2n$ nonisomorphic embeddings $T(u,v;\alpha)$.*

We remark that Theorem 2.5 improves [5, Theorem 1] by a factor of 3. In the following table we compare n_e, the number of equivalence classes of admissible permutations on n vertices, and hence by Theorem 2.4 the number of nonisomorphic triangulations $T(u,v;\alpha)$, with the bound given in Theorem 2.5. The values of n_e were found using a computer (see also [5, Table 2]) and they indicate that the bound in Theorem 2.5 is really tight.

n	5	7	11	13
n_e	2	13	16,687	1,537,183
$\left\lceil \frac{(n-2)!}{2n} \right\rceil$	1	9	16,495	1,535,262

3 Proofs

The triangles of $T(u,v;\alpha)$ with anti-clockwise rotation of vertices (u_i, v_j, w_k) are coloured white, and the triangles with clockwise rotation of vertices (u_i, v_j, w_k) are coloured black. From Figure 1 it will be seen that the white triangles are (u_i, v_{i+a_0}, w_0), (u_i, v_{i+a_1}, w_1), …, $(u_i, v_{i+a_{n-1}}, w_{n-1})$ and the black triangles are $(u_{i-a_{n-1}}, v_i, w_0)$, (u_{i-a_0}, v_i, w_1), …, $(u_{i-a_{n-2}}, v_i, w_{n-1})$, where $0 \le i \le n-1$. However, if β is a (non-trivial) rotation of α then the triangles of $T(u,v;\alpha)$ and $T(u,v;\beta)$ are labelled differently even though α and β are identical permutations.

Proof of Theorem 2.2.
In $T(u,v;\alpha)$ the white triangles are (u_i, v_{i+a_j}, w_j), while the black ones are $(u_{i-a_{j-1}}, v_i, w_j)$, $0 \le i,j \le n-1$. We start by relabelling the indices of w using the permutation $j \to a_j$. Denote the resulting set of white triangles by A^* and the resulting set of black triangles by B^*. Then the relabelling gives

$$A^* = \{(u_i, v_{i+a_j}, w_{a_j})\} \qquad B^* = \{(u_{i-a_{j-1}}, v_i, w_{a_j})\}, \tag{1}$$

where $0 \le i,j \le n-1$. Observe now that rotating α does not cause a relabelling of A^* or B^*. We split the proof into two parts, (i) and (ii).

(i) We begin with the cases when the white squares of $T(u,v;\alpha)$ are relabelled to form C_n. First consider the case $u = r$ and $v = c$. Denote by A' and B' the corresponding sets of white and black triangles, respectively. Then (1) gives

$$A' = \{(r_i, c_{i+a_j}, e_{a_j})\} \qquad B' = \{(r_{i-a_{j-1}}, c_i, e_{a_j})\},$$

where $0 \le i,j \le n-1$. Denote by A and B the sets, obtained from A' and B', respectively, by applying the permutation $i \to -i$ to the row labels. Then

$$A = \{(r_{-i}, c_{i+a_j}, e_{a_j})\} \qquad B = \{(r_{-i+a_{j-1}}, c_i, e_{a_j})\},$$

where $0 \le i,j \le n-1$. If we consider A as a Latin square, then $A(-i, i+a_j) = -i+i+a_j$, $0 \le i,j \le n-1$, so that $A = C_n$. On the other hand, for every k and i, $0 \le k,i \le n-1$, there is j such that $a_{j-1} = k+i$. Then $B(-i+a_{j-1}, i) = B(k,i) = B(k, a_{j-1}-k) = a_j$, so that $B = F_n^\alpha$. Hence, $T(r,c;\alpha)$ is represented by $C_n \bowtie F_n^\alpha$.

Since $C_n(i,j) = C_n(j,i)$ and $F_n^\alpha(i,j) = F_n^\alpha(j,i)$, $0 \le i,j \le n-1$, and since $T(c,r;\alpha)$ is obtained from $T(r,c;\alpha)$ by interchanging the rows for columns and vice versa, the embedding $T(c,r;\alpha)$ is also represented by $C_n \bowtie F_n^\alpha$.

Now consider the case $u = r$ and $v = e$. Denote by A and B the corresponding sets of white and black triangles, respectively. Then (1) gives

$$A = \{(r_i, c_{a_j}, e_{i+a_j})\} \qquad\qquad B = \{(r_{i-a_{j-1}}, c_{a_j}, e_i)\},$$

where $0 \leq i, j \leq n-1$. Thus, $A(i, a_j) = i + a_j$, which means that $A = C_n$. On the other hand, putting $k = i - a_{j-1}$ gives $B(k, a_j) = i = a_{j-1} + k$, so that $B = G_n^\alpha$. Hence, $T(r, e; \alpha)$ is represented by $C_n \bowtie G_n^\alpha$.

Next consider the case $u = e$ and $v = r$. Denote by A' and B' the corresponding sets of white and black triangles, respectively. Then (1) gives

$$A' = \{(r_{i+a_j}, c_{a_j}, e_i)\} \qquad\qquad B' = \{(r_i, c_{a_j}, e_{i-a_{j-1}})\},$$

where $0 \leq i, j \leq n-1$. Denote by A and B sets, obtained from A' and B', respectively, by applying the permutation $i \to -i$ to the row labels and to the entry labels. Then

$$A = \{(r_{-i-a_j}, c_{a_j}, e_{-i})\} \qquad\qquad B = \{(r_{-i}, c_{a_j}, e_{-i+a_{j-1}})\},$$

where $0 \leq i, j \leq n-1$. Thus, $A(-i-a_j, a_j) = -i$, which means that $A = C_n$. On the other hand, taking $k = -i$ gives $B(k, a_j) = a_{j-1} + k$, so that $B = G_n^\alpha$. Hence, $T(e, r; \alpha)$ is represented by $C_n \bowtie G_n^\alpha$.

Since C_n is symmetric, we have $C_n(i, j) = C_n(j, i)$, $0 \leq i, j \leq n-1$. Moreover, taking $\ell = a_j$ gives

$$H_n^\alpha(\ell, k) = H_n^\alpha(a_j, k) = a_{j-1} + k = G_n^\alpha(k, a_j) = G_n^\alpha(k, \ell).$$

Since $T(r, e; \alpha)$ is represented by $C_n \bowtie G_n^\alpha$, by interchanging rows for columns and vice versa it will be seen that $T(c, e; \alpha)$ is represented by $C_n \bowtie H_n^\alpha$. Similarly, since $T(e, r; \alpha)$ is represented by $C_n \bowtie G_n^\alpha$, $T(e, c; \alpha)$ is represented by $C_n \bowtie H_n^\alpha$.

(ii) We now turn to the cases when the black squares of $T(u, v; \alpha)$ are relabelled to form C_n. In the sets A^* and B^* from (1), relabel the indices of w by the permutation $a_j \to -a_{j-1}$ and denote by $\bar{A}$ and $\bar{B}$ the corresponding sets of white and black triangles. Then

$$\bar{A} = \{(u_i, v_{i+a_j}, w_{-a_{j-1}})\} \qquad \bar{B} = \{(u_{i-a_{j-1}}, v_i, w_{-a_{j-1}})\},$$

where $0 \leq i, j \leq n-1$. Recall that all the indices are considered in $\mathbb{Z}_n$. Substituting $-\ell + 1$ for j we get

$$\bar{A} = \{(u_i, v_{i+a_{-\ell+1}}, w_{-a_{-\ell}})\} \qquad \bar{B} = \{(u_{i-a_{-\ell}}, v_i, w_{-a_{-\ell}})\},$$

and setting $b_k = -a_{-k}$, $0 \le k \le n-1$, we obtain

$$\bar{A} = \{(u_i, v_{i-b_{\ell-1}}, w_{b_\ell})\} \qquad \bar{B} = \{(u_{i+b_\ell}, v_i, w_{b_\ell})\},$$

where $0 \le i, \ell \le n-1$. With $\beta = (b_0, b_1, \ldots, b_{n-1}) = -\alpha^{-1}$, the black (white) triangles of $T(u, v; \alpha)$ are exactly the white (black) triangles of $T(v, u; \beta)$. Now the result follows from part (i) of the proof. $\square$

Proof of Theorem 2.3.
The first step is to determine those permutations $\pi^r : i_t \to t$, $\pi^c : j_t \to t$ and $\pi^e : k_t \to k$ respectively of the row labels, column labels and entries for which $\pi = (\pi^r, \pi^c, \pi^e)$ fixes C_n. To do this we show that the three values i_0, j_0, i_1 (the preimages of $0, 0, 1$ respectively in π^r, π^c, π^r) determine π uniquely. Put $d = i_1 - i_0$.

First we find j_1. Since

$$C_n(i_0, j_1) = C_n(i_1, j_0) = C_n(i_0 + d, j_0) = i_0 + j_0 + d,$$

we get $j_1 = j_0 + d$. Similarly,

$$C_n(i_0, j_2) = C_n(i_1, j_1) = C_n(i_0 + d, j_0 + d) = i_0 + j_0 + 2d,$$

and so $j_2 = j_0 + 2d$. Proceeding in this way gives $j_t = j_0 + td$, $0 \le t \le n-1$. Since $\{j_t : 0 \le t \le n-1\} = \{0, 1, \ldots, n-1\}$ it follows that d must be coprime with n.

Interchanging the rows for columns and arguing as above gives $i_t = i_0 + td$, $0 \le t \le n-1$. It then follows that for every s and t, $0 \le s, t \le n-1$,

$$C_n(i_s, j_t) = C_n(i_0 + sd, j_0 + td) = i_0 + j_0 + (s+t)d = k_{s+t}.$$

In particular, $k_\ell = i_0 + j_0 + \ell d$, $0 \le \ell \le n-1$.

Hence, if π fixes C_n as described, then it is determined by i_0, j_0 and $d = i_1 - i_0$, where d is coprime with n. In such a case, since d is coprime with n, there is $g \in \mathbb{Z}_n$ such that $g \cdot d = 1$. Then $(g, n) = 1$ and we have $\pi^r(q) = g(q - i_0)$, $\pi^c(q) = g(q - j_0)$ and $\pi^e(q) = g(q - i_0 - j_0)$, $0 \le q \le t-1$.

We can now examine the effect of applying such a mapping π to the second square in one of our embeddings. By Theorem 2.2 it suffices to consider F_n^γ, G_n^γ and H_n^γ, where $\gamma \in \{\alpha, -\alpha^{-1}\}$. We discuss only the case $\gamma = \alpha$ here as the case $\gamma = -\alpha^{-1}$ is analogous.

First consider F_n^α. We have $F_n^\alpha(k, a_{j-1} - k) = a_j$, $0 \leq j, k \leq n-1$. Applying π to F_n^α gives the Latin square B where

$$B(g(k - i_0), g(a_{j-1} - k - j_0)) = g(a_j - i_0 - j_0).$$

Set $x = g(k - i_0)$ and $f = g(-i_0 - j_0)$. Then

$$B(x, f + ga_{j-1} - x) = f + ga_j.$$

Hence, $B = F_n^\beta$, where $\beta = f + g\alpha$.

Next consider G_n^α. We have $G_n^\alpha(k, a_j) = a_{j-1} + k$, $0 \leq j, k \leq n-1$. Applying π to G_n^α gives the Latin square B where

$$B(g(k - i_0), g(a_j - j_0)) = g(a_{j-1} + k - i_0 - j_0).$$

Set $x = g(k - i_0)$ and $f = g(-j_0)$. Then

$$B(x, f + ga_j) = f + ga_{j-1} + x.$$

Hence, $B = G_n^\beta$, where $\beta = f + g\alpha$.

Finally consider H_n^α. We have $H_n^\alpha(a_j, k) = a_{j-1} + k$, $0 \leq j, k \leq n-1$. Applying π to H_n^α gives the Latin square B where

$$B(g(a_j - i_0), g(k - j_0)) = g(a_{j-1} + k - i_0 - j_0).$$

Set $x = g(k - j_0)$ and $f = g(-i_0)$. Then

$$B(f + ga_j, x) = f + ga_{j-1} + x.$$

Hence, $B = H_n^\beta$, where $\beta = f + g\alpha$. $\square$

Proof of Theorem 2.4.

(a) Denote by $Z = \{z_0, z_1, \ldots, z_{n-1}\}$ the third part of the embedded graph in $T(x, y; \beta)$. Suppose that there is an isomorphism taking $T(u, v; \alpha)$ to $T(x, y; \beta)$ which does *not* map $\{U, V\}$ to $\{X, Y\}$. Then $\{U, V\}$ must be mapped to $\{X, Z\}$ or $\{Y, Z\}$. We will assume that $\{U, V\}$ is mapped to $\{X, Z\}$, as the other case can be solved analogously by replacing G by H in the argument below.

We will make use of our preceding results by representing $T(u, v; \alpha)$ and $T(x, y; \beta)$ as biembeddings of Latin squares. To do this, relabel x, y and z as r, c and e respectively so that $T(x, y; \beta)$ becomes $T(r, c; \beta)$. Since the original isomorphism maps $\{U,V\}$ to $\{X,Z\}$, (u,v) may be relabelled as either (r,e) or (e,r), so that $T(u,v;\alpha)$ becomes either $T(r,e;\alpha)$ or $T(e,r;\alpha)$.

By Theorem 2.2, $T(r,c;\beta)$ can be represented by $C_n \bowtie F_n^\beta$, and consequently either $T(r,e;\alpha)$ or $T(e,r;\alpha)$ can also be represented by $C_n \bowtie F_n^\beta$. However, by Theorem 2.3 and the remark which follows it, whenever the vertices of $T(r,e;\alpha)$ or $T(e,r;\alpha)$ are relabelled so that one colour class of triangles forms C_n, then the triangles in the other class form G_n^γ where γ is a permutation equivalent to α. Hence, $F_n^\beta = G_n^\gamma$ for some admissible permutation γ. Let us solve this last equation.

Put $t = F_n^\beta(0,0) = G_n^\gamma(0,0)$. Then

$$G_n^\gamma(k,0) = G_n^\gamma(0,0) + k = t + k.$$

For any k and i, choosing j so that $i = b_{j-1} - k$, we get

$$F_n^\beta(k,i) = F_n^\beta(k, b_{j-1} - k) = b_j.$$

Setting $i = 0$, so that $k = b_{j-1}$, gives

$$F_n^\beta(k,0) = F_n^\beta(b_{j-1},0) = b_j = G_n^\gamma(k,0) = t + k.$$

Hence, $b_j = t + b_{j-1}$ and consequently $\beta = (0, t, 2t, \ldots, (n-1)t)$. But then t must be coprime with n because β is admissible. It follows that $\beta = t(0,1,\ldots,n-1)$, so that β is equivalent to $\delta = (0,1,\ldots,n-1)$. Consequently, by applying Theorem 2.1(a), $T(x,y,\beta)$ (and therefore also $T(u,v;\alpha)$) is isomorphic to the unique regular triangulation by $K_{n,n,n}$ described in Example 2.1.

(b) If the triangulations $T(u,v;\alpha)$ and $T(x,y;\beta)$ have an isomorphism mapping $\{U,V\}$ to $\{X,Y\}$ then, by Theorem 2.1, α and β are equivalent. By part (a), if they have an isomorphism that does not map $\{U,V\}$ to $\{X,Y\}$, then β is equivalent to δ and, by reversing the roles of the two triangulations, α is also equivalent to δ. Hence in both cases, α and β are equivalent. Already by Theorem 2.1 we have the converse: if α and β are equivalent then $T(u,v;\alpha)$ and $T(x,y;\beta)$ are isomorphic. $\square$

Proof of Theorem 2.5.
By Theorem 2.4, the number of nonisomorphic triangulations $T(u,v;\alpha)$ is equal to the number of equivalence classes of cyclic permutations of n elements $\alpha = (a_0, a_1, \ldots, a_{n-1})$, for which $\{a_0, a_1, \ldots, a_{n-1}\} = \{0,1,\ldots,n-1\}$

and $(a_i - a_{i-1}, n) = 1$, $0 \le i \le n-1$. However, as n is a prime number, any cyclic permutation α of $\{0, 1, \ldots, n-1\}$ satisfies $(a_i - a_{i-1}, n) = 1$. Since α and β are equivalent whenever there are $f, g \in \mathbb{Z}_n$, $(n, g) = 1$, and $h \in \{-1, 1\}$, such that $\beta = f + g\alpha^h$, the equivalence class of permutations has at most $2n(n-1)$ elements. As there are $(n-1)!$ cyclic permutations of order n, the number of equivalence classes of cyclic permutations is at least $(n-1)!/2n(n-1)$. $\square$

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