Dihedral biembeddings and triangulations by complete and complete tripartite graphs

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Abstract

We construct biembeddings of some Latin squares which are Cayley tables of dihedral groups. These facilitate the construction of n^{an^2} nonisomorphic face 2-colourable triangular embeddings of the complete tripartite graph $K_{n,n,n}$ and the complete graph K_n for linear classes of values of n and suitable constants a. Previously the best known lower bounds for the number of such embeddings that are applicable to linear classes of values of n were of the form 2^{an^2}. We remark that trivial upper bounds are $n^{n^2/3}$ in the case of complete graphs K_n and n^{2n^2} in the case of complete tripartite graphs $K_{n,n,n}$.

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1 Introduction

Establishing the existence of a minimum genus surface embedding of each complete graph K_n was a crucial step in Ringel and Youngs' solution of the famous Heawood map colouring problem for surfaces of positive genus [21]. For some residue classes modulo 12 such embeddings necessarily have all their faces triangular. Until 1999 the maximum number of known nonisomorphic triangular embeddings of K_n for any particular n in either an orientable or nonorientable surface was a mere three [20]. In [15, 16, 17, 18, 19] a lower bound of the form 2^{an} for all sufficiently large n was established for the number of minimum genus embeddings of K_n. At the same time in [2, 10] it was proved that, for linear classes of values of n, there are at least 2^{an^2} such triangular embeddings for some constants $a > 0$. In fact these embeddings are face 2-colourable and the triangular faces in each colour class determine a Steiner triple system of order n. In order to obtain results of this latter type, the authors first constructed 2^{an^2} nonisomorphic face 2-colourable triangular embeddings of the complete tripartite graph $K_{n,n,n}$; each such embedding can be viewed as a pair of Latin squares.

With regard to an upper bound, by using the method of Lemma 5.2 of [4], it is easily shown that the number of distinct twofold triple systems of order n is at most $n^{n^2/3}$. As observed in [1], there is a one-to-one correspondence between twofold triple systems of order n and triangular embeddings of K_n in generalized pseudosurfaces. Consequently the number of nonisomorphic triangular surface embeddings of K_n cannot exceed $n^{n^2/3}$, and a slight refinement of the argument extends this bound to all minimum genus embeddings of K_n. Again using the method of Lemma 5.2 of [4], a trivial upper bound for the number of triangular embeddings of $K_{n,n,n}$ is n^{2n^2}.

More recently, the gap between these best known upper and lower bounds was substantially narrowed for an infinite, but rather sparse, set of values of n. (By "sparse" we mean that the number of suitable values not exceeding m is of order $\log m$.) It was shown that for these values of n and for a certain positive constant a, there are at least n^{an^2} nonisomorphic triangular embeddings of K_n in a nonorientable [6] as well as in an orientable surface [12]. As in the case of the 2^{an^2} lower bound, a major component of the proof was the establishment of a similar lower bound on the number of nonisomorphic face 2-colourable triangular embeddings of $K_{n,n,n}$. This bound was itself a considerable improvement on the previous best known lower bound, although it was also achieved for a rather sparse set of values of n. In the current paper we establish lower bounds of the form n^{an^2} for linear classes of values of n. This adds weight to the conjecture that the numbers of minimum genus embeddings of K_n and of $K_{n,n,n}$ are of this form for all values of n and not just for some sparsely distributed special values. Moreover, the constants a in our bounds are equal to the best so far achieved for much sparser values of n.

When a triangular embedding of $K_{n,n,n}$ is face 2-colourable, the triangular faces in each colour class determine a Latin square of order n by taking these faces as the (row, column, entry) triples, where the row labels, the column labels and the entries form the three sets of the partition. Hence, one approach to constructing (roughly) n^{n^2} nonisomorphic triangular embeddings of $K_{n,n,n}$ would be to find for every Latin square L of order n a mate L', such that L and L' form the two colour classes of a face 2-colourable triangular embedding of $K_{n,n,n}$. Focusing on the problem from this direction, in [11] it was shown that, with the single exception of the group C_2^2, each Cayley table of an Abelian group of order n appears as a colour class in a face 2-colourable triangular embedding of $K_{n,n,n}$. However, at present we are not able to find such a mate for a general Latin square L. Here, therefore, we use an alternative approach based on particular squares of order n having the "nice" property of containing a cubic number of Latin subsquares of order 2 (see Lemma 2.1 below). Then by applying a generalized product construction using different subsets of disjoint subsquares of order 2, we obtain a large number of nonisomorphic embeddings of certain complete tripartite graphs. This in turn leads to a large number of face 2-colourable triangular embeddings of certain complete graphs by use of a construction described in [2, 10].

All the surfaces we consider will be closed, connected 2-manifolds, without a boundary: that is, in the orientable case, S_g the sphere with g handles and, in the nonorientable case, N_γ the sphere with γ crosscaps. Given a triangular embedding of some simple graph G with vertex set $V(G)$, the *rotation* at a vertex $v \in V(G)$ is the cyclically ordered permutation of vertices adjacent to v, with the ordering determined by the embedding. Conversely, given a set of triangular faces with each edge appearing in precisely two faces, the faces may be sewn together along their common edges; if at each vertex v the resulting permutation of neighbouring vertices is a single cycle, then the faces form a triangulation of some surface with the cycle at v forming the rotation at v. A triangular embedding of a graph G_1 is said to be *isomorphic* to a triangular embedding of a graph G_2 if there is a bijection between the vertex sets of G_1 and G_2, preserving all incidences between vertices, edges and faces. Two triangular embeddings (which may or may not be isomorphic) of the same graph are said to be *differently labelled* if there exists a triple of vertex labels that corresponds to a triangular face in one of the embeddings but not in the other.

Ringel and Youngs' results [21, 22] establish that for $n \equiv 3$ or $7 \pmod{12}$ there exists a face 2-colourable triangular embedding of K_n in an orientable surface, and embeddings given in [21] and in [13] establish the existence of such embeddings in a nonorientable surface for $n \equiv 1$ or $3 \pmod 6$ when $n \geq 9$.

A *parallel class* in a triangular embedding of $K_{n,n,n}$ is a set of n triangular faces that cover all $3n$ vertices. A *partial parallel class* is a set of triangular faces in which each vertex appears at most once. When discussing face 2-colourable triangular embeddings of $K_{n,n,n}$, we may alternatively refer to them as *biembeddings* of the associated Latin squares and we may use the terms *triangles*, *triangular faces* or *triples* interchangeably to refer to the faces of the embeddings or to the triples of the Latin squares. The colour classes of a face 2-coloured embedding will be taken as black and white. We write $A \bowtie B$ to denote the fact that the Latin square A biembeds with the Latin square B, and we also use this notation to denote the biembedding itself, taking A white and B black. A parallel class in one colour class of the embedding corresponds to a *transversal* in the associated Latin square of side n, that is to say a set of n entries from the square that contains every entry symbol, and covers every row and every column. Similarly, a partial parallel class corresponds to a partial transversal. When convenient we may switch between usage of these equivalent terms. It was shown in [7], by a very easy argument, that a face 2-colourable triangular embedding of $K_{n,n,n}$, i.e. a biembedding of two Latin squares, is necessarily in an orientable surface.

A *Pasch configuration* in a triangular embedding of a graph consists of four triangles having the form $(a, b, c), (a, d, e), (f, b, e), (f, d, c)$. In a biembedding of two Latin squares $A \bowtie B$, a Pasch configuration in the white (respectively, black) colour class corresponds to a subsquare of order 2 in A (B). When convenient we may switch between usage of these two terms. We will use Pasch configurations to construct large numbers of embeddings.

We say that the Latin square B is *paratopic* to A (or is in the same *main class* as A) if A can be transformed to B by permuting each of the sets of row labels, column labels, and entries, and then permuting these 3 sets in any one of the 6 possible permutations. If $A \bowtie B$ where B is paratopic to

A, then we may say that A has a *self-embedding*.

Whenever n is odd, there is a face 2-colourable triangular embedding of $K_{n,n,n}$ with a parallel class in each colour class. Such an embedding is given by the pair of Latin squares C_n and $C_n + 1$ with rows and columns indexed by, and entries in, $\mathbb{Z}_n$; here $C_n(i,j) = i+j$ and $(C_n+1)(i,j) = i+j+1$ with arithmetic in $\mathbb{Z}_n$, see for example [7, 9].

We will identify a group G with its Cayley table, so that we may write $G \bowtie H$, meaning that the Latin square formed by a Cayley table of G biembeds with the Latin square H. For any remaining undefined items of terminology and for background information we refer the reader to [3, 14, 21].

2 Biembeddings

In [11] it was shown that, with the single exception of the group C_2^2, each Abelian group appears in a biembedding. In the course of proving this result, biembeddings were obtained covering the dihedral groups D_n for $n = 2^i$ when $i \geq 2$. We now show that, for $n \equiv 1$ or 5 (mod 6), the Latin square D_n formed from the Cayley table of the dihedral group of order $2n$ has a self-embedding.

Theorem 2.1 *If $n \equiv 1$ or $5 \pmod 6$ then $D_n \bowtie H_n$, for some H_n.*

Proof. The proof is by direct construction. In our squares of order $2n$, the row labels, column labels and entries will be taken as $0, 1, \ldots, n-1, 0', 1', \ldots, n-1'$, where $n-1'$ is written for $(n-1)'$ to save on excessive use of brackets; a similar gloss will be applied to other compound terms. The standard form for D_n is shown in Figure 1 of [11]. However, in the current context, the argument is simplified by reversing the order of the last $n-1$ rows. This gives the representation of D_n shown in Figure 1.

With $C_n(i,j) = i+j \pmod n$, this representation of D_n has the schematic form

$$D_n = \begin{array}{|c|c|} \hline C_n & C_n' \\ \hline -C_n' & -C_n \\ \hline \end{array}\,.$$

We will prove that, for suitable constants α, β, γ and δ, we may take

$$H_n = \begin{array}{|c|c|} \hline C_n + \alpha' & C_n + \beta \\ \hline -C_n + \gamma & -C_n + \delta' \\ \hline \end{array}\,,$$

	0	1	2	$\dots$	$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
0	0	1	2	$\dots$	$n-1$	$0'$	$1'$	$2'$	$\dots$	$n-1'$
1	1	2	3	$\dots$	0	$1'$	$2'$	$3'$	$\dots$	$0'$
2	2	3	4	$\dots$	1	$2'$	$3'$	$4'$	$\dots$	$1'$
$\vdots$				$\vdots$					$\vdots$	
$n-1$	$n-1$	0	1	$\dots$	$n-2$	$n-1'$	$0'$	$1'$	$\dots$	$n-2'$
$0'$	$0'$	$n-1'$	$n-2'$	$\dots$	$1'$	0	$n-1$	$n-2$	$\dots$	1
$1'$	$n-1'$	$n-2'$	$n-3'$	$\dots$	$0'$	$n-1$	$n-2$	$n-3$	$\dots$	0
$\vdots$				$\vdots$					$\vdots$	
$n-2'$	$2'$	$1'$	$0'$	$\dots$	$3'$	2	1	0	$\dots$	3
$n-1'$	$1'$	$0'$	$n-1'$	$\dots$	$2'$	1	0	$n-1$	$\dots$	2

Figure 1: A representation of D_n.

where $(C_n+\alpha)(i,j) = (i+j+\alpha) \pmod{n}$. To do this we compute sections of the rotations at typical row, column and entry vertices. All arithmetic encountered is to be taken in $\mathbb{Z}_n$.

So, consider first the rotation at the row vertex i $(0 \le i \le n-1)$. Starting at the column vertex j and proceeding to the entry vertex $i+j$ given by D_n, the following sequence of column and entry vertices is obtained.

$$\begin{array}{cccccc} \text{col.} & \text{entry} & \text{col.} & \text{entry} & \text{col.} & \cdots \\ j & i+j & j-\beta' & i+j-\beta' & j-(\alpha+\beta) & \cdots \end{array}$$

Provided that $\alpha+\beta$ is coprime with n, this rotation will form a single cycle of length $4n$. Similarly, the rotation at the row vertex i' $(0 \le i \le n-1)$ is given by

$$\begin{array}{cccccc} \text{col.} & \text{entry} & \text{col.} & \text{entry} & \text{col.} & \cdots \\ j & -i-j' & j+\delta' & -i-j-\delta & j+(\delta+\gamma) & \cdots \end{array}$$

Provided that $\delta+\gamma$ is coprime with n, this rotation will form a single cycle of length $4n$.

The column vertex j gives the sequence

$$\begin{array}{cccccc} \text{entry} & \text{row} & \text{entry} & \text{row} & \text{entry} & \cdots \\ k & k-j & k+\alpha' & -k-j-\alpha' & k+(\alpha+\gamma) & \cdots \end{array}$$

and the column vertex j' gives the sequence

$$\begin{array}{cccccc} \text{entry} & \text{row} & \text{entry} & \text{row} & \text{entry} & \cdots \\ k & -k-j' & k+\delta' & k-j+\delta & k+(\delta+\beta) & \cdots \end{array}$$

Both of these rotations are single cycles of length $4n$ provided that $\alpha+\gamma$ and $\delta+\beta$ are coprime with n.

The entry vertex k gives the sequence

$$\begin{array}{cccccc} \text{row} & \text{col.} & \text{row} & \text{col.} & \text{row} & \cdots \\ i & k-i & -2k+i+\gamma' & k-i-\gamma' & i+(\gamma-\beta) & \cdots \end{array}$$

and the entry vertex k' gives the sequence

$$\begin{array}{cccccc} \text{row} & \text{col.} & \text{row} & \text{col.} & \text{row} & \cdots \\ i & k-i' & -2k+i+\delta' & k-i-\delta & i+(\delta-\alpha) & \cdots \end{array}$$

Both of these rotations are single cycles of length $4n$ provided that $\gamma-\beta$ and $\delta-\alpha$ are coprime with n.

Hence if all of $\alpha+\beta, \delta+\gamma, \alpha+\gamma, \delta+\beta, \gamma-\beta$ and $\delta-\alpha$ are coprime with n then $D_n \bowtie H_n$. It is easy to show that if n is even or if $3|n$, then there is no solution for $(\alpha,\beta,\gamma,\delta)$. However, if $n \equiv 1$ or $5 \pmod 6$, then solutions do exist, examples of which are given by $(\alpha,\beta,\gamma,\delta) = (0,1,n-1,2)$ and $(1,1,2,2)$. □

Corollary 2.1.1 *If $n \equiv 1$ or $5 \pmod 6$ then D_n has a self-embedding $D_n \bowtie D_n^*$.*

Proof. Following the notation of the proof of Theorem 2.1, if we take $(\alpha,\beta,\gamma,\delta) = (1,1,2,2)$, then the resulting square H_n is in the same main class as D_n. To see this, observe that

$$H_n = \begin{array}{|c|c|}\hline C_n+1' & C_n+1 \\ \hline -C_n+2 & -C_n+2' \\ \hline \end{array}\,,$$

and an obvious permutation of the rows gives the array

$$\begin{array}{|c|c|}\hline C_n' & C_n \\ \hline -C_n & -C_n' \\ \hline \end{array}\,.$$

If the entries are then permuted using $\prod_{k=0}^{n-1}(k,k')$ we obtain the square D_n as shown in Figure 1. Hence, by taking $D_n^* = H_n$, we get a self-embedding of D_n. □

We remark that not all the solutions for H_n given by Theorem 2.1 give self-embeddings of D_n. An example of this occurs for H_{19} when $(\alpha, \beta, \gamma, \delta) = (0, 1, 3, 5)$. A Latin square L is (a paratopic copy of) a group Cayley table only if it satisfies the *quadrangle criterion* (see [5]), meaning that if $(e, f, w), (g, h, w), (i, f, x), (j, h, x), (i, k, y), (j, l, y), (e, k, z) \in L$ then $(g, l, z) \in L$. It is easy to show that this is not the case for this H_{19}, and so this square is not a copy of a group Cayley table.

Lemma 2.1 *The Latin square D_n has at least n^3 distinct Pasch configurations; if n is odd then the number is precisely n^3. If $n \equiv 1$ or $5 \pmod 6$ then the self-embedding $D_n \bowtie D_n^*$ from the previous corollary has a parallel class formed from n triples of D_n and n triples from D_n^*.*

Proof. Take D_n in the form shown in Figure 1. Choose any column j_1 from those labelled 0 to $n-1$, any row i_1 from those labelled 0 to $n-1$, and any other row i_2' from those labelled $0'$ to $n-1'$. The corresponding triples are $(i_1, j_1, i_1 + j_1)$ and $(i_2', j_1, -(i_2 + j_1)')$. Let j_2' be the column of row i_1 that contains the entry $-(i_2 + j_1)'$ so that $i_1 + j_2 = -(i_2 + j_1)$. The corresponding triple is $(i_1, j_2', -(i_2 + j_1)')$. The entry in row i_2', column j_2' is $-(i_2 + j_2) = i_1 + j_1$ and the corresponding triple is $(i_2', j_2', i_1 + j_1)$. Thus the four triples form a Pasch configuration. There are n^3 choices for (i_1, i_2, j_1), so there are at least n^3 Pasch configurations in D_n. If there are any further Pasch configurations then there must be a Pasch configuration in C_n, but it is easily seen that this is not the case when n is odd.

When $n \equiv 1$ or $5 \pmod 6$, the leading diagonal of C_n provides a partial transversal in D_n, and the leading diagonal of $-C_n + 2'$ provides a disjoint partial transversal in D_n^*. Together these form a parallel class in the embedding. □

We will apply a product construction for biembeddings of Latin squares with the biembedding $D_n \bowtie D_n^*$ as one of the ingredients. The result is a large number of nonisomorphic embeddings of the same graph. Our construction is a modification of one given in [10], and the modification is described in [6]. We start by giving an informal description of the original version and follow this with an explanation of the modification which uses Pasch configurations.

Take a face 2-coloured triangular embedding P of $K_{p,p,p}$ and another Q of $K_{q,q,q}$, and assume that the latter has a parallel class of triangular faces in one of the two colour classes, say black. Now take q copies of P, say P_i for $i = 1, 2, \ldots, q$. For each oriented white triangular face $W = (a, b, c)$ of P, we "bridge" the corresponding white triangles $W_i = (a_i, b_i, c_i)$ of the embeddings P_i. To do this we glue a copy of Q to these triangles in the following manner. We take a copy of Q and label the vertex parts with $\{a_i\}, \{b_i\}, \{c_i\}$ in such a way that the parallel class has oriented black triangles labelled (a_i, c_i, b_i). We then glue the black triangle (a_i, c_i, b_i) of Q onto the white triangle W_i of P_i, so that corresponding vertices and edges are identified and the interiors of the two triangles are removed at the same time. Repeating this process for every white triangle of P results in a face 2-coloured triangular embedding of $K_{pq,pq,pq}$ in an orientable surface.

In this construction, the bridging operation provides all the "missing" adjacencies between the q copies P_i. The bridge across the q triangles $W_i = (a_i, b_i, c_i)$ yields the adjacencies a_ib_j, a_ic_j, b_ic_j for $i, j = 1, 2, \ldots, q,\ i \neq j$. Now suppose that P contains a Pasch configuration $(a, b, c), (a, d, e), (f, b, e), (f, d, c)$. The four corresponding bridges provide the missing adjacencies $a_ib_j, a_ic_j, b_ic_j, a_id_j, a_ie_j, d_ie_j, f_ib_j, f_ie_j, b_ie_j, f_id_j, f_ic_j, d_ic_j$ for $i \neq j$. It is, however, possible to provide these adjacencies by an alternative arrangement of bridges. Concentrating for a moment on the adjacencies between P_1 and P_2, we may bridge (a_1, b_1, c_1) to (a_2, d_2, e_2), (a_1, d_1, e_1) to (a_2, b_2, c_2), (f_1, b_1, e_1) to (f_2, d_2, c_2), and (f_1, d_1, c_1) to (f_2, b_2, e_2) by suitable renaming of the vertices of the four bridges involved. The first bridge then provides the adjacencies $a_1d_2, a_1e_2, b_1a_2, b_1e_2, c_1a_2, c_1d_2$, the second provides $a_1b_2, a_1c_2, d_1a_2, d_1c_2, e_1a_2, e_1b_2$, the third provides $f_1d_2, f_1c_2, b_1f_2, b_1c_2, e_1f_2, e_1d_2$, and the fourth provides $f_1b_2, f_1e_2, d_1f_2, d_1e_2, c_1f_2, c_1b_2$. These 24 adjacencies are the same as the 24 adjacencies arising for the original bridging arrangement across the Pasch configuration between P_1 and P_2.

We will call the original arrangement of bridges *standard* and an arrangement of the type just described *non-standard*. We slightly tighten this definition of a non-standard arrangement below, but before doing so we give an example to help clarify the situation.

So consider the case $q = 3$ where the bridges are face 2-coloured triangular embeddings of $K_{3,3,3}$ in the torus. In the standard arrangement we may take the four bridges on a Pasch configuration as shown in Figure 2. The lightly shaded triangles are glued to the corresponding triangles on the surfaces defined by P_1, P_2 and P_3. In the non-standard arrangement the four bridges are relabelled as shown in Figure 3, and the gluing operation is carried out in the same manner.

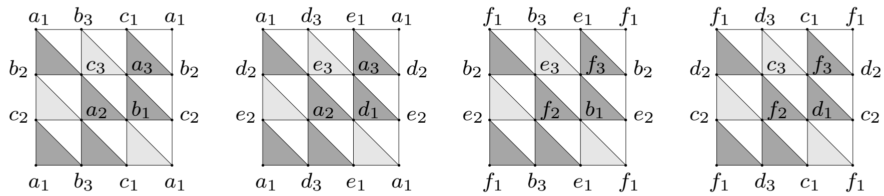

Figure 2: Standard bridging arrangement.

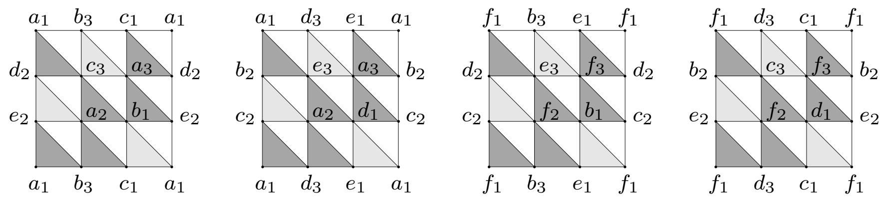

Figure 3: Non-standard bridging arrangement.

The embedding that results when a standard arrangement of these particular four bridges on a Pasch configuration is replaced by a non-standard arrangement will always be differently labelled from the original. For example, the triangle (a_1, b_1, c_2) appears in Figure 2, but not in Figure 3. To ensure that such different labelling occurs in the general case where the bridges are biembeddings of $K_{q,q,q}$, we now tighten our definition of a non-standard bridging arrangement on a Pasch configuration $\{\{a,b,c\},\{a,d,e\},\{f,b,e\},\{f,d,c\}\}$ by requiring that for some $i, j,\ j \neq i$, the four bridges contain white triangles of the form $(a_i, b_i, e_j), (a_i, d_i, c_j), (f_i, b_i, c_j)$ and (f_i, d_i, e_j). Since a standard bridge on (a,b,c) must contain a white triangle $(a_i, b_i, c_j),\ j \neq i$ (in fact one such triangle for each value of i), this requirement is easily satisfied by relabelling the black triangle (a_j, b_j, c_j) as (a_j, d_j, e_j), and then doing likewise for the other three bridges, when forming the non-standard bridge arrangement.

Lemma 2.2 *Suppose that P is a face 2-coloured triangular embedding of $K_{p,p,p}$ having a parallel class $\mathcal{T} = \mathcal{T}_w \cup \mathcal{T}_b$, where $\mathcal{T}_w$ comprises p_1 white triangles and $\mathcal{T}_b$ comprises p_2 black triangles (so that $p_1 + p_2 = p$). Suppose also that $q \geq 3$ is odd and $Q = C_q \bowtie (C_q + 1)$ where C_q defines the white faces and $C_q + 1$ the black with transversal $\mathcal{U} = \{(i, i, 2i+1) :\ i \in \mathbb{Z}_q\}$. Let R be an embedding of $K_{pq,pq,pq}$ formed by the product construction described above using the black transversal $\mathcal{U}$, but with the restriction that standard bridges*

are applied to all the white triangles of $\mathcal{T}_w$. Then R has the parallel class of black triangles
$\{(a_i, b_{i+2}, c_{i+1}) : (a, b, c) \in \mathcal{T}_w, i \in \mathbb{Z}_q\} \cup \{(x_i, y_i, z_i) : (x, y, z) \in \mathcal{T}_b, i \in \mathbb{Z}_q\}$.

Proof. Suppose that $W = (a, b, c) \in \mathcal{T}_w$. To apply a standard bridge to W, the triple $(i, i, 2i+1)$ of $\mathcal{U}$ is identified with the triple (a_i, b_i, c_i) of P, so that the row vertex i is relabelled a_i, the column vertex i is relabelled b_i, and the entry vertex $2i+1$ is relabelled c_i. Consequently each triple $(i, j, i+j+1)$ of $C_q + 1$ yields a black triangle $(a_i, b_j, c_{(i+j)/2})$ of R. Hence R has q black triangles (a_i, b_{i+2}, c_{i+1}) $(i \in \mathbb{Z}_q)$, and these form a partial parallel class in R. Furthermore, each black triangle $(x, y, z) \in \mathcal{T}_b$ results in q black triangles (x_i, y_i, z_i) $(i \in \mathbb{Z}_q)$ which form a disjoint partial parallel class in R. So altogether we have a parallel class of $p_1 q + p_2 q = pq$ black triangles $\{(a_i, b_{i+2}, c_{i+1}) : (a, b, c) \in \mathcal{T}_w, i \in \mathbb{Z}_q\} \cup \{(x_i, y_i, z_i) : (x, y, z) \in \mathcal{T}_b, i \in \mathbb{Z}_q\}$. □

Definition 2.1 *Given a face 2-coloured triangular embedding M of a graph G, a collection $\mathcal{C}$ of Pasch configurations in one of the colour classes will be called* independent *if no two of the Pasch configurations have a common triple.*

The following lemma was established in [6]. It will assist us in estimating the number of embeddings of some complete tripartite graphs.

Lemma 2.3 *Suppose that P is a face 2-coloured triangular embedding of $K_{p,p,p}$ and that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two different independent collections of Pasch configurations in the white colour class. For $i = 1, 2$, let R_i be the embedding that results when we apply standard $K_{q,q,q}$ bridges to each white face not in $\mathcal{C}_i$ and non-standard bridge arrangements to each Pasch configuration in $\mathcal{C}_i$. Then the embeddings R_1 and R_2 will be differently labelled.*

Lemma 2.4 *Suppose that n is odd. Then the number of different independent collections of Pasch configurations in D_n that do not contain any of the triples $(i, i, 2i)$ $(i \in \mathbb{Z}_n)$ is at least*

$$(1 + 4n)^{\lfloor \frac{n^2 - n}{4} \rfloor}.$$

Proof. As shown in Lemma 2.1, D_n has precisely n^3 Pasch configurations. It is also easy to see that every triple of D_n occurs in precisely n Pasch configurations, so the number of Pasch configurations that contain no triple $(i,i,2i)$ $(i \in \mathbb{Z}_n)$ is $n^3 - n^2$.

Denote by I_k the number of distinct independent collections of Pasch configurations in D_n that contain precisely k Pasch configurations but omit any Pasch configuration containing a triple $(i,i,2i)$ $(i \in \mathbb{Z}_n)$. Each Pasch configuration contains four triples. So, for $k-1 < \frac{n^3-n^2}{4n}$ we have

$$\begin{aligned} I_k &\geq (n^3-n^2)(n^3-n^2-4n)(n^3-n^2-8n)\cdots(n^3-n^2-4(k-1)n)/k! \\ &\geq (4n)^k N(N-1)(N-2)\cdots(N-(k-1))/k! \\ &= (4n)^k \binom{N}{k}, \end{aligned}$$

where $N = \lfloor \frac{n^2-n}{4} \rfloor$. Then, summing over $k = 0, 1, \ldots, N$ gives the number of distinct independent collections of Pasch configurations in D_n as at least

$$(1+4n)^N = (1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor}. \qquad \square$$

Theorem 2.2 *Suppose that* $n \equiv 1$ *or* 5 *(mod* 6*) and that* $q \geq 3$ *is odd. Then there are at least*

$$(1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor}$$

differently labelled face 2-colourable triangular embeddings of $K_{2nq,2nq,2nq}$ *all of which have a common parallel class of black triangular faces. Furthermore, there are at least*

$$\frac{(1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor}}{6((2nq)!)^3}$$

nonisomorphic face 2-colourable triangular embeddings of $K_{2nq,2nq,2nq}$.

Proof. Take the face 2-coloured triangular embedding of $K_{2n,2n,2n}$ given by $D_n \bowtie D_n^*$. By Lemma 2.4, the number of independent collections of Pasch configurations in D_n that miss all the triples $(i,i,2i)$ $(i \in \mathbb{Z}_n)$ is at least

$$(1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor}.$$

By applying Lemma 2.3 and bridging the white triangles, using standard bridges on the triangles given by $(i,i,2i)$ $(i \in \mathbb{Z}_n)$, we see that there is at least this number of differently labelled face 2-coloured triangular embeddings of $K_{2nq,2nq,2nq}$. And by Lemmas 2.1 and 2.2, all the resulting embeddings of $K_{2nq,2nq,2nq}$ have a common parallel class of black triangles.

The maximum possible size of an isomorphism class of such an embedding is $6((2nq)!)^3$, and so the number of isomorphism classes is at least

$$\frac{(1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor}}{6((2nq)!)^3}. \qquad \square$$

As mentioned in the Introduction, a face 2-colourable triangular embedding of a complete regular tripartite graph is necessarily orientable, so all the embeddings described in Theorem 2.2 and in Corollary 2.2.1 below are orientable.

Corollary 2.2.1 *For $r \equiv 6$ or 30 (mod 36), as $r \to \infty$ there are at least $r^{r^2(\frac{1}{144}-o(1))}$ nonisomorphic face 2-colourable triangular embeddings of $K_{r,r,r}$.*

Proof. In Theorem 2.2, take $q = 3$, write $r = 6n$, and note that $r! < r^r$. $\square$

This gives the first linear class of values r for which the number of nonisomorphic triangular embeddings of $K_{r,r,r}$ is known to be of the form r^{ar^2} for some positive constant a. Also, the constant $a = 1/144$ is equal to the best so far achieved in [8] for sparse values of r. We now use Theorem 2.2 to get a similar estimate for some complete graphs. The following theorem is taken from [6] and is obtained by applying a recursive construction to the embeddings from [13, 21, 22] mentioned in the Introduction.

Theorem 2.3 *Suppose that $m \equiv 3$ or 7 (mod 12) and that $r \equiv 0$ or 4 (mod 6). Suppose also that there are k differently labelled face 2-colourable triangular embeddings of $K_{r,r,r}$, all of which have a common parallel class of black triangular faces. Then we may construct $k^{(m-1)(m-3)/6}$ differently labelled face 2-colourable triangular embeddings of $K_{r(m-1)+1}$, all of which are nonorientable.*

Combining this with our results above, we obtain the following corollary.

Corollary 2.3.1 *Suppose that $q \geq 3$ and that either (a) $n \equiv 1 \pmod 6$ and $q \equiv 3$ or $5 \pmod 6$, or (b) $n \equiv 5 \pmod 6$ and $q \equiv 1$ or $3 \pmod 6$. Then, if $m \equiv 3$ or $7 \pmod{12}$, there are at least*

$$(1+4n)^{\lfloor \frac{n^2-n}{4} \rfloor \frac{(m-1)(m-3)}{6}}$$

differently labelled face 2-colourable triangular embeddings of $K_{2nq(m-1)+1}$, all of which are nonorientable. The number of nonisomorphic embeddings of this type is at least this number divided by $(2nq(m-1)+1)!$.

Proof. The first part follows by taking $r = 2nq$ in the theorem. The maximum size of an isomorphism class is $(2nq(m-1)+1)!$ and this gives the second part. □

Corollary 2.3.2 *If $s = 6n(m-1)+1$, where $n \equiv 1$ or $5 \pmod 6$ and $m \equiv 3$ or $7 \pmod{12}$, then for fixed m as $n \to \infty$, there are at least $s^{s^2(\frac{m-3}{864(m-1)} - o(1))}$ nonisomorphic nonorientable face 2-colourable triangular embeddings of K_s.*

Proof. In the previous corollary take $q = 3$ and note that $s! < s^s$. □

By taking, for example, $m = 7$, a bound of $s^{s^2(\frac{1}{1296} - o(1))}$ is obtained for $s \equiv 37$ or $181 \pmod{216}$. The linear classes produced by these corollaries are the first for which the number of nonisomorphic triangular embeddings of K_s is known to be of the form s^{as^2} for some positive constant a. Again, the constant $\frac{m-3}{864(m-1)}$ (respectively $\frac{1}{1296}$ in the case $m = 7$) is currently the best known, see [8].

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