Wiener index of iterated line graphs of trees homeomorphic to H

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Abstract

This is fourth paper out of five in which we completely solve a problem of Dobrynin, Entringer and Gutman. Let G be a graph. Denote by $L^i(G)$ its i-iterated line graph and denote by $W(G)$ its Wiener index. Moreover, denote by H a tree on six vertices, out of which two have degree 3 and four have degree 1. Let $j \geq 3$. In previous papers we proved that for every tree T, which is not homeomorphic to a path, claw $K_{1,3}$ and H, it holds $W(L^j(T)) > W(T)$. Here we prove that $W(L^4(T)) > W(T)$ for every tree T homeomorphic to H. As a consequence, we obtain that with the exception of paths and the claw $K_{1,3}$, for every tree T it holds $W(L^i(T)) > W(T)$ whenever $i \geq 4$.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. For any two of its vertices, say u and v, denote by $d_G(u, v)$ (or by $d(u, v)$ if no confusion is likely) the distance from u to v in G. The *Wiener index* (*Wiener number*) of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of G. Wiener index was introduced by Wiener in [22]. It is related to several properties of chemical

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molecules, see [14]. For this reason Wiener index is widely studied by chemists, although it has interesting applications also in computer networks, see e.g. [9]. The interest of mathematicians was attracted in 1970's, when it was reintroduced as *the transmission* and *the distance of a graph*; see [21] and [10], respectively. Recently, several special issues of journals were devoted to (mathematical properties) of Wiener index (see [12] and [13]). For surveys and some up-to-date papers related to Wiener index of trees and line graphs see [5, 6], [8, 18, 19, 20, 24] and [2, 3, 7, 11, 23], respectively.

By the definition, if G has a unique vertex, then $W(G) = 0$. In this case, we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices (and hence also the set of edges) of G is empty.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges share an endpoint in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0 \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1] we have the following statement.

Theorem 1.1 *Let T be a tree on n vertices. Then $W(L(T)) = W(T) - \binom{n}{2}$.*

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see e.g. [4]. In [5], the following problem was posed:

Problem 1.2 *Is there a tree T satisfying the equality $W(L^i(T)) = W(T)$ for some $i \geq 3$?*

As observed above, if T is a trivial tree, then $W(L^i(T)) = W(T)$ for every $i \geq 1$, although here the graph $L^i(T)$ is empty. The real question is if there is some other tree T and $i \geq 3$ such that $W(L^i(T)) = W(T)$. In this paper we solve Problem 1.2 for $i \geq 4$; see Corollary 1.4 below.

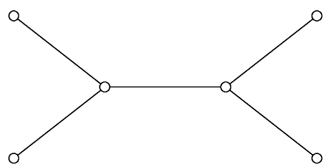

Figure 1: The tree H.

The essential part of this paper deals with trees homeomorphic to the graph H on 6 vertices, depicted in Figure 1, that is, the trees that have precisely two vertices of degree 3, four vertices of degree 1 and all other vertices of degree 2. (Recall that graphs G_1 and G_2 are *homeomorphic* if and only if the graphs obtained from G_1 and G_2, respectively, by repeatedly substituting the vertices of degree two together with the two incident edges with a single edge, are isomorphic.) In particular, we prove the following result:

Theorem 1.3 *Let T be a tree homeomorphic to* H. *Then $W(L^i(T)) > W(T)$ for every $i \geq 4$.*

In [16] we proved an analogue of Theorem 1.3 for trees different from a path which are not homeomorphic to neither $K_{1,3}$ nor H, and in [17] for trees homeomorphic (but not isomorphic) to $K_{1,3}$. By observing that $W(L^i(T)) < W(T)$ for $T \cong K_{1,3}$ or P_n, we can summarize our results for $i \geq 4$ as follows:

Corollary 1.4 *Let T be a tree and $i \geq 4$. Then the following holds:*

$$\begin{array}{ll} W(L^i(T)) < W(T) & \textit{if } T \textit{ is the claw } K_{1,3} \textit{ or a path } P_n \textit{ with } n \geq 2; \\ W(L^i(T)) = W(T) & \textit{if } T \textit{ is the trivial graph } P_1; \\ W(L^i(T)) > W(T) & \textit{otherwise.} \end{array}$$

We remark that the case $i = 3$, which resolves Problem 1.2, is dealt with in a forthcomming paper.

2 Preliminaries

In paper [15], it was shown that for every connected graph G, the function $i \mapsto W(L^i(G))$ is convex. This implies that in order to prove Theorem 1.3, it suffices to prove

$$W(L^4(T)) > W(T) \tag{1}$$

for every tree T homeomorphic to H.

By convexity of function $i \mapsto W(L^i(T))$, it follows that $W(L^4(T)) + W(L^2(T)) > 2W(L^3(T))$ and therefore $W(L^4(T)) - W(T) > 2W(L^3(T)) - W(L^2(T)) - W(T)$. Define the quantity $D(T)$ as

$$D(T) = 2W(L^3(T)) - W(L^2(T)) - W(T). \tag{2}$$

By the analysis above, (1) holds if the following lemma is true.

Lemma 2.1 *Let T be a tree homeomorphic to* H. *Then $D(T) > 0$.*

A *ray* (a *pendant path*) in T is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in G. Let T be a tree homeomorphic to H such that a longest ray of T has length at least 2. By pred(T) we denote a tree obtained from T by deleting the pendant vertex (and its incident edge) of a longest ray. Obviously, pred(T), standing for a predecessor of T, induces a partial ordering on trees homeomorphic to T. We prove Lemma 2.1 by induction applied to this partial ordering.

We start with the base of the induction, that is with trees, all rays of which have length 1. In [15] we proved that if T is a tree distinct from a path and the claw $K_{1,3}$ and all rays of T have length 1, then $W(L^3(T)) > W(T)$. By convexity of function $i \mapsto W(L^i(T))$ we get $W(L^3(T)) > W(L^2(T))$. Summing these two inequalities we obtain:

Proposition 2.2 *Let T be a tree, all rays of which have length* 1*, distinct from a path and the claw $K_{1,3}$. Then* $2W(L^3(T)) - W(L^2(T)) - W(T) = D(T) > 0$.

Now we describe the induction step. Let T be a tree with a longest ray of length at least 2, and let R^- be a longest ray in T. Remove the last vertex of R^- (i.e., the vertex of degree 1) and the edge incident with this vertex, and denote the resulting graph by T^-. Obviously, T^- is pred(T). By the induction hypothesis we assume that $D(T^-) > 0$ and our aim is to prove $D(T) > 0$. Define

$$\Delta T = D(T) - D(T^-). \tag{3}$$

To establish the induction step it suffices to prove:

Lemma 2.3 *Let T be a tree homeomorphic to* H *with a longest ray of length at least* 2 *and let T^- and ΔT be as defined above. Then* $\Delta T \geq 0$.

Now we introduce some necessary notation used in the following sections. If z is a vertex of a graph G, then its degree is denoted by d_z. By the definition, every vertex $w \in V(L(G))$ corresponds to an edge of G. Let us denote by $B_1(w)$ this edge of G. For two subgraphs S_1 and S_2 of G, by $d(S_1, S_2)$ we denote the shortest distance in G between a vertex of S_1 and a vertex of S_2. If S_1 and S_2 share an edge then we set $d(S_1, S_2) = -1$. Let w and z be two vertices of $L(G)$. Then $d_{L(G)}(w, z) = d(B_1(w), B_1(z)) + 1$ (see also [15]).

Let $u, v \in V(G)$, $u \neq v$. Denote by $\alpha_i(u, v)$ the number of pairs $w, z \in V(L(G))$, with $u \in V(B_1(w))$ and $v \in V(B_2(z))$, such that $d(B_1(w), B_1(z)) = d(u, v) - 1 + i$. Since $d(u, v) - 2 \leq d(B_1(w), B_1(z)) \leq d(u, v)$, we have $\alpha_i(u, v) = 0$ for all $i \notin \{-1, 0, 1\}$. In [15] we have the following statement:

Proposition 2.4 *Let G be a connected graph. Then*

$$W(L(G)) = \frac{1}{4} \sum_{u \neq v} \Big[d_u\, d_v\, d(u, v) - \alpha_{-1}(u, v) + \alpha_1(u, v) \Big] + \frac{1}{4} \sum_{u} \binom{d_u}{2},$$

where the first sum is taken over unordered pairs $u, v \in V(G)$ and the second one over $u \in V(G)$.

In the next section we find a formula for ΔT and in the last one we prove Lemma 2.3.

3 Formula for ΔT

From now on, we will work with line graphs of trees. To simplify the notation, define $LG = L(G)$ for an arbitrary graph G. Let us recall the structure of line graphs of trees. For any tree F, the graph LF consists of cliques. Denote by $\mathcal{C}(LF)$ the set of maximal cliques of LF. Then every vertex of LF belongs to at most two cliques from $\mathcal{C}(LF)$; each pair of cliques from $\mathcal{C}(LF)$ intersect in at most one vertex; and the cliques of $\mathcal{C}(LF)$ have a "tree structure", i.e., there are no distinct cliques $C_0, C_1, \ldots, C_{t-1}$, $t \geq 3$, such that C_i and C_{i+1} have nonempty intersection, $0 \leq i \leq t-1$, the addition being modulo t. Consequently, for every pair of vertices u, v of LF, there is a unique shortest path starting at u and terminating at v.

Before we state an exact formula for ΔT, we specify some vertices of T and LT related to T^- and LT^- via R^-. Denote by b' the last vertex of R^- and denote by a' its neighbour. Then $V(T^-) = V(T) \setminus \{b'\}$. Further, denote by b the edge $a'b'$ and denote by a the other edge of T incident with a'. Then ab is an edge of LT and $V(LT^-) = V(LT) \setminus \{b\}$, see Figure 2 below. In the next formulae, all the degrees and distances are considered in LT^- (rather than in LT). For $u \in V(LT^-) \setminus \{a\}$ define

$$\begin{aligned} h(u) &= \Big(d_u\Big((d_u-1)d_a - \frac{1}{2}\Big) - 1\Big)d(u,a) \\ &\quad + (d_u - 1)\Big(2d_u\,d_a - 2d_a - d_u - \frac{1}{2}\Big) - 2 - 2\phi(u,a), \end{aligned} \tag{4}$$

where

$$\phi(u,a) = \begin{cases} (d_a-1)(d_u-2) & \text{if } d(u,a) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1 *Let T be a tree with a longest ray of length at least 2. Further, let T^- and ΔT be as in Section 2 and let a be as defined above. Then we have*

$$\Delta T = \sum_u h(u) + d_a\left((d_a-1)(2d_a-1) - \frac{1}{2}\right) - 3,$$

where the sum is taken over the vertices u of $V(LT^-) \setminus \{a\}$ and the degrees and distances are considered in LT^-.

PROOF. For $i \in \{0, 2, 3\}$ denote

$$\Delta \mathrm{WL}^i = W(L^i(T)) - W(L^i(T^-)).$$

Then

$$\Delta T = 2\Delta \mathrm{WL}^3 - \Delta \mathrm{WL}^2 - \Delta \mathrm{WL}^0. \tag{5}$$

In (3) and (4) of [16] we derived

$$\begin{aligned} \Delta \mathrm{WL}^3 \;\; &= \;\; \sum_u \left[\binom{d_u}{2} d_a\, d(u,a) + (d_u - 1)\Big(d_u\, d_a - d_a - \frac{1}{2} d_u \Big) - \phi(u,a) \right] \\ &\quad + \frac{1}{2} d_a (d_a - 1)(2d_a - 1), \end{aligned} \tag{6}$$

and in (7) of [16] we have

$$\Delta \mathrm{WL}^0 = \sum_u \Big[d(u,a) + 2 \Big] + 3. \tag{7}$$

In both (6) and (7) the sum is taken over the vertices u of $V(LT^-) \setminus \{a\}$ and the degrees and distances are considered in LT^-. (We remark that in [16], instead of T and T^- we use T^* and T, respectively.) Hence, it remains to find $\Delta \mathrm{WL}^2$. Observe that

$$\Delta \mathrm{WL}^2 = W(L^2(T)) - W(L^2(T^-)) = W(L(LT)) - W(L(LT^-)).$$

Let $F \in \{T, T^-\}$. By Proposition 2.4, we have

$$W(L(LF)) = \frac{1}{4} \sum_{u \neq v} \Big[d_u\, d_v\, d(u,v) - \alpha_{-1}(u,v) + \alpha_1(u,v) \Big] + \frac{1}{4} \sum_u \binom{d_u}{2}, \tag{8}$$

where the degrees and distances are considered in LF, the first sum is taken over unordered pairs $u, v \in V(LF)$ and the second one over $u \in V(LF)$. We find $\alpha_{-1}(u,v)$ and $\alpha_1(u,v)$.

Let u and v be distinct vertices of LF. Recall that $\alpha_i(u,v)$ is the number of pairs $w, z \in V(L(LF))$, with $u \in V(B_1(w))$ and $v \in V(B_1(z))$, such that $d(B_1(w), B_1(z)) = d(u,v) - 1 + i$. As mentioned above, there is a unique shortest $u-v$ path in LF. Denote this path by $u{=}x_0, x_1, \ldots, x_t{=}v$. If $d(B_1(w), B_1(z)) = d(u,v) - 2$ then w is the edge ux_1 and z is the edge $x_{t-1}v$. Hence, $\alpha_{-1}(u,v) = 1$.

To find $\alpha_0(u,v)$ we distinguish two cases:

Case 1. $d(u,v) \geq 2$. In this case u and v do not belong to a common clique from $\mathcal{C}(LF)$. Since $d(B_1(w), B_1(z)) = d(u,v) - 1$, it holds either $x_1 \in V(B_1(w))$ or $x_{t-1} \in V(B_1(z))$, but not both. In the first case we obtain $(d_v - 1)$ pairs w, z and in the second $(d_u - 1)$ pairs w, z. Thus, $\alpha_0(u,v) = d_u + d_v - 2$.

Case 2. $d(u,v) = 1$. In this case u and v belong to a common clique. All pairs w, z mentioned in the previous case contribute to $\alpha_0(u,v)$, but we have to add pairs

w, z such that $v \notin V(B_1(w))$, $u \notin V(B_1(z))$ and $d(B_1(w), B_1(z)) = d(u, v) - 1 = 0$. For these pairs the edges $B_1(w)$ and $B_1(z)$ share a vertex distinct from u and v. Denote by c the order of the clique from $\mathcal{C}$ containing both u and v. Then $\alpha_0(u, v) = d_u + d_v - 2 + c - 2$.

We have $d_u\, d_v$ pairs $w, z \in V(L(LF))$, such that $u \in V(B_1(w))$ and $v \in V(B_1(z))$. Hence,

$$\alpha_1(u, v) = \begin{cases} (d_u - 1)(d_v - 1) & \text{if } d(u, v) \geq 2; \\ (d_u - 1)(d_v - 1) - (c - 2) & \text{if } d(u, v) = 1. \end{cases}$$

Now we evaluate $\Delta\mathrm{WL}^2 = W(L(LT)) - W(L(LT^-))$. The graph LT has one more vertex than LT^-, namely the vertex b of degree 1, and the degree of a is $d_a + 1$ in LT (observe that by d_a we denoted the degree of a in LT^-). Therefore, all the terms of (8) for pairs u, v which do not contain neither a nor b, cancell out in $W(L^2(LT)) - W(L^2(LT^-))$. But we need to subtract the terms for pairs u, a in LT^-, to add the terms for pairs u, a and u, b in LT, for each $u \in V(LT^-) \setminus \{a\}$, and finally to add the term for pair a, b. As regards the second sum in (8), we have to subtract the term corresponding to a in LT^- and add the terms corresponding to a and b in LT, the later one being 0 as the degree of b is 1 in LT. If $d(u, v) = 1$ then the order of the clique from $\mathcal{C}$ containing both u and v is denoted by $c(u, v)$. If $d(u, v) \geq 2$ we set $c(u, v) = 2$. Observe that for every $u \in V(LT^-)$ we have $c(u, b) = 2$. We obtain

$$\begin{aligned} \Delta\mathrm{WL}^2 &= -\frac{1}{4}\sum_u \Big[d_u\, d_a\, d(u, a) - 1 + (d_u - 1)(d_a - 1) - (c(u, a) - 2)\Big] \\ &\quad +\frac{1}{4}\sum_u \Big[d_u\,(d_a + 1) d(u, a) - 1 + (d_u - 1) d_a - (c(u, a) - 2)\Big] \\ &\quad +\frac{1}{4}\sum_u \Big[d_u\, 1(d(u, a) + 1) - 1 + 0 - (c(u, b) - 2)\Big] \\ &\quad +\frac{1}{4}\Big[1(d_a + 1)1 - 1 + 0 - (c(a, b) - 2)\Big] - \frac{1}{4}\binom{d_a}{2} + \frac{1}{4}\binom{d_a + 1}{2} \\ &= \frac{1}{2}\sum_u \Big[d_u\, d(u, a) + d_u - 1\Big] + \frac{1}{2} d_a, \end{aligned} \tag{9}$$

where $u \in V(LT^-) \setminus \{a\}$. Now substituting (6), (9) and (7) into (5) we obtain the result. □

4 Proof of Lemma 2.3

With the notation as in the previous sections, denote by $l + 2$ the length of R^-. Since we assume that the longest ray of T has length at least 2 (see the definition of

ΔT), we have $l \geq 0$. Further, all the rays of T^- have length at most $l+2$, and the ray terminating at a' has length $l+1$. Consequently, all rays of LT^- have length at most $l+1$ and the ray terminating at a has length l, see Figure 2 below.

We prove Lemma 2.3 in two steps. First we prove it in the case $l=0$ (Lemma 4.1) and then in the case $l \geq 1$ (Lemma 4.2). If $l=0$ then a' is adjacent to a vertex of degree at least 3 in T^-, so that in LT^- we have $d_a \geq 2$. On the other hand, if $l \geq 1$ then $d_a = 1$.

Let v be an endvertex of a ray, say R, in LT. Then $d_v = 1$. By $\overline{v}$ we denote the first vertex of R, i.e., a closest vertex to v whose degree is at least 3. Observe that if u and v are distinct vertices of degree 1 in LT, then $\overline{u} \neq \overline{v}$.

In the next lemma we do not need to restrict ourselves to trees homeomorphic to H. However, we require that R^- has length 2, and as this length equals $l+2$, we require $l=0$.

Lemma 4.1 *Let T be a tree different from a path, in which the longest ray has length exactly* 2. *Let ΔT be as in Section 2. Then $\Delta T \geq 0$.*

Proof. With the notation as in the beginning of Section 3, we find a lower bound for $\sum_u h(u)$, where $u \in V(LT) \setminus \{a\}$. Consider three cases:

Case 1. $d_u = 1$. Then $d(u,a) > 1$, so that $h(u) = -\frac{3}{2}d(u,a) - 2$ by (4).

Case 2. $d_u = 2$. Since $(d_a - 1)(d_u - 2) = 0$, we have $\phi(u,a) = 0$ also in this case. By (4) we have $h(u) = (2d_a - 2)d(u,a) + (2d_a - \frac{5}{2}) - 2 \geq 0$, as $d_a \geq 2$.

Case 3. $d_u \geq 3$. By (4) we have

$$
\begin{aligned}
h(u) &\geq \Big(d_u\Big((d_u-1)d_a - \tfrac{1}{2}\Big) - 1\Big)d(u,a) \\
&\quad + (d_u - 1)\Big(2d_u\, d_a - 2d_a - d_u - \tfrac{1}{2}\Big) - 2 - 2(d_a - 1)(d_u - 2) \\
&> \tfrac{19}{2}d(u,a) + (d_u - 1)\Big(2d_u\, d_a - 2d_a - d_u - \tfrac{1}{2} - 2d_a + 2\Big) - 2 \\
&\geq \tfrac{19}{2}d(u,a) + (d_u - 1)\Big(d_a(\tfrac{4}{3}d_u - 4) + d_u(\tfrac{2}{3}d_a - 1) + \tfrac{3}{2}\Big) - 2 \\
&\geq \tfrac{19}{2}d(u,a) + (d_u - 1)\tfrac{5}{2} - 2 \\
&\geq \tfrac{19}{2}d(u,a) + 3
\end{aligned}
$$

as $d_u \geq 3$ and $d_a \geq 2$.

Hence,

$$
h(u) \geq \begin{cases} -\frac{3}{2}d(u,a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ \frac{19}{2}d(u,a) + 3 & \text{if } d_u \geq 3. \end{cases}
$$

Since $l = 0$, all rays of T^- have length at most 2, and consequently all rays of LT^- have length at most 1. Hence, if $d_u = 1$ then $d(u, \overline{u}) = 1$ in LT^-. Thus,

$$
h(u) + h(\overline{u}) \geq -\tfrac{3}{2}d(u,a) - 2 + \tfrac{19}{2}d(\overline{u},a) + 3
$$

$$\begin{aligned} &\geq -\tfrac{3}{2}d(\overline{u},a) - \tfrac{7}{2} + \tfrac{19}{2}d(\overline{u},a) + 3 \\ &\geq 8d(\overline{u},a) - \tfrac{1}{2} \\ &\geq 0. \end{aligned} \tag{10}$$

Denote by V_1 the set of vertices of degree 1 in $V(LT^-)$. Notice that $a \notin V_1$. Since $\overline{u} \neq \overline{v}$ whenever $u, v \in V_1$, $u \neq v$, by (10) we have

$$\sum_u h(u) \geq \sum_{u \in V_1} \Big(h(u) + h(\overline{u}) \Big) \geq 0.$$

Finally, since $d_a\left[(d_a - 1)(2d_a - 1) - \frac{1}{2}\right] - 3 \geq 0$ if $d_a \geq 2$, we have

$$\Delta T = \sum_u h(u) + d_a \left((d_a - 1)(2d_a - 1) - \frac{1}{2} \right) - 3 \geq 0,$$

by Proposition 3.1. □

If $l \geq 1$, that is if R^- has length at least 3, then $h(u) < 0$ even if $d_u = 2$. Thus, our estimations need to be more tight in this case. For this reason we concentrate only to trees homeomorphic to H.

Lemma 4.2 *Let $l \geq 1$. Further, let T be a tree homeomorphic to* H, *in which the longest ray has length $l + 2$. Let ΔT be as in Section 2. Then $\Delta T \geq 0$.*

Proof. We use the notation as in Section 3. Since $d_a = 1$, by Proposition 3.1 we have

$$\Delta T = \sum_u h(u) - \frac{7}{2}, \tag{11}$$

where

$$h(u) = \left(d_u \left(d_u - \frac{3}{2} \right) - 1 \right) d(u,a) + (d_u - 1)\left(d_u - \frac{5}{2} \right) - 2,$$

see (4). Hence,

$$h(u) = \begin{cases} -\frac{3}{2}d(u,a) - 2 & \text{if } d_u = 1, \\ -\frac{5}{2} & \text{if } d_u = 2, \\ \frac{7}{2}d(u,a) - 1 & \text{if } d_u = 3, \\ 9d(u,a) + \frac{5}{2} & \text{if } d_u = 4. \end{cases} \tag{12}$$

Observe that LT^- has no vertex of degree greater than 4.

Suppose that there is a ray R' in T^- starting at c'. Denote by c the vertex of LT^- corresponding to the first edge of R', and denote by $R(c)$ the set of vertices of LR'. We find $\sum_u h(u)$ where $u \in R(c)$, $u \neq a$. We distinguish three cases:

Case 1. $c = \overline{a}$. Then $d(c, a) = l$. As $l \geq 1$ and T^- is a tree homeomorphic to H, $R(c)$ has one vertex of degree 3, namely c, and $l-1$ vertices of degree 2. By (12), we have

$$\sum_{u \in R(c) \setminus \{a\}} h(u) = \frac{7}{2} d(c, a) - 1 - (l-1)\frac{5}{2} = \frac{7}{2} l - 1 - \frac{5}{2} l + \frac{5}{2} = l + \frac{3}{2}.$$

Case 2. $c \neq \overline{a}$ *and* $|R(c)| = 1$. Since $R(c)$ has a unique vertex, namely c, the length of R' is 1. Since T^- is homeomorphic to H, the edge of T^- corresponding to c has one endvertex of degree 3 and the other endvertex of degree 1. Thus $d_c = 2$, so that by (12) we have

$$\sum_{u \in R(c)} h(u) = -\frac{5}{2}.$$

Case 3. $c \neq \overline{a}$ *and* $|R(c)| \geq 2$. Since the length of R' is at most $l+2$, the length of ray LR' is at most $l+1$. Since T^- is a tree homeomorphic to H, $R(c)$ has one vertex of degree 3, at most l vertices of degree 2, and one vertex of degree 1. By (12), we have

$$\sum_{u \in R(c)} h(u) \geq \frac{7}{2} d(c, a) - 1 - l\frac{5}{2} - \frac{3}{2}\Big(d(c, a) + l + 1\Big) - 2 = 2d(c, a) - 4l - \frac{9}{2}.$$

Denote $S(c) = \sum_{u \in R(c) \setminus \{a\}} h(u)$. By the previous analysis we have

$$S(c) \geq \begin{cases} l + \frac{3}{2} & \text{if } c = \overline{a}, \\ -\frac{5}{2} & \text{if } c \neq \overline{a} \text{ and } |R(c)| = 1, \\ 2d(c, a) - 4l - \frac{9}{2} & \text{if } c \neq \overline{a} \text{ and } |R(c)| \geq 2. \end{cases} \tag{13}$$

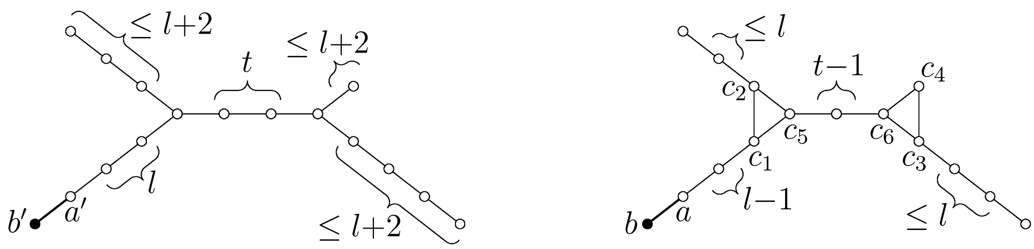

Figure 2: Left: trees T and T^-; right: graphs LT and LT^-.

In Figure 2 we have T, that is, a tree homeomorphic to H, and also its line graph. Denote by P' the path connecting the two vertices of degree 3 in T, and denote by P the line graph of P'. Further, denote by $c_1, c_2, \ldots, c_6$ vertices of LT corresponding to edges incident with vertices of degree 3 in T, see Figure 2. We assume that $c_1 = \overline{a}$,

$c_1c_2 \in E(LT)$ and $c_5, c_6 \in V(P)$. Denote by t the length of P. Then $t \geq 0$, and in the case $t = 0$ we have $c_5 = c_6$. Denote $\mathrm{SP} = \sum_{u \in V(P)} h(u)$. Then

$$\sum_{u \in V(LT^-) \setminus \{a\}} h(u) = \sum_{i=1}^{4} S(c_i) + \mathrm{SP}. \tag{14}$$

We distinguish three cases:

Case 1. $t = 0$. Then $c_5 = c_6$ and the degree of c_5 is 4. Since $l \geq 1$, $d(c_2, a) = d(c_5, a) = l + 1$ and $d(c_3, a) = d(c_4, a) = l + 2$, by (13) and (12) we have

$$\begin{array}{rcl} S(c_1) & \geq & l + \frac{3}{2} \\ S(c_2) & \geq & \min\{-\frac{5}{2}, -2l - \frac{5}{2}\} = -2l - \frac{5}{2} \\ S(c_3) & \geq & \min\{-\frac{5}{2}, -2l - \frac{1}{2}\} = -2l - \frac{1}{2} \\ S(c_4) & \geq & \min\{-\frac{5}{2}, -2l - \frac{1}{2}\} = -2l - \frac{1}{2} \\ \mathrm{SP} & = & 9(l + 1) + \frac{5}{2} = 9l + \frac{23}{2}. \end{array}$$

By (14) we have $\sum_{u \in V(LT^-) \setminus \{a\}} h(u) = 4l + \frac{19}{2}$, and by (11) we conclude $\Delta T \geq 4l + \frac{19}{2} - \frac{7}{2} \geq 0$.

Case 2. $t \geq 1$ *and* $t \geq l - 1$. Then both c_5 and c_6 have degree 3 and P has $t - 1$ vertices of degree 2. Since $l \geq 1$, $d(c_2, a) = d(c_5, a) = l + 1$, $d(c_6, a) = l + t + 1$ and $d(c_3, a) = d(c_4, a) = l + t + 2$, by (13) and (12) we have

$$\begin{array}{rcl} S(c_1) & \geq & l + \frac{3}{2} \\ S(c_2) & \geq & \min\{-\frac{5}{2}, -2l - \frac{5}{2}\} = -2l - \frac{5}{2} \\ S(c_3) & \geq & \min\{-\frac{5}{2}, 2t - 2l - \frac{1}{2}\} \geq \min\{-\frac{5}{2}, 2(l-1) - 2l - \frac{1}{2}\} = -\frac{5}{2} \\ S(c_4) & \geq & \min\{-\frac{5}{2}, 2t - 2l - \frac{1}{2}\} \geq \min\{-\frac{5}{2}, 2(l-1) - 2l - \frac{1}{2}\} = -\frac{5}{2} \\ \mathrm{SP} & = & \frac{7}{2}(l + 1) - 1 - \frac{5}{2}(t - 1) + \frac{7}{2}(l + t + 1) - 1 = 7l + t + \frac{15}{2}. \end{array}$$

By (14) we have $\sum_{u \in V(LT^-) \setminus \{a\}} h(u) = 6l + t + \frac{3}{2}$, and by (11) we conclude $\Delta T \geq 6l + t + \frac{3}{2} - \frac{7}{2} \geq 0$.

Case 3. $t \geq 1$ *and* $t \leq l - 2$. Then again, both c_5 and c_6 have degree 3 and P has $t - 1$ vertices of degree 2, $d(c_2, a) = d(c_5, a) = l + 1$, $d(c_6, a) = l + t + 1$ and $d(c_3, a) = d(c_4, a) = l + t + 2$. Since $l \geq 1$, by (13) and (12) we have

$$\begin{array}{rcl} S(c_1) & \geq & l + \frac{3}{2} \\ S(c_2) & \geq & \min\{-\frac{5}{2}, -2l - \frac{5}{2}\} = -2l - \frac{5}{2} \\ S(c_3) & \geq & \min\{-\frac{5}{2}, 2t - 2l - \frac{1}{2}\} = 2t - 2l - \frac{1}{2} \\ S(c_4) & \geq & \min\{-\frac{5}{2}, 2t - 2l - \frac{1}{2}\} = 2t - 2l - \frac{1}{2} \\ \mathrm{SP} & = & 7l + t + \frac{15}{2}. \end{array}$$

By (14) we have $\sum_{u \in V(LT^-)\setminus\{a\}} h(u) = 2l + 5t + \frac{11}{2}$, and by (11) we conclude $\Delta T \geq 2l + 5t + \frac{11}{2} - \frac{7}{2} \geq 0$.

Hence, in every case $\Delta T \geq 0$ as required. □

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