Complete solution of equation $W(L^3(T)) = W(T)$ for Wiener index of iterated line graphs of trees

M. Knor*, M. Mačaj†, P. Potočnik‡, R. Škrekovski§

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Abstract

Let G be a graph. Denote by $L^i(G)$ its i-iterated line graph and denote by $W(G)$ its Wiener index. In [14] we show that there is an infinite class $\mathcal{T}$ of trees T satisfying $W(L^3(T)) = W(T)$, which disproves a conjecture of Dobrynin and Entringer. In this paper we prove that except of the trees of $\mathcal{T}$, there is no non-trivial tree T satisfying $W(L^3(T)) = W(T)$. Consequently, for a tree T and $i \geq 3$, the equation $W(L^i(T)) = W(T)$ holds if and only if $T \in \mathcal{T}$ and $i = 3$.

Keywords: Wiener index, tree, iterated line graph.

1 Introduction

Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For any two vertices u, v let $d(u, v)$ be the distance from u to v. The *Wiener index* of G, $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken over unordered pairs of vertices of G. The Wiener index was introduced by Wiener in [22]. Since it is related to several properties of chemical

*Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68, Bratislava, Slovakia, knor@math.sk.

†Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská Dolina, 842 15 Bratislava, Slovakia, martin.macaj@fmph.uniba.sk.

‡Faculty of Mathematics and Physics, University of Ljubljana; also affiliated with IAM, University of Primorska, Koper, Slovenia, primoz.potocnik@fmf.uni-lj.si.

§Faculty of Information Studies, 8000 Novo Mesto & Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana & FAMNIT, University of Primorska, 6000 Koper, Slovenia, skrekovski@gmail.com.

molecules (see [13]), it is widely studied by chemists. The interest of mathematicians was attracted in 1970's, when it was reintroduced as *the transmission* and *the distance of a graph*; see [21] and [9], respectively. Recently, several special issues of journals were devoted to (mathematical properties) of the Wiener index (see [11] and [12]). For surveys and some up-to-date papers related to the Wiener index of trees and line graphs see [5, 6], [8, 19, 20, 24] and [2, 3, 7, 10, 23], respectively.

By the definition, if G has a unique vertex, then $W(G) = 0$. In this case, we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices of G is empty.

The line graph of G, $L(G)$, has vertex set identical with the set of edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

The Wiener index of the line graph of a tree T can easily be computed from $W(T)$ by using the following result from [1]:

Theorem 1.1 *Let T be a tree on n vertices. Then $W(L(T)) = W(T) - \binom{n}{2}$.*

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see e.g. [4]. In [5], the following problem was posed:

Problem 1.2 *Is there any tree T satisfying equality $W(L^i(T)) = W(T)$ for some $i \geq 3$?*

As observed above, if T is a trivial tree, then $W(L^i(T)) = W(T)$ for every $i \geq 1$, although here the graph $L^i(T)$ is empty. The real question is, if there is a nontrivial tree T and $i \geq 3$ such that $W(L^i(T)) = W(T)$.

In papers [15, 16, 17, 18] (see [18, Corollary 1.4]) we solved Problem 1.2 for $i \geq 4$:

Theorem 1.3 *Let T be a tree and $i \geq 4$. Then we have*

$$\begin{array}{ll} W(L^i(T)) = W(T) & \textit{if } T \textit{ is trivial,} \\ W(L^i(T)) < W(T) & \textit{if } T \textit{ is a nontrivial path or the claw } K_{1,3}, \\ W(L^i(T)) > W(T) & \textit{otherwise.} \end{array}$$

In this paper we consider Problem 1.2 for the remaining case $i = 3$. Let H_0 be the tree on six vertices, out of which two have degree 3 and four have degree 1. In [16, Corollary 1.6], we proved:

Theorem 1.4 *Let T be a tree which is not homeomorphic to a path, claw $K_{1,3}$ or H_0, and let $i \geq 3$. Then $W(L^i(T)) > W(T)$.*

(Recall that two graphs G_1 and G_2 are homeomorphic if and only if there is a third graph H, such that both G_1 and G_2 can be obtained from H by means of edge subdivision.)

By Theorem 1.4, to solve Problem 1.2 for $i = 3$, it suffices to consider paths and trees homeomorphic to the claw $K_{1,3}$ and H_0.

First, let us concentrate to paths. Denote by P_n a path on n vertices. If $n \geq 2$, then $W(P_n) > W(P_{n-1})$, since P_{n-1} is a subgraph embedded isometrically in P_n. Since $L(P_n) = P_{n-1}$ if $n \geq 2$, while $L(P_1)$ is an empty graph, we have $W(L^i(P_n)) \neq W(P_n)$ for every $i \geq 1$ if P_n is a nontrivial path.

Similarly, there is no solution of Problem 1.2 among trees homeomorphic to the claw $K_{1,3}$; namely, in Section 3 we prove the following:

Theorem 1.5 *Let T be a tree homeomorphic to $K_{1,3}$. Then $W(L^3(T)) \neq W(T)$.*

However, there is a non-trivial solution of Problem 1.2 among trees homeomorphic to H_0. Denote by $H_{a,b,c,d,e}$ a specific tree homeomorphic H_0, defined as follows: In $H_{a,b,c,d,e}$, the two vertices of degree 3 are joined by a path of length $e+1$, $e \geq 0$. Hence, this path has e vertices of degree 2. Further, at one vertex of degree 3 there start two pendant paths of lengths a and b, where $a, b \geq 1$, and at the other vertex of degree 3 there start another two pendant paths of lengths c and d, where $c, d \geq 1$. Thus $H_{a,b,c,d,e}$ has $a + b + c + d + e + 2$ vertices (see Figure 1 for $H_{3,3,4,2,2}$). By symmetry, we may assume that $a \geq b$, $c \geq d$ and $b \geq d$. That is, we assume that the shortest pendant path in $H_{a,b,c,d,e}$ has length d.

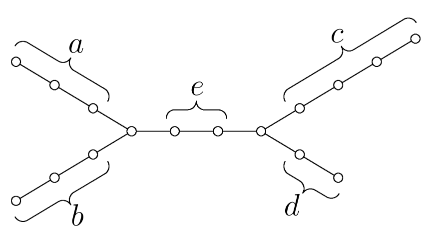

Figure 1: The graph $H_{a,b,c,d,e}$.

In Section 3, we prove the following:

Theorem 1.6 *The equation $W(L^3(H_{a,b,c,d,e})) = W(H_{a,b,c,d,e})$ holds if and only if $d = e = 1$ and there are $i, j \in \mathbb{Z}$, $i \geq j$, such that*

$$\begin{aligned} a &= 128 + 3i^2 + 3j^2 - 3ij + i \\ b &= 128 + 3i^2 + 3j^2 - 3ij + j \\ c &= 128 + 3i^2 + 3j^2 - 3ij + i + j. \end{aligned} \tag{1}$$

We remark that the "if" part of Theorem 1.6 was already proved in [14]. The smallest tree satisfying (1) is $H_{128,128,128,1,1}$ on 388 vertices obtained when $i = j = 0$.

We can summarize our results regarding Problem 1.2 in the following theorem:

Theorem 1.7 *Let T be a tree and $i \geq 3$. Then we have*

(i) $W(L^i(T)) = W(T)$ if T is trivial or $i = 3$ and T is $H_{a,b,c,1,1}$, where a, b, c satisfy (1);

(ii) $W(L^i(T)) \neq W(T)$ if $i = 3$ and T is homeomorphic to $K_{1,3}$ or H_0 with the exception of trees mentioned in (i);

(iii) $W(L^i(T)) < W(T)$ if T is a nontrivial path or the claw $K_{1,3}$;

(iv) $W(L^i(T)) > W(T)$ otherwise.

It is obvious that trees mentioned in (ii) either satisfy $W(L^3(T)) < W(T)$ or $W(L^3(T)) > W(T)$. If $T \neq K_{1,3}$, in some cases we prove $W(L^3(T)) > W(T)$, but in the others using congruences we can only show $W(L^3(T)) \neq W(T)$, see below.

In the next section we present a lemma, with the help of which we prove Theorems 1.5 and 1.6 in Sections 3 and 4, respectively.

2 Preliminaries

A degree of a vertex, say v, is denoted by $\deg(v)$, or when convenient, by d_v. Analogously as a vertex of $L(G)$ corresponds to an edge of G, a vertex of $L^2(G)$ corresponds to a path of length two in G. For $x \in V(L^2(G))$ we denote the corresponding path by $B_2(x)$. For two subgraphs S_1 and S_2 of G, the shortest distance in G between a vertex of S_1 and a vertex of S_2 is denoted by $d(S_1, S_2)$. If S_1 and S_2 share an edge, then we set $d(S_1, S_2) = -1$.

Let x and y be two vertices of $L^2(G)$, such that u is the center of $B_2(x)$, the vertex v is the center of $B_2(y)$ and $u \neq v$. Then

$$d_{L^2(G)}(x, y) = d(B_2(x), B_2(y)) + 2.$$

Let $u, v \in V(G)$, $u \neq v$. Let $\beta_i(u, v)$ denote the number of pairs $x, y \in V(L^2(G))$, with u being the center of $B_2(x)$ and v being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u, v) - 2 + i$. Since $d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v)$, we have $\beta_i(u, v) = 0$ for all $i \notin \{0, 1, 2\}$. Moreover, $\sum_{i=0}^{2} \beta_i(u, v) = \binom{\mathrm{d}_u}{2}\binom{\mathrm{d}_v}{2}$.

Let

$$h(u, v) = \left(\binom{\mathrm{d}_u}{2}\binom{\mathrm{d}_v}{2} - 1\right) d(u, v) + \beta_1(u, v) + 2\beta_2(u, v). \tag{2}$$

In [14, Lemma 2.2] we have the following statement:

Lemma 2.1 *Let G be a connected graph. Then*

$$W(L^2(G)) - W(G) = \sum_{u \neq v} h(u,v) + \sum_{u} \left[3\binom{\mathrm{d}_u}{3} + 6\binom{\mathrm{d}_u}{4} \right],$$

where the first sum is taken over unordered pairs of vertices $u, v \in V(G)$, such that either $\mathrm{d}_u \neq 2$ or $\mathrm{d}_v \neq 2$, and the second one is taken over $u \in V(G)$.

Observe that $W(P_n) = \Big((n-1) + \ldots + 1\Big) + \Big((n-2) + \ldots + 1\Big) + \ldots + 1 = \binom{n+1}{3}$. Using this fact, one can show that $W(H_{a,b,c,d,e})$ is a polynomial of third degree in a, b, c, d and e, and so is also $W(L^3(H_{a,b,c,d,e}))$ (the situation with the claw being similar). However, if we calculate $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e})$ with the help of Lemma 2.1, we obtain a polynomial the degree of which is at most 2, since the pairs of vertices u, v with $\mathrm{d}_u = \mathrm{d}_v = 2$ do not contribute to $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e})$ (for detailed calculation see the proofs below).

3 Proof of Theorem 1.5

Proof of Theorem 1.5. Let $C_{a,b,c}$ be a tree homeomorphic to the claw $K_{1,3}$ in which the paths connecting the vertices of degree 1 with the vertex of degree 3 have lengths a, b and c, where $a \geq b \geq c \geq 1$. The tree $C_{a,b,c}$ has exactly $a+b+c+1$ vertices, see Figure 2 for $C_{4,3,2}$.

We prove Theorem 1.5 by counting the distances in $L(C_{a,b,c})$ instead of in $C_{a,b,c}$ and $L^3(C_{a,b,c})$. In $L(C_{a,b,c})$ we distinguish 6 vertices x_1, x_2, x_3, y_1, y_2 and y_3. The vertices x_1, x_2 and x_3 correspond to the pendant edges of $C_{a,b,c}$, while the vertices y_1, y_2 and y_3 correspond to the edges incident with the vertex of degree 3 in $C_{a,b,c}$, see Figure 2 for $L(C_{4,3,2})$. Observe that if $c = 1$ ($b = 1$ or $a = 1$), then $x_3 = y_3$ ($x_2 = y_2$ or $x_1 = y_1$), and in such a case, $\deg(x_3) = 2$ ($\deg(x_2) = 2$ or $\deg(x_1) = 2$, respectively).

In what follows, the graph $L(C_{a,b,c})$ is denoted by LC. Further, for $i \in \{1,2,3\}$, let V_i be the set of vertices of $V(LC)$ of degree i. For $x \in V_1$ and $y \in V_3$, define

$$\begin{aligned}
S^1(x) &= \sum_u h(u,x) && \text{where } u \in V(LC) \setminus V_1,\\
M^1 &= \sum_{u \neq v} h(u,v) && \text{where } u, v \in V_1,\\
S^3(y) &= \sum_u h(u,y) && \text{where } u \in V_2,\\
M^3 &= \sum_{u \neq v} h(u,v) && \text{where } u, v \in V_3,\\
D &= \sum_u \left[3\tbinom{u}{3} + 6\tbinom{u}{4}\right] && \text{where } u \in V_3.
\end{aligned}$$

Observe that $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3} S^3(y) + M^3$ sums $h(u,v)$ for all pairs $\{u, v\}$ of vertices such that either $\deg(u) \neq 2$ or $\deg(v) \neq 2$.

Denote $P = W(L^3(C_{a,b,c})) - W(C_{a,b,c})$. Since $C_{a,b,c}$ has $a+b+c+1$ vertices, we have $W(C_{a,b,c}) = W(LC) + \binom{a+b+c+1}{2}$, by Theorem 1.1. Thus, by Lemma 2.1, we have

$$\begin{aligned} P &= W(L^2(LC)) - W(LC) - \tbinom{a+b+c+1}{2} \\ &= \sum_{x\in V_1} S^1(x) + M^1 + \sum_{y\in V_3} S^3(y) + M^3 + D - \tbinom{a+b+c+1}{2}. \end{aligned} \tag{3}$$

This naturally splits the problem into four cases according to the size of V_1. In each of these cases we evaluate S^1's, M^1, S^3's, M^3 and D, and we solve the equation $P = 0$. To avoid fractions, in some cases we solve the equation $2P = 0$ instead of $P = 0$.

Figure 2: The graphs $C_{a,b,c}$ and $L(C_{a,b,c}) = LC$.

Case 1. $a, b, c \geq 2$, that is, $|V_1| = 3$.

We start with evaluating $S^1(x)$, where $x \in V_1$. Since $\deg(x) = 1$, we have $\beta_j(u, x) = 0$, $0 \leq j \leq 2$. Hence, $h(u, x) = -d(u, x)$, see (2). The sum of distances from x_1 to all interior vertices of $x_1 - x_2$ path is $1+2+\ldots+(a+b-2) = \binom{a+b-1}{2}$ (see Figure 2). The sum of distances from x_1 to all interior vertices of $x_1 - x_3$ path, not included in the previous calculation, is $a + (a+1) + \ldots + (a+c-2) = \binom{a+c-1}{2} - \binom{a}{2}$. In this way we get $S^1(x_1)$ and analogously we calculate $S^1(x_2)$ and $S^1(x_3)$:

$$\begin{aligned} S^1(x_1) &= -\tbinom{a+b-1}{2} - \tbinom{a+c-1}{2} + \tbinom{a}{2}, \\ S^1(x_2) &= -\tbinom{a+b-1}{2} - \tbinom{b+c-1}{2} + \tbinom{b}{2}, \\ S^1(x_3) &= -\tbinom{a+c-1}{2} - \tbinom{b+c-1}{2} + \tbinom{c}{2}. \end{aligned}$$

Now $h(x_1, x_2) = -(a+b-1)$. Using the symmetry we obtain

$$M^1 = -(a+b-1) - (a+c-1) - (b+c-1).$$

In $S^3(y)$ we sum $h(u, y)$, where $\deg(u) = 2$ and $\deg(y) = 3$. Hence, $\binom{\mathrm{d}_u}{2}\binom{\mathrm{d}_y}{2} - 1 = 2$. Since $\beta_0(u, y) = 2$, $\beta_1(u, y) = 1$ and $\beta_2(u, y) = 0$, we have $h(u, y) = 2d(u, y) + 1$. Thus, the sum of $h(u, y_1)$'s for interior vertices u of $y_1 - x_1$ path is $2(1 + 2 + \ldots + (a-2)) + (a-2) = 2\binom{a-1}{2} + (a-2)$ (see Figure 2). Analogously, the sum of $h(u, y_1)$'s for interior vertices of $y_2{-}x_2$ path is $2(2{+}3{+}\ldots{+}(b{-}1)){+}(b{-}2) = 2\binom{b}{2}{-}2{+}(b{-}2) = 2\binom{b}{2} + (b{-}4)$. In this way we get

$$\begin{aligned}
S^3(y_1) &= 2\tbinom{a-1}{2} + (a{-}2) + 2\tbinom{b}{2} + (b{-}4) + 2\tbinom{c}{2} + (c{-}4),\\
S^3(y_2) &= 2\tbinom{a}{2} + (a{-}4) + 2\tbinom{b-1}{2} + (b{-}2) + 2\tbinom{c}{2} + (c{-}4),\\
S^3(y_3) &= 2\tbinom{a}{2} + (a{-}4) + 2\tbinom{b}{2} + (b{-}4) + 2\tbinom{c-1}{2} + (c{-}2).
\end{aligned}$$

Consider $h(y_1, y_2)$. Here $\binom{\mathrm{d}_{y_1}}{2}\binom{\mathrm{d}_{y_2}}{2} - 1 = 8$, $\beta_0(y_1, y_2) = 4$, $\beta_1(y_1, y_2) = 5$ and $\beta_2(y_1, y_2) = 0$ (see Figure 2). This means that $h(y_1, y_2) = 8{+}5 = 13$, and analogously also $h(y_1, y_3) = 13$ and $h(y_2, y_3) = 13$. Hence

$$M^3 = 13 + 13 + 13.$$

Finally, since LC has exactly three vertices of degree 3 and no vertex of higher degree, we have

$$D = \sum_u \left[3\tbinom{\mathrm{d}_u}{3} + 6\tbinom{\mathrm{d}_u}{4}\right] = 3\left[3\tbinom{3}{3}\right] = 9.$$

By (3), expanding the terms (using a computer package, for instance), we get

$$\begin{aligned}
P &= (a^2{+}b^2{+}c^2) - 3(ab{+}ac{+}bc) + (a{+}b{+}c) + 21\\
&= (a{+}b{+}c)^2 - 5(ab{+}ac{+}bc) + (a{+}b{+}c) + 21.
\end{aligned}$$

Now substitute $x = (a{+}b{+}c)$ and consider the equation $P = 0$ over $\mathbb{Z}_5$. We get

$$x^2 + x + 1 = 0,$$

which has no solution in $\mathbb{Z}_5$. Consequently, $P = 0$ has no integer solution and $W(L^3(C_{a,b,c})) - W(C_{a,b,c}) \neq 0$ in this case.

Case 2. $a, b \geq 2$, $c = 1$, that is, $|V_1| = 2$.

In this case the vertex $x_3 = y_3$ has degree 2, so we do not need to find $S^1(x_3)$ and $S^3(y_3)$, see (3), but we must include the distances to x_3 in $S^1(x_1)$, $S^1(x_2)$, $S^3(y_1)$ and $S^3(y_2)$. Analogously as in the previous case we have

$$\begin{aligned}
S^1(x_1) &= -\tbinom{a+b-1}{2} - a,\\
S^1(x_2) &= -\tbinom{a+b-1}{2} - b,\\
M^1 &= -(a{+}b{-}1),\\
S^3(y_1) &= 2\tbinom{a-1}{2} + (a{-}2) + 2\tbinom{b}{2} + (b{-}4) + 2 + 1,\\
S^3(y_2) &= 2\tbinom{a}{2} + (a{-}4) + 2\tbinom{b-1}{2} + (b{-}2) + 2 + 1,\\
M^3 &= 13,\\
D &= 2 \cdot 3\tbinom{3}{3} = 6.
\end{aligned}$$

By (3), expanding the terms we get

$$\begin{aligned} 2P &= (a^2+b^2) - 6ab - 5(a+b) + 30 \\ &= (a+b)^2 - 8ab - 5(a+b) + 30. \end{aligned} \tag{4}$$

Now consider the equation $2P = 0$ over $\mathbb{Z}_5$. We get $(a'+b')^2 + 2a'b' = 0$. It is a matter of routine to check that the only solution in $\mathbb{Z}_5$ is $a' = b' = 0$. Hence, in (4) we have $25 \mid (a+b)^2$, $25 \mid 8ab$ and $25 \mid 5(a+b)$. Since $25 \nmid 30$, (4) has no integer solution. Thus, $P = 0$ has no solution also in this case.

Case 3. $a \geq 2$, $b = c = 1$, that is, $|V_1| = 1$.

The vertices $x_2 = y_2$ and $x_3 = y_3$ have degree 2, so we do not need to find $S^1(x_2)$, $S^1(x_3)$, $S^3(y_2)$ and $S^3(y_3)$. We have

$$\begin{aligned} S^1(x_1) &= -\tbinom{a}{2} - a - a, \\ M^1 &= 0, \\ S^3(y_1) &= 2\tbinom{a-1}{2} + (a-2) + 2 + 1 + 2 + 1, \\ M^3 &= 0, \\ D &= 3\tbinom{3}{3} = 3. \end{aligned}$$

By (3), expanding the terms we get

$$P = -6a + 6 < 0$$

as $a \geq 2$. Thus, $P = 0$ has no solution in this case.

Case 4. $a = b = c = 1$, that is, $|V_1| = 0$.

In this case $C_{a,b,c} = K_{1,3}$ and $L^i(K_{1,3})$ is a cycle of length 3 for every $i \geq 1$. Since $W(G) = 3$ if G is a cycle of length 3, while $W(K_{1,3}) = 9$, we have $W(L^3(C_{1,1,1})) - W(C_{1,1,1}) \neq 0$ also in this case. □

4 Proof of Theorem 1.6

Proof of Theorem 1.6. We proceed analogously as in the proof of Theorem 1.5. That is, we prove Theorem 1.6 by counting the distances in $L(H_{a,b,c,d,e})$ instead of those in $H_{a,b,c,d,e}$ and $L^3(H_{a,b,c,d,e})$. In $L(H_{a,b,c,d,e})$ we distinguish 10 vertices $x_1, x_2, \ldots, x_4$ and $y_1, y_2, \ldots, y_6$. The vertices $x_1, \ldots, x_4$ correspond to pendant edges of $H_{a,b,c,d,e}$, while the vertices $y_1, \ldots, y_6$ correspond to edges incident with vertices of degree 3 in $H_{a,b,c,d,e}$ (see Figure 3). Observe that if $e = 0$, then $y_5 = y_6$ and $\deg(y_5) = 4$. If $d = 1$ ($c = 1$, $b = 1$ or $a = 1$), then $x_4 = y_4$ ($x_3 = y_3$, $x_2 = y_2$ or $x_1 = y_1$), and in such a case $\deg(x_4) = 2$ ($\deg(x_3) = 2$, $\deg(x_2) = 2$ or $\deg(x_1) = 2$, respectively).

In what follows, the graph $L(H_{a,b,c,d,e})$ is denoted by LH. Further, for $i \in \{1,2,3,4\}$, let V_i be the set of vertices of $V(LH)$ of degree i. For $x \in V_1$ and $y \in V_3 \cup V_4$, define

$$\begin{aligned}
S^1(x) &= \sum_u h(u,x) && \text{where } u \in V(LH) \setminus V_1,\\
M^1 &= \sum_{u \neq v} h(u,v) && \text{where } u,v \in V_1,\\
S^3(y) &= \sum_u h(u,y) && \text{where } u \in V_2,\\
M^3 &= \sum_{u \neq v} h(u,v) && \text{where } u,v \in V_3 \cup V_4,\\
D &= \sum_u \left[3\tbinom{u}{3} + 6\tbinom{u}{4}\right] && \text{where } u \in V_3 \cup V_4.
\end{aligned}$$

Observe that once again, $\sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3$ sums $h(u,v)$ for all pairs $\{u,v\}$ of vertices such that either $\deg(u) \neq 2$ or $\deg(v) \neq 2$.

Denote $P = W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e})$. Since $H_{a,b,c,d,e}$ has $a+b+c+d+e+2$ vertices, we have $W(H_{a,b,c,d,e}) = W(LH) + \binom{a+b+c+d+e+2}{2}$, by Theorem 1.1. Thus, by Lemma 2.1, we have

$$\begin{aligned}
P &= W(L^2(LH)) - W(LH) - \tbinom{a+b+c+d+e+2}{2}\\
&= \sum_{x \in V_1} S^1(x) + M^1 + \sum_{y \in V_3 \cup V_4} S^3(y) + M^3 + D - \tbinom{a+b+c++d+e+2}{2}. \qquad (5)
\end{aligned}$$

If $e = 0$, then we have one vertex of degree 4 in LH, while if $e \geq 1$, then the greatest degree of a vertex in LH is 3. By symmetry, we distinguish eleven cases. In the first five cases we have $e \geq 1$ and in the next five we have $e = 0$. In each of these first ten cases (the last case will be solved in a different way) we evaluate S^1's, M^1, S^3's, M^3 and D and we solve the equation $P = 0$. To avoid fractions, in some cases we solve the equation $2P = 0$.

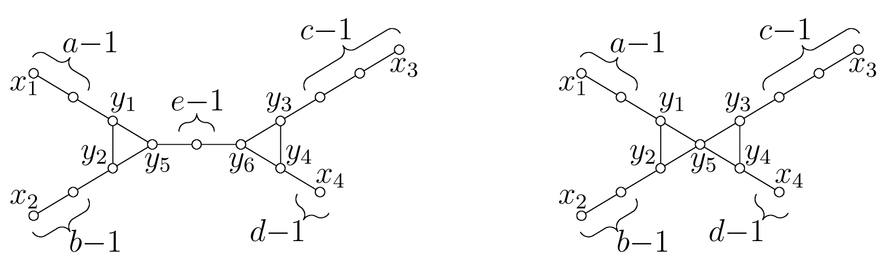

Figure 3: The graph $LH = L(H_{a,b,c,d,e})$ for $e \geq 1$ and $e = 0$.

Case 1. $a,b,c,d \geq 2$, $e \geq 1$.

We start with evaluating $S^1(x)$, where $x \in V_1$. Since $\deg(x) = 1$, we have $\beta_j(u,x) = 0$, $0 \le j \le 2$. Hence, $h(u,x) = -d(u,x)$. The sum of distances from x_1 to all interior vertices of $x_1 - x_2$ path is $1+2+\ldots+(a+b-2) = \binom{a+b-1}{2}$ (see Figure 3). The sum of distances from x_1 to all interior vertices of $x_1 - x_3$ path, not included in the previous calculation, is $\binom{a+e+c}{2} - \binom{a}{2}$. Finally, the sum of distances from x_1 to all interior vertices of $x_1 - x_4$ path, not included previously, is $\binom{a+e+d}{2} - \binom{a+e+1}{2}$. In this way we get $S^1(x_1)$ and analogously we calculate $S^1(x_2)$, $S^1(x_3)$ and $S^1(x_4)$:

$$\begin{aligned}
S^1(x_1) &= -\tbinom{a+b-1}{2} - \tbinom{a+e+c}{2} + \tbinom{a}{2} - \tbinom{a+e+d}{2} + \tbinom{a+e+1}{2},\\
S^1(x_2) &= -\tbinom{a+b-1}{2} - \tbinom{b+e+c}{2} + \tbinom{b}{2} - \tbinom{b+e+d}{2} + \tbinom{b+e+1}{2},\\
S^1(x_3) &= -\tbinom{a+e+c}{2} - \tbinom{b+e+c}{2} + \tbinom{e+c+1}{2} - \tbinom{c+d-1}{2} + \tbinom{c}{2},\\
S^1(x_4) &= -\tbinom{a+e+d}{2} - \tbinom{b+e+d}{2} + \tbinom{e+d+1}{2} - \tbinom{c+d-1}{2} + \tbinom{d}{2}.
\end{aligned}$$

Now $h(x_1,x_2) = -(a+b-1)$ and $h(x_1,x_3) = -(a+e+c)$. Using the symmetry we obtain

$$M^1 = -(a+b-1) - (a+e+c) - (a+e+d) - (b+e+c) - (b+e+d) - (c+d-1).$$

In $S^3(y)$ we sum $h(u,y)$, where $\deg(u) = 2$ and $\deg(y) = 3$. Hence, $\binom{\mathrm{d}_u}{2}\binom{\mathrm{d}_y}{2} - 1 = 2$. Since $\beta_0(u,y) = 2$, $\beta_1(u,y) = 1$ and $\beta_2(u,y) = 0$, we have $h(u,y) = 2d(u,y) + 1$. Thus, the sum of $h(u,y_1)$ for interior vertices u of $y_1 - x_1$ path is $2(1+2+\ldots+(a-2)) + (a-2) = 2\binom{a-1}{2} + (a-2)$ (see Figure 3). Analogously, the sum of $h(u,y_1)$ for interior vertices of $y_2 - x_2$ path is $2(2+3+\ldots+(b-1)) + (b-2) = 2\binom{b}{2} + (b-4)$; the sum of $h(u,y_1)$ for interior vertices of $y_5 - y_6$ path is $2(2+3+\ldots+e) + (e-1) = 2\binom{e+1}{2} + (e-3)$; and the sum of $h(u,y_1)$ for interior vertices of $y_3 - x_3$ path is $2((e+3)+(e+4)+\ldots+(e+c)) + (c-2) = 2\binom{e+c+1}{2} - 2\binom{e+3}{2} + (c-2)$. In this way we get

$$\begin{aligned}
S^3(y_1) &= 2\tbinom{a-1}{2} + (a-2) + 2\tbinom{b}{2} + (b-4) + 2\tbinom{e+1}{2} + (e-3)\\
&\quad + 2\tbinom{e+c+1}{2} - 2\tbinom{e+3}{2} + (c-2) + 2\tbinom{e+d+1}{2} - 2\tbinom{e+3}{2} + (d-2),\\
S^3(y_2) &= 2\tbinom{a}{2} + (a-4) + 2\tbinom{b-1}{2} + (b-2) + 2\tbinom{e+1}{2} + (e-3)\\
&\quad + 2\tbinom{e+c+1}{2} - 2\tbinom{e+3}{2} + (c-2) + 2\tbinom{e+d+1}{2} - 2\tbinom{e+3}{2} + (d-2),\\
S^3(y_3) &= 2\tbinom{a+e+1}{2} - 2\tbinom{e+3}{2} + (a-2) + 2\tbinom{b+e+1}{2} - 2\tbinom{e+3}{2} + (b-2)\\
&\quad + 2\tbinom{e+1}{2} + (e-3) + 2\tbinom{c-1}{2} + (c-2) + 2\tbinom{d}{2} + (d-4),\\
S^3(y_4) &= 2\tbinom{a+e+1}{2} - 2\tbinom{e+3}{2} + (a-2) + 2\tbinom{b+e+1}{2} - 2\tbinom{e+3}{2} + (b-2)\\
&\quad + 2\tbinom{e+1}{2} + (e-3) + 2\tbinom{c}{2} + (c-4) + 2\tbinom{d-1}{2} + (d-2),\\
S^3(y_5) &= 2\tbinom{a}{2} + (a-4) + 2\tbinom{b}{2} + (b-4) + 2\tbinom{e}{2} + (e-1)\\
&\quad + 2\tbinom{e+c}{2} - 2\tbinom{e+2}{2} + (c-2) + 2\tbinom{e+d}{2} - 2\tbinom{e+2}{2} + (d-2),\\
S^3(y_6) &= 2\tbinom{a+e}{2} - 2\tbinom{e+2}{2} + (a-2) + 2\tbinom{b+e}{2} - 2\tbinom{e+2}{2} + (b-2)\\
&\quad + 2\tbinom{e}{2} + (e-1) + 2\tbinom{c}{2} + (c-4) + 2\tbinom{d}{2} + (d-4).
\end{aligned}$$

Consider $h(y_i, y_j)$, where $1 \le i < j \le 6$. Here $\binom{\mathrm{d}_{y_i}}{2}\binom{\mathrm{d}_{y_j}}{2} - 1 = 8$ and $\beta_0(y_i, y_j) = 4$. If y_i and y_j lie in a common triangle, then $\beta_1(y_i, y_j) = 5$ and $\beta_2(y_i, y_j) = 0$, while if y_i and y_j do not lie in a common triangle, then $\beta_1(y_i, y_j) = 4$ and $\beta_2(y_i, y_j) = 1$. This means that $h(y_1, y_2) = 13$, $h(y_1, y_3) = 8(e+2)+6 = 8e+22$, $h(y_1, y_6) = 8(e+1)+6 = 8e+14$ and $h(y_5, y_6) = 8e+6$. Hence

$$\begin{aligned} M^3 &= \Big(13 + (8e{+}22) + (8e{+}22) + 13 + (8e{+}14)\Big) + \Big((8e{+}22) + (8e{+}22) \\ &\quad + 13 + (8e{+}14)\Big) + \Big(13 + (8e{+}14) + 13\Big) + \Big((8e{+}14) + 13\Big) + \Big((8e{+}6)\Big). \end{aligned}$$

Finally,

$$D = \sum_u \Big[3\tbinom{\mathrm{d}_u}{3} + 6\tbinom{\mathrm{d}_u}{4}\Big] = 6\Big[3\tbinom{3}{3}\Big] = 18.$$

By (5), expanding the terms (using a computer package, for instance), we get

$$\begin{aligned} 2P &= 7(a^2{+}b^2{+}c^2{+}d^2{+}e^2) - 6(ab{+}ac{+}ad{+}bc{+}bd{+}cd) + 4(ae{+}be{+}ce{+}de) \\ &\quad + 5(a{+}b{+}c{+}d) + 65e + 234 \\ &= 7(a{+}b{+}c{+}d{+}e)^2 - 20(ab{+}ac{+}ad{+}bc{+}bd{+}cd) - 10(ae{+}be{+}ce{+}de) \\ &\quad + 5(a{+}b{+}c{+}d) + 65e + 234. \end{aligned}$$

Now substitute $x = (a{+}b{+}c{+}d{+}e)$ and consider the equation $2P = 0$ over $\mathbb{Z}_5$. We get

$$2x^2 + 4 = 0,$$

which has no solution in $\mathbb{Z}_5$. Consequently, $P = 0$ has no integer solution and $W(L^3(H_{a,b,c,d,e})) - W(H_{a,b,c,d,e}) \neq 0$ in this case.

Case 2. $a, b, c \ge 2$, $d = 1$, $e \ge 1$.

In this case the vertex $x_4 = y_4$ has degree 2, so we do not need to find $S^1(x_4)$ and $S^3(y_4)$, but we must include the distances to x_4 in $S^1(x_1)$, $S^1(x_2)$, $S^1(x_3)$, $S^3(y_1)$, $S^3(y_2)$, $S^3(y_3)$, $S^3(y_5)$ and $S^3(y_6)$. Analogously as in the previous case we have

$$\begin{aligned} S^1(x_1) &= -\tbinom{a+b-1}{2} - \tbinom{a+e+c}{2} + \tbinom{a}{2} - (a{+}e{+}1), \\ S^1(x_2) &= -\tbinom{a+b-1}{2} - \tbinom{b+e+c}{2} + \tbinom{b}{2} - (b{+}e{+}1), \\ S^1(x_3) &= -\tbinom{a+e+c}{2} - \tbinom{b+e+c}{2} + \tbinom{e+c+1}{2} - c, \\ M^1 &= -(a{+}b{-}1) - (a{+}e{+}c) - (b{+}e{+}c), \\ S^3(y_1) &= 2\tbinom{a-1}{2} + (a{-}2) + 2\tbinom{b}{2} + (b{-}4) + 2\tbinom{e+1}{2} + (e{-}3) \\ &\quad + 2\tbinom{e+c+1}{2} - 2\tbinom{e+3}{2} + (c{-}2) + 2(e{+}2) + 1, \\ S^3(y_2) &= 2\tbinom{a}{2} + (a{-}4) + 2\tbinom{b-1}{2} + (b{-}2) + 2\tbinom{e+1}{2} + (e{-}3) \\ &\quad + 2\tbinom{e+c+1}{2} - 2\tbinom{e+3}{2} + (c{-}2) + 2(e{+}2) + 1, \end{aligned}$$

$$\begin{aligned}
S^3(y_3) &= 2\binom{a+e+1}{2} - 2\binom{e+3}{2} + (a-2) + 2\binom{b+e+1}{2} - 2\binom{e+3}{2} + (b-2) \\
&\quad + 2\binom{e+1}{2} + (e-3) + 2\binom{c-1}{2} + (c-2) + 2 + 1, \\
S^3(y_5) &= 2\binom{a}{2} + (a-4) + 2\binom{b}{2} + (b-4) + 2\binom{e}{2} + (e-1) \\
&\quad + 2\binom{e+c}{2} - 2\binom{e+2}{2} + (c-2) + 2(e+1) + 1, \\
S^3(y_6) &= 2\binom{a+e}{2} - 2\binom{e+2}{2} + (a-2) + 2\binom{b+e}{2} - 2\binom{e+2}{2} + (b-2) \\
&\quad + 2\binom{e}{2} + (e-1) + 2\binom{c}{2} + (c-4) + 2 + 1, \\
M^3 &= \Big(13 + (8e+22) + 13 + (8e+14)\Big) + \Big((8e+22) + 13 + (8e+14)\Big) \\
&\quad + \Big((8e+14) + 13\Big) + \Big((8e+6)\Big), \\
D &= 5 \cdot 3\binom{3}{3} = 15.
\end{aligned}$$

By (5), expanding the terms we get

$$P = 3(a^2+b^2+c^2+e^2) - 3(ab+ac+bc) + (ae+be) + 2ce - 2(a+b) - c + 28e + 97.$$

Since $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2+b^2+c^2) - 2(ab+ac+bc) \geq 0$, we have $3(a^2+b^2+c^2) - 3(ab+ac+bc) \geq 0$. Hence, if $e \geq 2$, then

$$P \geq 3e^2 + (e-2)(a+b) + c(2e-1) + 28e + 97 > 0.$$

This means that if $P = 0$ then $e = 1$. For $e = 1$ we obtain

$$P = 3(a^2+b^2+c^2) - 3(ab+ac+bc) - a - b + c + 128.$$

Substituting $a = 128 + x$, $b = 128 + y$ and $c = 128 + z$ we get

$$P = 3(x^2+y^2+z^2) - 3(xy+xz+yz) - x - y + z.$$

Now we solve the equation $P = 0$. This gives

$$3(x^2+y^2+z^2) - 3(xy+xz+yz) = x + y - z = 3t$$

or equivalently

$$\tfrac{3}{2}\Big((x-y)^2 + (y-z)^2 + (z-x)^2\Big) = x + y - z = 3t,$$

where t is nonnegative integer. Since x, y and z were defined using a, b and c, the differences $(z-y)$ and $(z-x)$ are integer numbers. Set $i = (z-y)$ and $j = (z-x)$. Then $(x-y) = (z-y) - (z-x) = i - j$, so that

$$2t = (x-y)^2 + (y-z)^2 + (z-x)^2 = (i-j)^2 + (-i)^2 + j^2 = 2i^2 + 2j^2 - 2ij$$

and consequently $3t = 3i^2 + 3j^2 - 3ij = x + y - z$. This gives

$$\begin{array}{rcl} x &=& 3t + (z-y) = 3i^2 + 3j^2 - 3ij + i, \\ y &=& 3t + (z-x) = 3i^2 + 3j^2 - 3ij + j, \\ z &=& x + y - 3t = 3i^2 + 3j^2 - 3ij + i + j, \end{array}$$

which is equivalent to (1).

In [14] we proved that for every triple a, b, c satisfying (1) and $e = 1$ it holds $P = 0$ (that is, $W(L^3(H_{a,b,c,1,1})) = W(H_{a,b,c,1,1})$). Thus, $P = 0$ in this case if and only if $e = 1$ and a, b, c satisfy (1).

Case 3. $a, c \geq 2$, $b = d = 1$, $e \geq 1$.

The vertices $x_2 = y_2$ and $x_4 = y_4$ have degree 2, so we do not need to find $S^1(x_2)$, $S^1(x_4)$, $S^3(y_2)$ and $S^3(y_4)$. We have

$$\begin{array}{rcl} S^1(x_1) &=& -a - \binom{a+e+c}{2} - (a+e+1), \\ S^1(x_3) &=& -\binom{a+e+c}{2} - (e+c+1) - c, \\ M^1 &=& -(a+e+c), \\ S^3(y_1) &=& 2\binom{a-1}{2} + (a-2) + 2 + 1 + 2\binom{e+1}{2} + (e-3) \\ && + 2\binom{e+c+1}{2} - 2\binom{e+3}{2} + (c-2) + 2(e+2) + 1, \\ S^3(y_3) &=& 2\binom{a+e+1}{2} - 2\binom{e+3}{2} + (a-2) + 2(e+2) + 1 \\ && + 2\binom{e+1}{2} + (e-3) + 2\binom{c-1}{2} + (c-2) + 2 + 1, \\ S^3(y_5) &=& 2\binom{a}{2} + (a-4) + 2 + 1 + 2\binom{e}{2} + (e-1) \\ && + 2\binom{e+c}{2} - 2\binom{e+2}{2} + (c-2) + 2(e+1) + 1, \\ S^3(y_6) &=& 2\binom{a+e}{2} - 2\binom{e+2}{2} + (a-2) + 2(e+1) + 1 \\ && + 2\binom{e}{2} + (e-1) + 2\binom{c}{2} + (c-4) + 2 + 1, \\ M^3 &=& \Big((8e+22) + 13 + (8e+14)\Big) + \Big((8e+14) + 13\Big) + \Big((8e+6)\Big), \\ D &=& 4 \cdot 3\binom{3}{3} = 12. \end{array}$$

By (5), expanding the terms we get

$$2P = 5(a^2+c^2+e^2) - 6ac + 2(ae+ce) - 11(a + c) + 45e + 148.$$

Since $4(a-c)^2 = 4a^2+4c^2-8ac \geq 0$ and $(a+c-6)^2 = a^2+c^2+2ac-12(a+c)+36 \geq 0$, we get

$$\begin{array}{rcl} 2P &\geq& a^2 + c^2 + 5e^2 + 2ac + 2(ae+ce) - 11(a + c) + 45e + 148 \\ &\geq& 5e^2 + 2(ae+ce) + (a + c) + 45e + 112 > 0. \end{array}$$

Thus, the equation $P = 0$ has no solution in this case.

Case 4. $a,b\geq 2$, $c=d=1$, $e\geq 1$.

The vertices $x_3=y_3$ and $x_4=y_4$ have degree 2, so we do not need to find $S^1(x_3)$, $S^1(x_4)$, $S^3(y_3)$ and $S^3(y_4)$. We have

$$\begin{aligned}
S^1(x_1) &= -\tbinom{a+b-1}{2}-\tbinom{a+e+2}{2}+\tbinom{a}{2}-(a+e+1),\\
S^1(x_2) &= -\tbinom{a+b-1}{2}-\tbinom{b+e+2}{2}+\tbinom{b}{2}-(b+e+1),\\
M^1 &= -(a+b-1),\\
S^3(y_1) &= 2\tbinom{a-1}{2}+(a-2)+2\tbinom{b}{2}+(b-4)+2\tbinom{e+1}{2}+(e-3)\\
&\quad +2(e+2)+1+2(e+2)+1,\\
S^3(y_2) &= 2\tbinom{a}{2}+(a-4)+2\tbinom{b-1}{2}+(b-2)+2\tbinom{e+1}{2}+(e-3)\\
&\quad +2(e+2)+1+2(e+2)+1,\\
S^3(y_5) &= 2\tbinom{a}{2}+(a-4)+2\tbinom{b}{2}+(b-4)+2\tbinom{e}{2}+(e-1)\\
&\quad +2(e+1)+1+2(e+1)+1,\\
S^3(y_6) &= 2\tbinom{a+e}{2}-2\tbinom{e+2}{2}+(a-2)+2\tbinom{b+e}{2}-2\tbinom{e+2}{2}+(b-2)\\
&\quad +2\tbinom{e}{2}+(e-1)+2+1+2+1,\\
M^3 &= \Big(13+13+(8e+14)\Big)+\Big(13+(8e+14)\Big)+\Big((8e+6)\Big),\\
D &= 4\cdot 3\tbinom{3}{3}=12.
\end{aligned}$$

By (5), expanding the terms we get

$$2P=5(a^2+b^2+e^2)-6ab-13(a+b)+47e+148.$$

Since $4(a-b)^2=4a^2+4b^2-8ab\geq 0$ and $(a+b-7)^2=a^2+b^2+2ab-14(a+b)+49\geq 0$, we get

$$\begin{aligned}
2P &\geq a^2+b^2+5e^2+2ab-13(a+b)+47e+148\\
&\geq 5e^2+(a+b)+47e+99>0.
\end{aligned}$$

Thus, the equation $P=0$ has no solution in this case.

Case 5. $a\geq 2$, $b=c=d=1$, $e\geq 1$.

The vertices $x_2=y_2$, $x_3=y_3$ and $x_4=y_4$ have degree 2, so we have

$$\begin{aligned}
S^1(x_1) &= -a-\tbinom{a+e+2}{2}-(a+e+1),\\
M^1 &= 0,\\
S^3(y_1) &= 2\tbinom{a-1}{2}+(a-2)+2+1+2\tbinom{e+1}{2}+(e-3)+2(e+2)+1+2(e+2)+1,\\
S^3(y_5) &= 2\tbinom{a}{2}+(a-4)+2+1+2\tbinom{e}{2}+(e-1)+2(e+1)+1+2(e+1)+1,\\
S^3(y_6) &= 2\tbinom{a+e}{2}-2\tbinom{e+2}{2}+(a-2)+2(e+1)+1+2\tbinom{e}{2}+(e-1)+2+1+2+1,\\
M^3 &= \Big(13+(8e+14)\Big)+\Big((8e+6)\Big),\\
D &= 3\cdot 3\tbinom{3}{3}=9.
\end{aligned}$$

By (5), expanding the terms we get

$$P = 2a^2 + 2e^2 - 10a + 17e + 48.$$

Since $(a-5)^2 = a^2 - 10a + 25 \geq 0$, we get

$$P \;\geq\; a^2 + 2e^2 + 17e + 23 > 0.$$

Thus, the equation $P = 0$ has no solution in this case.

Case 6. $a, b, c, d \geq 2$, $e = 0$.

In this case, and also in the next four, we have $y_5 = y_6$ and the degree of y_5 is 4 (see Figure 3). This does not affect $S^1(x_i)$, M^1 and $S^3(y_j)$, where $1 \leq i, j \leq 4$. Hence, analogously as above we get

$$\begin{aligned}
S^1(x_1) &= -\tbinom{a+b-1}{2} - \tbinom{a+c}{2} + \tbinom{a}{2} - \tbinom{a+d}{2} + \tbinom{a+1}{2},\\
S^1(x_2) &= -\tbinom{a+b-1}{2} - \tbinom{b+c}{2} + \tbinom{b}{2} - \tbinom{b+d}{2} + \tbinom{b+1}{2},\\
S^1(x_3) &= -\tbinom{a+c}{2} - \tbinom{b+c}{2} + \tbinom{c+1}{2} - \tbinom{c+d-1}{2} + \tbinom{c}{2},\\
S^1(x_4) &= -\tbinom{a+d}{2} - \tbinom{b+d}{2} + \tbinom{d+1}{2} - \tbinom{c+d-1}{2} + \tbinom{d}{2},\\
M^1 &= -(a+b-1) - (a+c) - (a+d) - (b+c) - (b+d) - (c+d-1),\\
S^3(y_1) &= 2\tbinom{a-1}{2} + (a-2) + 2\tbinom{b}{2} + (b-4) + 2\tbinom{c+1}{2} + (c-8) + 2\tbinom{d+1}{2} + (d-8),\\
S^3(y_2) &= 2\tbinom{a}{2} + (a-4) + 2\tbinom{b-1}{2} + (b-2) + 2\tbinom{c+1}{2} + (c-8) + 2\tbinom{d+1}{2} + (d-8),\\
S^3(y_3) &= 2\tbinom{a+1}{2} + (a-8) + 2\tbinom{b+1}{2} + (b-8) + 2\tbinom{c-1}{2} + (c-2) + 2\tbinom{d}{2} + (d-4),\\
S^3(y_4) &= 2\tbinom{a+1}{2} + (a-8) + 2\tbinom{b+1}{2} + (b-8) + 2\tbinom{c}{2} + (c-4) + 2\tbinom{d-1}{2} + (d-2),
\end{aligned}$$

where we simplified expressions as $2\binom{a+0+1}{2} - 2\binom{0+3}{2} + (a-2)$ to $2\binom{a+1}{2} + (a-8)$.

Now we discuss the terms containing $h(u, y_5)$. In $S^3(y_5)$ we sum $h(u, y_5)$, where $\deg(u) = 2$ and $\deg(y_5) = 4$. Hence $\binom{\mathrm{d}_u}{2}\binom{\mathrm{d}_{y_5}}{2} - 1 = 5$. Since $\beta_0(u, y_5) = 3$, $\beta_1(u, y_5) = 3$ and $\beta_2(u, y_5) = 0$, we have $h(u, y_5) = 5d(u, y_i) + 3$. Thus, the sum of $h(u, y_5)$ for interior vertices u of $y_1 - x_1$ path is $5(2+3+\ldots+(a-1)) + 3(a-2) = 5\binom{a}{2} - 5 + 3(a-2)$ (see Figure 3). In this way we get

$$\begin{aligned}
S^3(y_5) &= 5\tbinom{a}{2} - 5 + 3(a-2) + 5\tbinom{b}{2} - 5 + 3(b-2) + 5\tbinom{c}{2} - 5 + 3(c-2)\\
&\quad + 5\tbinom{d}{2} - 5 + 3(d-2).
\end{aligned}$$

Now consider $h(y_i, y_5)$, $1 \leq i \leq 4$. Here $\binom{\mathrm{d}_{y_i}}{2}\binom{\mathrm{d}_{y_5}}{2} - 1 = 17$ and $\beta_0(y_i, y_5) = 2 \cdot 3 = 6$. Since y_i and y_5 allways lie in a common triangle, we have $\beta_1(y_i, y_5) = 11$ and $\beta_2(y_i, y_5) = 1$ (see Figure 3). Thus, $h(y_i, y_5) = 17 \cdot 1 + 11 + 2 \cdot 1 = 30$. As regards $h(y_i, y_j)$, where $1 \leq i < j \leq 4$, analogously as above we get $h(y_1, y_2) = 13$ and $h(y_1, y_3) = 8e + 22 = 22$. Hence

$$M^3 = (13+22+22+30) + (22+22+30) + (13+30) + 30.$$

Finally,

$$D = \sum_u \Big[3\binom{\mathrm{d}_u}{3} + 6\binom{\mathrm{d}_u}{4}\Big] = 4\Big[3\binom{3}{3}\Big] + \Big[3\binom{4}{3} + 6\binom{4}{4}\Big] = 12 + 18.$$

By (5), expanding the terms we get

$$\begin{aligned} P &= 4(a^2{+}b^2{+}c^2{+}d^2) - 3(ab{+}ac{+}ad{+}bc{+}bd{+}cd) + 3(a{+}b{+}c{+}d) + 137 \\ &= 4(a{+}b{+}c{+}d)^2 - 11(ab{+}ac{+}ad{+}bc{+}bd{+}cd) + 3(a{+}b{+}c{+}d) + 137. \end{aligned}$$

Substitute $x = (a{+}b{+}c{+}d)$ and consider the equation $P = 0$ over $\mathbb{Z}_{11}$. We get

$$4x^2 + 3x + 5 = 0,$$

which has no solution in $\mathbb{Z}_{11}$. Consequently, $P = 0$ has no integer solution and $W(L^3(H_{a,b,c,d,0})) - W(H_{a,b,c,d,0}) \neq 0$ in this case.

Case 7. $a, b, c \geq 2$, $d = 1$, $e = 0$.

In this case the vertex $x_4 = y_4$ has degree 2, so we do not need to find $S^1(x_4)$ and $S^3(y_4)$. Analogously as in the previous case we have

$$\begin{aligned} S^1(x_1) &= -\binom{a{+}b{-}1}{2} - \binom{a{+}c}{2} + \binom{a}{2} - (a{+}1), \\ S^1(x_2) &= -\binom{a{+}b{-}1}{2} - \binom{b{+}c}{2} + \binom{b}{2} - (b{+}1), \\ S^1(x_3) &= -\binom{a{+}c}{2} - \binom{b{+}c}{2} + \binom{c{+}1}{2} - c, \\ M^1 &= -(a{+}b{-}1) - (a{+}c) - (b{+}c), \\ S^3(y_1) &= 2\binom{a{-}1}{2} + (a{-}2) + 2\binom{b}{2} + (b{-}4) + 2\binom{c{+}1}{2} + (c{-}8) + 4 + 1, \\ S^3(y_2) &= 2\binom{a}{2} + (a{-}4) + 2\binom{b{-}1}{2} + (b{-}2) + 2\binom{c{+}1}{2} + (c{-}8) + 4 + 1, \\ S^3(y_3) &= 2\binom{a{+}1}{2} + (a{-}8) + 2\binom{b{+}1}{2} + (b{-}8) + 2\binom{c{-}1}{2} + (c{-}2) + 2 + 1, \\ S^3(y_5) &= 5\binom{a}{2} - 5 + 3(a{-}2) + 5\binom{b}{2} - 5 + 3(b{-}2) + 5\binom{c}{2} - 5 + 3(c{-}2) + 5 + 3, \\ M^3 &= (13{+}22{+}30) + (22{+}30) + 30, \\ D &= 3\Big(3\binom{3}{3}\Big) + \Big(3\binom{4}{3} + 6\binom{4}{4}\Big) = 9 + 18. \end{aligned}$$

By (5), expanding the terms we get

$$2P = 7(a^2{+}b^2{+}c^2) - 6(ab{+}ac{+}bc) - 3(a{+}b) - c + 232.$$

Since $3(a{-}b)^2 + 3(b{-}c)^2 + 3(c{-}a)^2 = 6(a^2{+}b^2{+}c^2) - 6(ab{+}ac{+}bc) \geq 0$ and also $(a{-}2)^2 + (b{-}2)^2 + (c{-}1)^2 = (a^2{+}b^2{+}c^2) - 4(a{+}b) - 2c + 9 \geq 0$, we get

$$\begin{aligned} 2P &\geq (a^2{+}b^2{+}c^2) - 3(a{+}b) - c + 232 \\ &\geq a + b + c + 223 > 0. \end{aligned}$$

Thus, the equation $P=0$ has no solution in this case.

Case 8. $a, c \geq 2$, $b=d=1$, $e=0$.
The vertices $x_2=y_2$ and $x_4=y_4$ have degree 2, so we have

$$
\begin{array}{rcl}
S^1(x_1) & = & -a-\binom{a+c}{2}-(a+1), \\
S^1(x_3) & = & -\binom{a+c}{2}-(c+1)-c, \\
M^1 & = & -(a+c), \\
S^3(y_1) & = & 2\binom{a-1}{2}+(a-2)+2+1+2\binom{c+1}{2}+(c-8)+4+1, \\
S^3(y_3) & = & 2\binom{a+1}{2}+(a-8)+4+1+2\binom{c-1}{2}+(c-2)+2+1, \\
S^3(y_5) & = & 5\binom{a}{2}-5+3(a-2)+5+3+5\binom{c}{2}-5+3(c-2)+5+3, \\
M^3 & = & (22+30)+30, \\
D & = & 2\Big(3\binom{3}{3}\Big)+\Big(3\binom{4}{3}+6\binom{4}{4}\Big)=6+18.
\end{array}
$$

By (5), expanding the terms we get

$$P=3(a^2+c^2)-3ac-5(a+c)+92.$$

Since $2(a-c)^2=2(a^2+c^2)-4ac \geq 0$ and $(a-3)^2+(c-3)^2=(a^2+c^2)-6(a+c)+18 \geq 0$, we get

$$
\begin{array}{rcl}
P & \geq & (a^2+c^2)+ac-5(a+c)+92 \\
 & \geq & ac+(a+c)+74>0.
\end{array}
$$

Thus, the equation $P=0$ has no solution in this case.

Case 9. $a, b \geq 2$, $c=d=1$, $e=0$.
The vertices $x_3=y_3$ and $x_4=y_4$ have degree 2, so we have

$$
\begin{array}{rcl}
S^1(x_1) & = & -\binom{a+b-1}{2}-(a+1)-(a+1)-a, \\
S^1(x_2) & = & -\binom{a+b-1}{2}-(b+1)-(b+1)-b, \\
M^1 & = & -(a+b-1), \\
S^3(y_1) & = & 2\binom{a-1}{2}+(a-2)+2\binom{b}{2}+(b-4)+4+1+4+1, \\
S^3(y_2) & = & 2\binom{a}{2}+(a-4)+2\binom{b-1}{2}+(b-2)+4+1+4+1, \\
S^3(y_5) & = & 5\binom{a}{2}-5+3(a-2)+5\binom{b}{2}-5+3(b-2)+5+3+5+3, \\
M^3 & = & (13+30)+30, \\
D & = & 2\Big(3\binom{3}{3}\Big)+\Big(3\binom{4}{3}+6\binom{4}{4}\Big)=6+18.
\end{array}
$$

By (5), expanding the terms we get

$$P=3(a^2+b^2)-3ab-6(a+b)+92.$$

Since $2(a-b)^2 = 2(a^2+b^2)-4ab \geq 0$ and $(a-3)^2+(b-3)^2 = (a^2+b^2)-6(a+b)+18 \geq 0$, we get

$$\begin{aligned} P &\geq (a^2+b^2) + ab - 6(a+b) + 92 \\ &\geq ab + 74 > 0. \end{aligned}$$

Thus, the equation $P = 0$ has no solution in this case.

Case 10. $a \geq 2$, $b = c = d = 1$, $e = 0$.
The vertices $x_2 = y_2$, $x_3 = y_3$ and $x_4 = y_4$ have degree 2, so we have

$$\begin{aligned} S^1(x_1) &= -\tbinom{a+1}{2} - (a+1) - (a+1) - a, \\ M^1 &= 0, \\ S^3(y_1) &= 2\tbinom{a-1}{2} + (a-2) + 2 + 1 + 4 + 1 + 4 + 1, \\ S^3(y_5) &= 5\tbinom{a}{2} - 5 + 3(a-2) + 5 + 3 + 5 + 3 + 5 + 3, \\ M^3 &= 30, \\ D &= 3\tbinom{3}{3} + \Big(3\tbinom{4}{3} + 6\tbinom{4}{4}\Big) = 3 + 18. \end{aligned}$$

By (5), expanding the terms we get

$$2P = 5a^2 - 19a + 130.$$

Since $5(a-2)^2 = 5a^2 - 20a + 20$, we get

$$2P \geq a + 110.$$

Thus, the equation $P = 0$ has no solution in this case.

Case 11. $a = b = c = d = 1$, $e \geq 0$.
In [15, Theorem 1.5] we proved that $W(L^i(T)) > W(T)$ for every $i \geq 3$ and for every tree T which is different from a path and the claw $K_{1,3}$ and in which no leaf is adjacent to a vertex of degree 2. By this statement, for $H = H_{1,1,1,1,e}$ we have $W(L^3(H)) > W(H)$, which completes the proof. □

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