

On a conjecture about Wiener index in iterated line graphs of trees

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Abstract

Let G be a graph. Denote by $L^i(G)$ its i -iterated line graph and denote by $W(G)$ its Wiener index. Dobrynin and Melnikov conjectured that there exists no nontrivial tree T and $i \geq 3$, such that $W(L^i(T)) = W(T)$. We prove this conjecture for trees which are not homeomorphic to the claw $K_{1,3}$ and H , where H is a tree consisting of 6 vertices, 2 of which have degree 3.

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1 Introduction

Let G be a graph. For any two of its vertices, say u and v , by $d(u, v)$ we denote the distance from u to v in G . The *Wiener index* of G , $W(G)$, is defined as

$$W(G) = \sum_{u \neq v} d(u, v),$$

where the sum is taken through all unordered pairs of vertices of G . Wiener index was introduced by Wiener in 1947, see [15]. In the next decades, it was intensively studied by chemists, as it is related to many physical properties of organical molecules, see [9]. Graph theoretists reintroduced this parameter as the distance in

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1970 and transmission in 1984, see [6] and [14], respectively. Recently, graph theoretic aspects of Wiener index are intensively studied, see e.g. [7] and [8], or surveys [3] and [4].

By definition, if G has a unique vertex, that is, if $G = K_1$, then $W(G) = 0$. In this case we say that the graph G is *trivial*. We set $W(G) = 0$ also when the set of vertices (and therefore also the set of edges) of G is empty.

The line graph of G , $L(G)$, has vertex set identical with the set of edges of G . Two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G . Iterated line graphs are defined inductively as follows:

$$L^i(G) = \begin{cases} G & \text{if } i = 0, \\ L(L^{i-1}(G)) & \text{if } i > 0. \end{cases}$$

In [1], the following theorem was proved.

Theorem 1.1 [1] *If T is a tree on n vertices, then $W(L(T)) = W(T) - \binom{n}{2}$.*

Since $\binom{n}{2} > 0$ if $n \geq 2$, there is no nontrivial tree for which $W(L(T)) = W(T)$. However, there are trees T satisfying $W(L^2(T)) = W(T)$, see for example [2]. In [5] the following conjecture was posed (see also [3]).

Conjecture 1.2 [5] *Let T be a nontrivial tree and $i \geq 3$. Then $W(L^i(T)) \neq W(T)$.*

Denote by P_n a path on n vertices. If $n \geq 2$, then $W(P_n) > W(P_{n-1})$. Since $L(P_n) = P_{n-1}$ if $n \geq 2$, while $L(P_1)$ is an empty graph, it follows that $W(L^i(P_n)) < W(P_n)$ for every $i \geq 1$ provided that $n \geq 2$. Hence, Conjecture 1.2 is trivially true for paths of length at least 1.

In [11], we prove that for every graph G the function $W(L^i(G))$ is convex in variable i . The following corollary is a straightforward consequence of this fact.

Corollary 1.3 *Let T be a tree such that $W(L^3(T)) > W(T)$. Then for every $i \geq 3$ the inequality $W(L^i(T)) > W(T)$ holds.*

Let G be a graph. A *pendant path* (or a *ray* for short) R' in G is a (directed) path, the first vertex of which has degree at least 3, its last vertex has degree 1, and all of its internal vertices (if any exist) have degree 2 in G . Observe that if R' has length t , $t \geq 2$, then the edges of R' correspond to vertices of a ray R in $L(G)$ of length $t - 1$. In [11] we proved the following theorem.

Theorem 1.4 [11] *Let T be a tree distinct from a path and the claw $K_{1,3}$ such that all of its rays have length 1. Then $W(L^3(T)) > W(T)$.*

Here we extend this statement to trees with arbitrarily long rays. Denote by H a tree on 6 vertices, two of which have degree 3 and four of which have degree 1. (That is, H is the graph which “looks” like the letter H.) The main result of this paper is the following theorem.

Theorem 1.5 *Let T be a tree not homeomorphic to a path, claw $K_{1,3}$ and H . Then $W(L^3(T)) > W(T)$.*

Recall that graphs G_1 and G_2 are homeomorphic if and only if the graphs obtained from them by repeatedly removing a vertex of degree 2 (and making its two neighbours adjacent) are isomorphic. Combining Corollary 1.3 and Theorem 1.5 we obtain the following corollary, which proves Conjecture 1.2 for the trees T satisfying the assumption in Theorem 1.5.

Corollary 1.6 *Let T be a tree not homeomorphic to a path, claw $K_{1,3}$ and H . Then $W(L^i(T)) > W(T)$ for every $i \geq 3$.*

We remark that trees homeomorphic to the claw $K_{1,3}$ and the graph H are considered in forthcoming papers, see [12, 13].

For a tree T , let $D(T) = W(L^3(T)) - W(T)$. We prove $D(T) > 0$ by induction on the length of the longest ray in T . By Theorem 1.4, $D(T) > 0$ if the longest ray has length 1. Now we describe the induction step:

We suppose that $D(T) > 0$ for all trees rays of which have length at most $l + 1$. We would like to extend this statement to trees with rays of length at most $l + 2$. Let a' be the last vertex of a ray of length $l + 1$ in T , $l \geq 0$. Since we extend only one ray in turn, namely the ray terminating at a' , we assume that all rays of T have lengths at most $l + 2$. Add to T one new vertex b' and the edge $a'b'$, and denote the resulting tree by T^* . Denote by a the edge of T containing a' and denote by b the edge $a'b'$. Then ab is an edge of $L(T^*)$ and the degree of b is 1 in $L(T^*)$. Moreover, a is an endvertex of a ray of length l in $L(T)$ and b is an endvertex of a ray of length $l + 1$ in $L(T^*)$; see Figure 1. By assumption, all rays of $L(T)$ have lengths at most $l + 1$. Let

$$\Delta T = D(T^*) - D(T).$$

In the next section we present an exact formula for ΔT . In Section 3 we prove $\Delta T \geq 0$ and this will establish Theorem 1.5 (for more detailed explanation see the proof of Theorem 1.5).

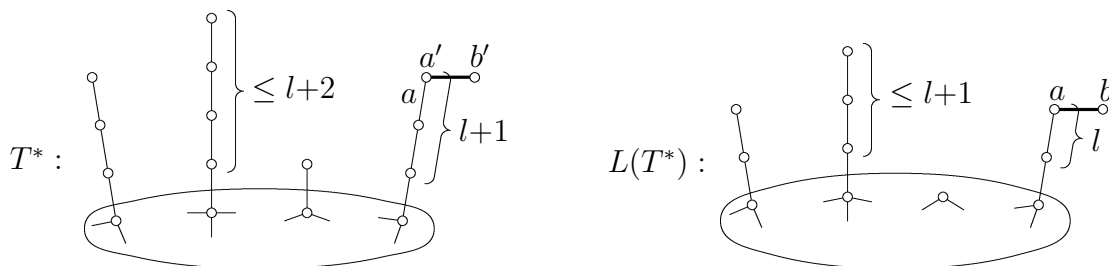


Figure 1: Description of the induction step for $l = 2$.

Now we introduce notation used throughout the paper. Let G be a graph. By $V(G)$ and $E(G)$ we denote its sets of vertices and edges, respectively. Since we work repeatedly with line graphs of trees, we simplify the notation and write LG rather than $L(G)$. The degree of a vertex z is denoted by d_z . If there are more graphs containing the vertex z , then d_z denotes the degree of z in LT (whatever the meaning of T at that point will be). Similarly, by $d(z, w)$ we denote the distance from z to w , and this distance is preferably considered in LT (rather than T). When no confusion is likely, any path starting at u and terminating at v will be denoted by $u - v$.

2 Preliminaries

Analogously as a vertex of $L(G)$ corresponds to an edge of G , a vertex of $L^2(G)$ corresponds to a path of length two in G . For $x \in V(L^2(G))$, we denote by $B_2(x)$ the corresponding path in G . For two subgraphs S_1 and S_2 of G , by $d(S_1, S_2)$ we denote the shortest distance in G between a vertex of S_1 and a vertex of S_2 . If S_1 and S_2 share s edges, then we set $d(S_1, S_2) = -s$.

Let x and y be two vertices of $L^2(G)$, such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Then $d_{L^2(G)}(x, y) = d(B_2(x), B_2(y)) + 2$; see [10, 11].

Let $u, v \in V(G)$, $u \neq v$. Denote by $\beta_i(u, v)$ the number of pairs $x, y \in V(L^2(G))$, with u being the center of $B_2(x)$ and v being the center of $B_2(y)$, such that $d(B_2(x), B_2(y)) = d(u, v) - 2 + i$. Since $d(u, v) - 2 \leq d(B_2(x), B_2(y)) \leq d(u, v)$, we see that $\beta_i(u, v) = 0$ for all $i \notin \{0, 1, 2\}$. In [11] the following statement was proved:

Proposition 2.1 *Let G be a connected graph. Then*

$$\begin{aligned} W(L^2(G)) &= \sum_{u \neq v} \left[\binom{d_u}{2} \binom{d_v}{2} d(u, v) + \beta_1(u, v) + 2\beta_2(u, v) \right] \\ &+ \sum_u \left[3 \binom{d_u}{3} + 6 \binom{d_u}{4} \right], \end{aligned}$$

where the first sum runs through all unordered pairs of distinct vertices $u, v \in V(G)$ and the second one runs through all $u \in V(G)$.

We apply Proposition 2.1 to line graphs of trees. Let us recall the structure of these graphs. For any tree F , the graph LF consists of cliques in the following sense: Denote by $\mathcal{C}(LF)$ the set of maximal cliques of LF . Then every vertex of LF belongs to at most two cliques from $\mathcal{C}(LF)$; each pair of cliques from $\mathcal{C}(LF)$ intersects in at most one vertex; and the cliques of $\mathcal{C}(LF)$ have a “tree structure”,

that is, there are no cliques C_0, C_1, \dots, C_{t-1} , $t \geq 3$, such that C_i and C_{i+1} have nonempty intersection, $0 \leq i \leq t-1$, the addition being modulo t .

We start with an exact formula for ΔT . For $u \in V(LT) \setminus \{a\}$, let

$$h_{LT}(u) = \left(\binom{d_u}{2} d_a - 1 \right) d(u, a) + (d_u - 1) \left(d_u d_a - d_a - \frac{1}{2} d_u \right) - 2 - \phi(u, a), \quad (1)$$

where

$$\phi(u, a) = \begin{cases} (d_a - 1)(d_u - 2) & \text{if } d(u, a) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2 *For a nontrivial tree, the following equality holds:*

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2} d_a (d_a - 1) (2d_a - 1) - 3,$$

where the sum is taken over all vertices $u \in V(LT) \setminus \{a\}$.

PROOF Let F be a tree and let u and v be distinct vertices of LF . Consider vertices $x, y \in V(L^2(LF))$ such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Due to the clique structure of LF , there is a unique shortest $u - v$ path in LF . Denote this path by $u = a_0, a_1, \dots, a_t = v$. If $d(B_2(x), B_2(y)) = d(u, v) - 2$, then we must have $a_1 \in V(B_2(x))$ and $a_{t-1} \in V(B_2(y))$. There are $(d_u - 1)$ ways to choose the other endvertex of $B_2(x)$, and there are $(d_v - 1)$ ways to choose the other endvertex of $B_2(y)$. Hence, $\beta_0(u, v) = (d_u - 1)(d_v - 1)$.

Now we find $\beta_1(u, v)$. We distinguish two cases: $d(u, v) \geq 2$ and $d(u, v) = 1$.

Suppose first $d(u, v) \geq 2$. In this case u and v do not belong to a common clique from $\mathcal{C}(LF)$. If $d(B_2(x), B_2(y)) = d(u, v) - 1$, then either $a_1 \in V(B_2(x))$ or $a_{t-1} \in V(B_2(y))$, but not both. In the first case we obtain $(d_u - 1) \binom{d_v - 1}{2}$ pairs x, y and in the second case $\binom{d_u - 1}{2} (d_v - 1)$ pairs x, y . Thus

$$\beta_1(u, v) = (d_u - 1) \binom{d_v - 1}{2} + \binom{d_u - 1}{2} (d_v - 1).$$

Suppose now that $d(u, v) = 1$. In this case, u and v belong to a common clique. All pairs x, y mentioned in the previous case contribute to $\beta_1(u, v)$, but we have to add pairs x, y such that $v \notin V(B_2(x))$, $u \notin V(B_2(y))$ and $d(B_2(x), B_2(y)) = d(u, v) - 1 = 0$. For these pairs, the paths $B_2(x)$ and $B_2(y)$ share at least one of their endvertices. Denote by $\alpha_{LF}(u, v)$ the number of these extra pairs. Then

$$\beta_1(u, v) = (d_u - 1) \binom{d_v - 1}{2} + \binom{d_u - 1}{2} (d_v - 1) + \alpha_{LF}(u, v).$$

Since we do not need to evaluate $\alpha_{LF}(u, v)$ in general, we postpone this computation until later. To simplify the notation, we set $\alpha_{LF}(u, v) = 0$ for all pairs u, v such that $d(u, v) \geq 2$.

We have $\binom{d_u}{2} \binom{d_v}{2}$ pairs $x, y \in V(L^2(LF))$ such that u is the center of $B_2(x)$ and v is the center of $B_2(y)$. Since

$$\begin{aligned} \binom{d_u}{2} \binom{d_v}{2} &= (d_u - 1)(d_v - 1) + (d_u - 1) \binom{d_v - 1}{2} + \binom{d_u - 1}{2} (d_v - 1) \\ &\quad + \binom{d_u - 1}{2} \binom{d_v - 1}{2}, \end{aligned}$$

we obtain $\beta_2(u, v) = \binom{d_u - 1}{2} \binom{d_v - 1}{2} - \alpha_{LF}(u, v)$. By Proposition 2.1, it follows that

$$\begin{aligned} W(L^2(LF)) &= \sum_{u \neq v} \left[\binom{d_u}{2} \binom{d_v}{2} d(u, v) + (d_u - 1) \binom{d_v - 1}{2} \right. \\ &\quad \left. + \binom{d_u - 1}{2} (d_v - 1) + 2 \binom{d_u - 1}{2} \binom{d_v - 1}{2} - \alpha_{LF}(u, v) \right] \\ &\quad + \sum_u \left[3 \binom{d_u}{3} + 6 \binom{d_u}{4} \right]. \end{aligned} \tag{2}$$

Now we evaluate $W(L^3(T^*)) - W(L^3(T)) = W(L^2(LT^*)) - W(L^2(LT))$; see the notation following Corollary 1.6. The graph LT^* has one more vertex than LT , namely the vertex b of degree 1, and the degree of a increased by 1 to $d_a + 1$ in LT^* . Therefore, all the terms of (2) for pairs u, v which do not contain neither a nor b cancel out in $W(L^2(LT^*)) - W(L^2(LT))$. However, we need to subtract the terms for pairs u, a in LT and to add the terms for pairs u, a in LT^* , $u \in V(LT) \setminus \{a\}$. We can ignore the terms containing b in LT^* , since the degree of b is 1, and thus b cannot be a center of $B_2(y)$ for any $y \in V(L^2(LT^*))$. (Observe that all terms of (2) are 0 if one of the vertices has degree 1.) As regards the second sum in (2), we have to subtract the term corresponding to a in LT and add the terms corresponding to a and b in LT^* , the second one being 0 since the degree of b is 1 in LT^* . Denote by $\Delta\alpha(u, v)$ the difference $\alpha_{LT^*}(u, v) - \alpha_{LT}(u, v)$ and denote by ΔWL^2 the difference

$W(L^2(LT^*)) - W(L^2(LT))$. By (2), it follows that

$$\begin{aligned}
\Delta WL^2 &= - \sum_u \left[\binom{d_u}{2} \binom{d_a}{2} d(u, a) + (d_u - 1) \binom{d_a - 1}{2} + \right. \\
&\quad \left. + \binom{d_u - 1}{2} (d_a - 1) + 2 \binom{d_u - 1}{2} \binom{d_a - 1}{2} - \alpha_{LT}(u, a) \right] \\
&\quad + \sum_u \left[\binom{d_u}{2} \binom{d_a + 1}{2} d(u, a) + (d_u - 1) \binom{d_a}{2} + \right. \\
&\quad \left. + \binom{d_u - 1}{2} d_a + 2 \binom{d_u - 1}{2} \binom{d_a}{2} - \alpha_{LT^*}(u, a) \right] \\
&\quad - 3 \binom{d_a}{3} - 6 \binom{d_a}{4} + 3 \binom{d_a + 1}{3} + 6 \binom{d_a + 1}{4} \\
&= \sum_u \left[\binom{d_u}{2} d_a d(u, a) + (d_u - 1)(d_a - 1) \right. \\
&\quad \left. + \binom{d_u - 1}{2} + 2 \binom{d_u - 1}{2} (d_a - 1) - \Delta\alpha(u, v) \right] \\
&\quad + \frac{1}{4} d_a (d_a - 1) \left[-2(d_a - 2) - (d_a - 2)(d_a - 3) \right. \\
&\quad \left. + 2(d_a + 1) + (d_a + 1)(d_a - 2) \right] \\
&= \sum_u \left[\binom{d_u}{2} d_a d(u, a) + (d_u - 1) \left(d_u d_a - d_a - \frac{1}{2} d_u \right) - \Delta\alpha(u, a) \right] \\
&\quad + \frac{1}{2} d_a (d_a - 1) (2d_a - 1). \tag{3}
\end{aligned}$$

Now we determine $\Delta\alpha(u, a)$. For $u \in V(LT) \setminus \{a\}$, the distance from u to a in LT is the same as in LT^* . Therefore $\Delta\alpha(u, a) = \alpha_{LT^*}(u, a) - \alpha_{LT}(u, a) = 0 - 0 = 0$ if $d(u, a) \geq 2$. If $d(u, a) = 1$, then in order to evaluate $\alpha_{LT^*}(u, a) - \alpha_{LT}(u, a)$ we need to count pairs x, y such that $b \in V(B_2(y))$. Denote by C the clique of $\mathcal{C}(LT)$ containing both a and u . The order of C is $d_a + 1$. We distinguish two cases.

- *Both endvertices of $B_2(x)$ are in C* : We have $\binom{d_a - 1}{2}$ choices for $B_2(x)$ in this case since $a \notin V(B_2(x))$. For each of these choices there are two choices for $B_2(y)$ such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Therefore there are $2 \binom{d_a - 1}{2}$ pairs x, y contributing to $\Delta\alpha(u, v)$ in this case.
- *Only one endvertex of $B_2(x)$ is in C* : For this vertex we have $d_a - 1$ choices, since $a \notin V(B_2(x))$, and for the other endvertex of $B_2(x)$ we have $d_u - d_a$

choices. In this case, for every x there is a unique y such that $B_2(x)$ and $B_2(y)$ share an endvertex and $b \in V(B_2(y))$. Thus there are $(d_a - 1)(d_u - d_a)$ pairs x, y contributing to $\Delta\alpha(u, v)$ in this case.

Hence,

$$\Delta\alpha(u, v) = 2 \binom{d_a - 1}{2} + (d_a - 1)(d_u - d_a) = (d_a - 1)(d_u - 2) = \phi(u, a). \quad (4)$$

Now we evaluate $W(T^*) - W(T)$. If F is a tree with n_0 vertices, then $W(LF) = W(F) - \binom{n_0}{2}$, by Theorem 1.1. Denote by n_1 the number of vertices of LF . Since $n_1 = n_0 - 1$, we have $W(F) = W(LF) + \binom{n_1+1}{2}$. Denote by n the number of vertices of LT . Then

$$\begin{aligned} W(T^*) - W(T) &= W(LT^*) + \binom{n+2}{2} - W(LT) - \binom{n+1}{2} \\ &= W(LT^*) - W(LT) + n + 1. \end{aligned}$$

In $W(LT^*) - W(LT)$, all terms for pairs u, v which do not contain b will cancel out. Therefore

$$\begin{aligned} W(T^*) - W(T) &= \sum_u d(u, b) + d(a, b) + n + 1 \\ &= \sum_u (d(u, a) + 1) + 1 + \sum_u 1 + 2 \\ &= \sum_u (d(u, a) + 2) + 3. \end{aligned} \quad (5)$$

where the sum goes once again through $n - 1$ vertices $u \in V(LT) \setminus \{a\}$.

Since $\Delta T = D(T^*) - D(T) = W(L^3(T^*)) - W(T^*) - W(L^3(T)) + W(T) = \Delta WL^2 - (W(T^*) - W(T))$, combining (3), (4) and (5) we obtain the required result.

□

3 Proof of Theorem 1.5

We prove that $\Delta T \geq 0$ for every tree T which is not homeomorphic to a path, claw $K_{1,3}$ or the graph H . Let l, a', b', a, b, T^* and ΔT be as in the discussion following Corollary 1.6. As explained there, we proceed by induction on l .

First we prove that $\Delta T \geq 0$ for the case $l = 0$. In this case, a' is adjacent to a vertex of degree at least 3 in T , implying that in LT we have $d_a \geq 2$.

Let v be an endvertex of a ray (a pendant path) R in LT , that is, $d_v = 1$. By \bar{v} we denote the first vertex of R , i.e., a vertex at shortest distance to v whose degree is at least 3. Due to the clique structure of LT described after Proposition 2.1, we have:

Observation 3.1 *If u and v are distinct vertices of degree 1 in LT , then $\bar{u} \neq \bar{v}$.*

We use Observation 3.1 repeatedly in the following proofs.

Lemma 3.2 *Let T be a tree different from a path in which all rays have length at most $l + 2$, and let $l = 0$. Then $\Delta T \geq 0$.*

PROOF We find a lower bound for $\sum_u h_{LT}(u)$. Consider four cases:

- $d_u = 1$: Then $d(u, a) > 1$, and thus $h_{LT}(u) = -d(u, a) - 2$ by (1).
- $d_u = 2$: Since $(d_a - 1)(d_u - 2) = 0$, we see that $\phi(u, a) = 0$ in this case as well. By (1) we obtain

$$h_{LT}(u) = (d_a - 1)d(u, a) + d_a - 3 \geq d_a - 1 + d_a - 3 = 2d_a - 4 \geq 0$$

since $d_a \geq 2$.

- $d_u \geq 3$ and $d(u, a) \geq 2$: By (1) it follows that

$$\begin{aligned} h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1 \right) d(u, a) + (d_u - 1) \left(d_u d_a - d_a - \frac{1}{2} d_u \right) - 2 \\ &\geq 5d(u, a) + (d_u - 1) \frac{1}{2} \left[d_a (d_u - 2) + d_u (d_a - 1) \right] - 2 \\ &\geq 5d(u, a) + 5 - 2 \\ &\geq d(u, a) + 11 \end{aligned}$$

as $d_u \geq 3$, $d_a \geq 2$ and $d(u, a) \geq 2$.

- $d_u \geq 3$ and $d(u, a) = 1$: By (1) it follows that

$$\begin{aligned} h_{LT}(u) &= \left(\binom{d_u}{2} d_a - 1 \right) d(u, a) + (d_u - 1) \left(d_u d_a - d_a - \frac{1}{2} d_u \right) - 2 \\ &\quad - (d_a - 1)(d_u - 2) \\ &\geq 5d(u, a) + d_u^2 d_a - \frac{1}{2} d_u^2 - 3d_u d_a + \frac{3}{2} d_u + 3d_a - \frac{3}{2} - \frac{5}{2} \\ &= 5d(u, a) + \frac{1}{2} \left[(2d_a - 1) (d_u (d_u - 3) + 3) - 5 \right] \\ &\geq d(u, a) + 6 \end{aligned}$$

as $d_u \geq 3$, $d_a \geq 2$ and $d(u, a) = 1$.

Hence,

$$h_{LT}(u) \geq \begin{cases} -d(u, a) - 2 & \text{if } d_u = 1, \\ 0 & \text{if } d_u = 2, \\ d(u, a) + 6 & \text{if } d_u \geq 3. \end{cases} \quad (6)$$

Since $l = 0$, all rays of T have length at most 2, implying that all rays of LT have length at most 1. Hence, if $d_u = 1$, then $d(u, \bar{u}) = 1$ in LT . Thus

$$h_{LT}(u) + h_{LT}(\bar{u}) \geq -d(u, a) - 2 + d(\bar{u}, a) + 6 = -d(\bar{u}, a) - 3 + d(\bar{u}, a) + 6 \geq 0.$$

Denote by V_1 the set of vertices of degree 1 in $V(LT) \setminus \{a\}$. By Observation 3.1, we have $\bar{u} \neq \bar{v}$ whenever $u, v \in V_1$, $u \neq v$. Therefore, by (6) it follows that

$$\sum_u h_{LT}(u) \geq \sum_{u \in V_1} (h_{LT}(u) + h_{LT}(\bar{u})) \geq 0.$$

Since $d_a \geq 2$, we have $\frac{1}{2}d_a(d_a - 1)(2d_a - 1) \geq 3$, implying that

$$\Delta T = \sum_u h_{LT}(u) + \frac{1}{2}d_a(d_a - 1)(2d_a - 1) - 3 \geq 0,$$

by Proposition 2.2. □

Now we prove that $\Delta T \geq 0$ for all $l \geq 1$, that is, from now on we consider $l \geq 1$. In this case $\phi(u, a) = 0$ since $d_a = 1$, which simplifies the expression for $h_{LT}(u)$ in (1). The problem is that $h_{LT}(u) < 0$ even if $d_u = 2$, suggesting that we need sharper estimations. We prove $\Delta T \geq 0$ by induction on the number of vertices of degree at least 3 in T .

Let G be a graph. A path of length at least one in G is *interior path* if its endvertices have degrees both at least 3, its interior vertices (if any) have degree 2 in G , and its edges are bridges of G . In the next lemma we show that it suffices to prove $\Delta T \geq 0$ for trees whose interior paths have lengths at most 2, i.e., we reduce the class of trees for which we need to prove $\Delta T \geq 0$.

Lemma 3.3 *Let T^s be obtained from T by subdividing one edge of an interior path of length t , $t \geq 2$, and let $l \geq 1$. Then $\Delta T^s \geq \Delta T$.*

PROOF Denote by P' the interior path of T , whose edge was subdivided to obtain T^s . Since P' has length $t \geq 2$, the edges of P' form an interior path P of length $t - 1 \geq 1$ in LT . Obviously, LT^s can be obtained from LT by subdividing one edge of P . Denote by e the endvertex of P , which has among the vertices of P the greatest distance from a . Let LT^s be obtained from LT by subdividing that edge of P which is incident to e . Denote the new vertex by w . Observe that for every vertex $u \in V(LT)$, the degree of u in LT is the same as its degree in LT^s .

Since the degree of a is the same in LT^s as in LT , namely 1, by Proposition 2.2 it suffices to show that $\sum_{u \in V(LT^s) \setminus \{a\}} h_{LT^s}(u) \geq \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u)$. We distinguish three cases.

- u is a vertex of LT such that e does not lie on $u - a$ path in LT : Then $d_{LT^s}(u, a) = d_{LT}(u, a)$, implying that $h_{LT^s}(u) = h_{LT}(u)$ and $h_{LT^s}(u) - h_{LT}(u) = 0$, see (1).
- u is a vertex of LT , such that e lays on $u - a$ path in LT : Then $d_{LT^s}(u, a) = d_{LT}(u, a) + 1$, implying that $h_{LT^s}(u) - h_{LT}(u) = \binom{d_u}{2} - 1$ since $d_a = 1$; see (1). Thus $h_{LT^s}(u) - h_{LT}(u) = -1$ if $d_u = 1$, $h_{LT^s}(u) - h_{LT}(u) = 0$ if $d_u = 2$ and $h_{LT^s}(u) - h_{LT}(u) \geq 2$ if $d_u \geq 3$.
- $u = w$: Since the degree of w is 2 in LT^s , we see that $h_{LT^s}(w) = -2$, by (1).

Every vertex u of degree 1 in LT is an endvertex of a ray starting at vertex \bar{u} of degree at least 3. By Observation 3.1, if u and v are distinct vertices of degree 1 in LT , then $\bar{u} \neq \bar{v}$. Denote by V_e the set of vertices u of LT such that $d_u = 1$ and e lays on $u - a$ path. Observe that $e \neq \bar{u}$ for any $u \in V_e$.

Denote $\Delta h = \sum_{u \in V(LT^s) \setminus \{a\}} h_{LT^s}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u)$. By the analysis above only vertices of $V_e \cup \{w\}$ contribute negative value to Δh . Therefore

$$\begin{aligned} \Delta h &\geq \sum_{u \in V_e} \left[\left(h_{LT^s}(u) - h_{LT}(u) \right) + \left(h_{LT^s}(\bar{u}) - h_{LT}(\bar{u}) \right) \right] \\ &\quad + \left(h_{LT^s}(e) - h_{LT}(e) \right) + h_{LT^s}(w) \\ &\geq \sum_{u \in V_e} (-1 + 2) + 2 - 2 \geq 0. \end{aligned}$$

Hence $\Delta T^s \geq \Delta T$. □

Let F be a tree with a ray of length $l + 1$ terminating in the edge a . Denote by S_{LF} the set of first edges of rays of F . Then S_{LF} is also a set of vertices of LF . These vertices have degree at least 3, with the exception when the corresponding edge is incident to vertices of degree 1 and 3 in F . Let $u \in S_{LF}$. If there is a ray in LF starting at u , then denote by $R_{LF}(u)$ the set of vertices (other than a) of this ray; otherwise set $R_{LF}(u) = \{u\}$. Since $l \geq 1$, there is a ray in LF starting at \bar{a} and $\bar{a} \neq s$. Observe also that $R_{LF}(u) \cap R_{LF}(v) = \emptyset$ whenever $u, v \in S_{LF}$, $u \neq v$.

Lemma 3.4 *Let F be a tree rays of which have length at most $l+2$, $l \geq 1$. Moreover, one ray of F has length exactly $l + 1$ and this ray terminates in the edge a . Let*

$c \in S_{LF}$ be a vertex of a clique from $\mathcal{C}(LF)$ of order $r \geq 3$. Then

$$\sum_{u \in R_{LF}(c)} h_{LF}(u) \geq \begin{cases} \binom{r}{2} l + \binom{r-1}{2} & \text{if } c = \bar{a}, \\ \binom{r-1}{2} d(c, a) + \binom{r-2}{2} - 2 & \text{if } c \neq \bar{a} \text{ and } |R_{LF}(c)| = 1, \\ \binom{r}{2} d(c, a) - 3l + \binom{r-1}{2} - 5 & \text{if } c \neq \bar{a} \text{ and } |R_{LF}(c)| \geq 2. \end{cases}$$

PROOF We distinguish three cases.

- $c = \bar{a}$: Then $R_{LF}(c)$ has one vertex of degree r , namely c with $d(c, a) = l$, and $l - 1$ vertices of degree 2. Since the degree of a is 1, by (1) we have

$$\begin{aligned} \sum_{u \in R_{LF}(c)} h_{LF}(u) &= \left(\binom{r}{2} - 1 \right) d(c, a) + (r - 1) \binom{r-2}{2} - 2 + (l - 1)(-2) \\ &= \left(\binom{r}{2} - 3 \right) l + \binom{r-1}{2}. \end{aligned}$$

- $c \neq \bar{a}$ and $|R_{LF}(c)| = 1$: Since the degree of c is $r - 1$, by (1) we obtain

$$\sum_{u \in R_{LF}(c)} h_{LF}(u) = h_{LF}(c) = \left(\binom{r-1}{2} - 1 \right) d(c, a) + (r - 2) \binom{r-3}{2} - 2.$$

- $c \neq \bar{a}$ and $|R_{LF}(c)| \geq 2$: Then $R_{LF}(c)$ has one vertex of degree r , namely c , one vertex of degree 1 at distance at most $d(c, a) + l + 1$ from a and at most l vertices of degree 2 since all rays of LF have length at most $l + 1$. By (1) it follows that

$$\begin{aligned} \sum_{u \in R_{LF}(c)} h_{LF}(u) &\geq \left(\binom{r}{2} - 1 \right) d(c, a) + (r - 1) \binom{r-2}{2} - 2 \\ &\quad - (d(c, a) + l + 1) - 2 + l(-2) \\ &= \left(\binom{r}{2} - 2 \right) d(c, a) - 3l + \binom{r-1}{2} - 5. \end{aligned}$$

□

Before we state the lemmas necessary for the basis of induction, we give the proof of induction step. That is, we prove that if $\Delta T^h \geq 0$ for every tree T^h homeomorphic to T rays of which have lengths at most $l + 2$, then $\Delta T^{gh} \geq 0$ for all trees T^{gh} homeomorphic to T^g rays of which have lengths at most $l + 2$, where T^g is

obtained from T by inserting a star at the end of one ray of T (of course, we cannot attach this star to a').

Let R' be a ray of T which does not terminate at a' . Remove R' from T and replace it by a path PR' of length i , $1 \leq i \leq 2$. Denote by c' the vertex of degree 1 in PR' . Now attach to c' exactly $j - 1$ rays, each of length at most $l + 2$, and denote the resulting graph by $T_{i,j}$, $j \geq 3$. In the next two lemmas we prove that $\Delta T_{i,j} \geq 0$.

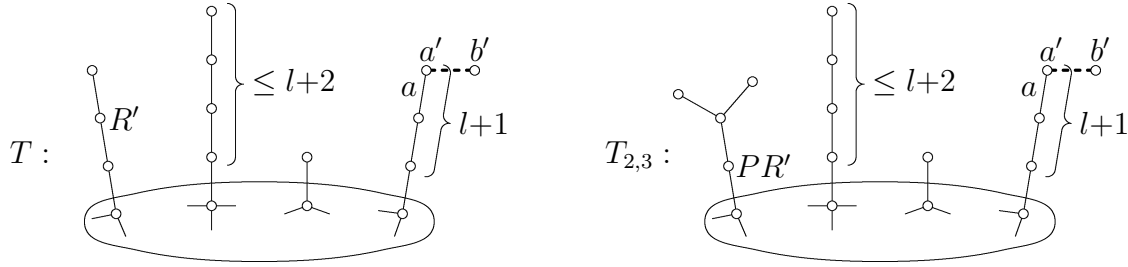


Figure 2: The trees T and $T_{2,3}$ in the case $l = 2$.

Lemma 3.5 *Suppose that $\Delta T^h \geq 0$ for all trees T^h homeomorphic to T rays of which have lengths at most $l + 2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l + 1$. Then $\Delta T_{i,3} \geq 0$ for all $i \in \{1, 2\}$.*

PROOF Since $\Delta T^h \geq 0$ for all trees homeomorphic to T rays of which have length at most $l + 2$, we may assume that the length of R' is exactly $l + 2$, $l \geq 1$. Then the edges of R' form a ray R in LT of length $l + 1$. Denote by e the first vertex of R . By (1) it follows that

$$\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -2l - (d(e, a) + l + 1) - 2 = -d(e, a) - 3l - 3$$

since $R_{LT}(e)$ has l vertices of degree 2 and one vertex of degree 1 at distance $d(e, a) + l + 1$ from a . We distinguish two cases.

- $i = 1$: Then PR' has length 1 and the unique edge of PR' corresponds to the vertex e in $LT_{1,3}$. In $LT_{1,3}$ the degree of e is $d_e + 3 - 2 = d_e + 1$ since e is in two cliques from $\mathcal{C}(LT_{1,3})$, one of them has order d_e and the other one has order 3. Denote by c any one of the other two vertices of this clique of order 3. Since $d(c, a) \geq l + 2$, we see that $d(c, a) - 3l - 4 \geq -2l - 2$. Hence, by Lemma 3.4,

$$\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{1,3}}(c)| = 1, \\ -2l - 2 & \text{if } |R_{LT_{1,3}}(c)| \geq 2. \end{cases}$$

Since $-2l - 2 \leq -2$, we have $\sum_{u \in R_{LT_{1,3}}(c)} h_{LT_{1,3}}(u) \geq -2l - 2$.

Denote

$$\Delta h = \sum_{u \in V(LT_{1,3}) \setminus \{a\}} h_{LT_{1,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In Δh all terms cancel out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing e , the vertex e itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that

$$\begin{aligned} \Delta h &\geq 2(-2l - 2) + \left(\binom{d_e + 1}{2} - 1 \right) d(e, a) + d_e \left(\frac{d_e + 1}{2} - 1 \right) - 2 \\ &\quad - \left(\binom{d_e}{2} - 1 \right) d(e, a) - (d_e - 1) \left(\frac{d_e}{2} - 1 \right) + 2 + (d(e, a) + 3l + 3) \\ &\geq (d_e + 1)d(e, a) + (d_e - 1) - l - 1 \\ &\geq 4d(e, a) - l + 1 \geq 0 \end{aligned}$$

since $d_e \geq 3$ and $d(e, a) \geq l + 1$. By Proposition 2.2, we have $\Delta T_{1,3} - \Delta T = \Delta h \geq 0$, implying that $\Delta T_{1,3} \geq \Delta T \geq 0$.

- $i = 2$: Then PR' has length 2. One edge of PR' corresponds to e , while the other corresponds to a vertex of degree 3, say f , in $LT_{2,3}$. Observe that the degree of e is d_e in $LT_{2,3}$ and the degree of f is 3 in $LT_{2,3}$. Analogously as in the previous case, denote by c any one of the two vertices of the triangle containing f , $c \neq f$. Since $d(c, a) = d(e, a) + 2 \geq l + 3$, we see that $d(c, a) - 3l - 4 \geq -2l - 1$. Hence, by Lemma 3.4

$$\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq \begin{cases} -2 & \text{if } |R_{LT_{2,3}}(c)| = 1, \\ -2l - 1 & \text{if } |R_{LT_{2,3}}(c)| \geq 2. \end{cases}$$

Since $l \geq 1$, it follows that $-2l - 1 \leq -2$, implying that $\sum_{u \in R_{LT_{2,3}}(c)} h_{LT_{2,3}}(u) \geq -2l - 1$. Denote

$$\Delta h = \sum_{u \in V(LT_{2,3}) \setminus \{a\}} h_{LT_{2,3}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u).$$

In Δh all terms cancel out, except the terms corresponding to vertices of rays starting at the clique of order 3 containing f , the vertex f itself, and the vertices of $R_{LT}(e) \setminus \{e\}$. By (1) it follows that

$$\begin{aligned} \Delta h &\geq 2(-2l - 1) + (2d(f, a) - 1) + (d(e, a) + 3l + 3) \\ &\geq 3d(e, a) - l + 2 \geq 0 \end{aligned}$$

since $d(f, a) = d(e, a) + 1$ and $d(e, a) \geq l + 1$. By Proposition 2.2, we have $\Delta T_{2,3} - \Delta T = \Delta h \geq 0$, implying that $\Delta T_{2,3} \geq \Delta T \geq 0$.

In both cases the inequality $\Delta T_{i,3} \geq 0$ holds, which completes the proof. \square

Now we extend the previous lemma to trees $T_{i,j}$ with higher j .

Lemma 3.6 *Suppose that $\Delta T^h \geq 0$ for all trees T^h homeomorphic to T rays of which have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Then $\Delta T_{i,j} \geq 0$ for all $j \geq 4$ and $i \in \{1, 2\}$.*

PROOF We use the notation of the proof of Lemma 3.5. Analogously as in the proof of Lemma 3.5, assume that the length of R' is $l+2$, $l \geq 1$. Then again

$$\sum_{u \in R_{LT}(e) \setminus \{e\}} h_{LT}(u) = -d(e, a) - 3l - 3.$$

Let c be one of the $j-1$ vertices of the clique of order j obtained from the edges incident to c' , other than e (in the case $i=1$) or f (in the case $i=2$). By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq \begin{cases} \binom{j-1}{2} d(c, a) + \binom{j-2}{2} - 2 & \text{if } |R_{LT_{i,j}}(c)| = 1, \\ \binom{j}{2} d(c, a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R_{LT_{i,j}}(c)| \geq 2. \end{cases}$$

Since $j \geq 4$ and $d(c, a) \geq l+2 \geq 3$, in any case we have $\sum_{u \in R_{LT_{i,j}}(c)} h_{LT_{i,j}}(u) \geq 0$. Now if $i=1$, then $h_{LT_{i,j}}(e) - h_{LT}(e) \geq 0$ since the degree of e is $d_e + j - 2$ in $T_{i,j}$; see (1). On the other hand, if $i=2$, then $h_{LT_{i,j}}(e) = h_{LT}(e)$ while $h_{LT_{i,j}}(f) \geq 0$, since the degree of f is $j \geq 4$ in $T_{i,j}$. Hence

$$\Delta h = \sum_{u \in V(LT_{i,j}) \setminus \{a\}} h_{LT_{i,j}}(u) - \sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u) \geq (j-1) \cdot 0 + 0 + d(e, a) + 3l + 3 \geq 0.$$

By Proposition 2.2, we have $\Delta T_{i,j} - \Delta T = \Delta h \geq 0$, showing that $\Delta T_{i,j} \geq \Delta T \geq 0$. \square

Now we prove $\Delta T \geq 0$ for the basis of induction. In all graphs in this basis, a' is an endvertex of a ray of length $l+1$ and a is the edge incident with a' .

Lemma 3.7 *Let T be a tree homeomorphic to a star $K_{1,k}$, $k \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Then $\Delta T \geq 0$.*

PROOF Here $|S_{LT}| = k$ and $\cup_{u \in S_{LT}} R_{LT}(u) = V(LT) \setminus \{a\}$ where $R_{LT}(u) \cap R_{LT}(v) = \emptyset$ if $u \neq v$. Thus $\sum_u h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u))$. We prove that $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$.

Choose $c \in S_{LT}$. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \begin{cases} \binom{k}{2} l + \binom{k-1}{2} & \text{if } c = \bar{a}, \\ \binom{k-1}{2} d(c, a) + \binom{k-2}{2} - 2 & \text{if } c \neq \bar{a} \text{ and } |R_{LT}(c)| = 1, \\ \binom{k}{2} d(c, a) - 3l + \binom{k-1}{2} - 5 & \text{if } c \neq \bar{a} \text{ and } |R_{LT}(c)| \geq 2. \end{cases}$$

Since $d(c, a) = l + 1$ in the last two cases, $k \geq 4$ and $l \geq 1$, in all three cases we conclude $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 1$.

We have $\sum_u h_{LT}(u) = \sum_{c \in S_{LT}} (\sum_{u \in R_{LT}(c)} h_{LT}(u)) \geq k \cdot 1 \geq 4$. Since $d_a = 1$, we see that $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, implying that $\Delta T \geq 0$. \square

Denote by $H_{i,j}$ a tree having $i+j$ vertices, where both i and j are at least 3. One of these vertices has degree i , another one has degree j and the remaining $i+j-2$ vertices have degrees 1. Obviously, the vertices of degrees i and j must be adjacent in $H_{i,j}$ and $H = H_{3,3}$.

Lemma 3.8 *Let T be a tree homeomorphic to $H_{3,j}$, $j \geq 4$, in which all rays have lengths at most $l+2$, $l \geq 1$, and in which the ray terminating in the edge a has length $l+1$. Suppose that the interior path of $H_{3,j}$ has length at most 2 and moreover suppose that the first vertex of the ray terminating in a has degree 3. Then $\Delta T \geq 0$.*

PROOF Denote $e = \bar{a}$ and let P' be the unique interior path of T . If P' has length 1, then the unique vertex of LP' (denote it by v) has degree $3+j-2 \geq 5$, while if P' has length 2, then one of the vertices of LP' has degree 3 and the other (denote it by v) has degree $j \geq 4$. Since by (1), $h_{LT}(u) \geq 0$ if $d_u \geq 3$ and $h_{LT}(u) \geq 5d(u, a) + 1$ if $d_u \geq 4$, the vertices of LP' contribute to $\sum_{u \in V(LT) \setminus \{a\}} h_{LT}(u)$ by at least $5d(v, a) + 1 \geq 5l + 6$ since $d(v, a) \geq l + 1$.

Denote by c any one of the $j-1$ vertices of the clique of order j from $\mathcal{C}(H_{3,j})$, which is not in LP' . By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq \begin{cases} \binom{j-1}{2} d(c, a) + \binom{j-2}{2} - 2 & \text{if } |R_{LT}(c)| = 1, \\ \binom{j}{2} d(c, a) - 3l + \binom{j-1}{2} - 5 & \text{if } |R_{LT}(c)| \geq 2. \end{cases}$$

Since $j \geq 4$ and $d(c, a) \geq l + 2 \geq 3$, in any case we have $\sum_{u \in R_{LT}(c)} h_{LT}(u) \geq 0$.

Now consider the rays attached to the clique of order 3 from $\mathcal{C}(H_{3,j})$. By Lemma 3.4

$$\sum_{u \in R_{LT}(e)} h_{LT}(u) = \left(\binom{3}{2} - 3 \right) l + \binom{3-1}{2} = 1.$$

Denote by f that vertex of the clique of order 3 from $\mathcal{C}(H_{3,j})$ which is different from e and which is not in LP' . By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(f)} h_{LT}(u) \geq \begin{cases} -2 & \text{if } |R_{LT}(f)| = 1, \\ d(f, a) - 3l - 4 & \text{if } |R_{LT}(f)| \geq 2. \end{cases}$$

Since $d(f, a) = l + 1$ and $l \geq 1$, in any case we have $\sum_{u \in R_{LT}(f)} h_{LT}(u) \geq -2l - 3$.

Now summing up the inequalities above we obtain

$$\sum_u h_{LT}(u) \geq (5l + 6) + (j - 1) \cdot 0 + 1 + (-2l - 3) = 3l + 4 \geq 3.$$

Since $d_a = 1$, we have $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, showing that $\Delta T \geq 0$. \square

Denote by $Y_{i,j}$, $1 \leq i, j \leq 2$, a tree having three vertices of degree 3, namely y'_1 , y'_2 and y'_3 . All the other vertices of $Y_{i,j}$ have degree at most 2. There are two interior paths in $Y_{i,j}$, namely $y'_1 - y'_2$ and $y'_2 - y'_3$, and their lengths are i and j , respectively. Moreover, there are five rays in $Y_{i,j}$. Two such rays start at y'_1 , one starts at y'_2 and two start at y'_3 . Of course, one of these rays has length exactly $l + 1$ and it terminates in a' .

Lemma 3.9 *Let T be the tree $Y_{i,j}$, $1 \leq i, j \leq 2$, in which all rays have lengths at most $l + 2$, $l \geq 1$. Then $\Delta T \geq 0$.*

PROOF Denote by x_1, x_2, x_3, x_4 and x_5 the five vertices of S_{LT} corresponding to the first edges of rays starting at y'_1, y'_1, y'_2, y'_3 and y'_3 , respectively. Since the degrees of y'_1, y'_2 and y'_3 are 3 in T , all x_1, x_2, \dots, x_5 are vertices of cliques of order 3 in LT . Let $x_t = \bar{a}$, $1 \leq t \leq 5$. By Lemma 3.4

$$\sum_{u \in R_{LT}(x_t)} h_{LT}(u) = 1.$$

For all other x_r , $1 \leq r \leq 5$ and $r \neq t$, by Lemma 3.4 we obtain

$$\sum_{u \in R_{LT}(x_r)} h_{LT}(u) \geq \min\{-2, d(x_r, a) - 3l - 4\}.$$

Since $l \geq 1$, this minimum equals $d(x_r, a) - 3l - 4$ if $d(x_r, a) \leq l + 4$. If $d(x_r, a) = l + 5$ then $\sum_{u \in R_{LT}(x_r)} h_{LT}(u) \geq \min\{-2, -2l + 1\} \geq -2l$.

Now we consider vertices corresponding to the edges of interior paths. If such a path has length 1, then its unique edge corresponds to a vertex, say e , the degree of which is 4 in LT . By (1) it follows that

$$h_{LT}(e) = 5d(e, a) + 1.$$

On the other hand, if such a path has length 2, then its edges correspond to two vertices, say e and f , both of degree 3. Suppose that e is closer to a than f . By (1) it follows that

$$h_{LT}(e) + h_{LT}(f) = 2d(e, a) - 1 + 2d(f, a) - 1 = 4d(e, a).$$

In what follows, we list contributions to $\sum_u h_{LT}(u)$ first by vertices of rays starting at x_1, x_2, \dots, x_5 and then by the vertices corresponding to edges of paths $y'_1 - y'_2$ and $y'_2 - y'_3$. By symmetry, there are two cases to consider. First, suppose that $t = 1$, that is, $\bar{a} = x_1$. We distinguish 4 subcases.

- $i = j = 1$: Then $d(x_2, a) = l + 1$, $d(x_3, a) = l + 2$ and $d(x_4, a) = d(x_5, a) = l + 3$. Since $l \geq 1$, we see that

$$\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (-2l - 2) + 2(-2l - 1) + (5l + 6) + (5l + 11) \geq 2l + 11 \geq 3.$$

- $i = 1$ and $j = 2$: Analogously as above we obtain

$$\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (-2l - 2) + 2(-2l) + (5l + 6) + (4l + 8) \geq l + 10 \geq 3.$$

- $i = 2$ and $j = 1$: Then

$$\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (-2l - 1) + 2(-2l) + (4l + 4) + (5l + 16) \geq l + 17 \geq 3.$$

- $i = j = 2$: Here $d(x_4, a) = d(x_5, a) = l + 5$. Hence

$$\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (-2l - 1) + 2(-2l) + (4l + 4) + (4l + 12) \geq 13 \geq 3.$$

Now suppose that $t = 3$, that is, $\bar{a} = x_3$. By symmetry, it suffices to consider 3 subcases.

- $i = j = 1$: Then $d(x_1, a) = d(x_2, a) = l + 2$ and also $d(x_4, a) = d(x_5, a) = l + 2$. Since $l \geq 1$, we see that

$$\sum_u h_{LT}(u) \geq 2(-2l - 2) + 1 + 2(-2l - 2) + (5l + 6) + (5l + 6) \geq 2l + 5 \geq 3.$$

- $i = 1$ and $j = 2$: Then the following holds

$$\sum_u h_{LT}(u) \geq 2(-2l - 2) + 1 + 2(-2l - 1) + (5l + 6) + (4l + 4) \geq l + 5 \geq 3.$$

- $i = j = 2$: Then

$$\sum_u h_{LT}(u) \geq 2(-2l - 1) + 1 + 2(-2l - 1) + (4l + 4) + (4l + 4) \geq 5 \geq 3.$$

Since $\Delta T = \sum_u h_{LT}(u) - 3$ by Proposition 2.2, we conclude that $\Delta T \geq 0$. \square

Now we prove $\Delta T \geq 0$ for the last graph of the basis of induction. Denote by X_k , $k \geq 4$, the tree having two vertices of degree 3, namely y'_1 and y'_2 , and one vertex of degree k , namely y'_3 . All other vertices of X_k have degree at most 2. There are two interior paths in X_k , namely $y'_1 - y'_2$ and $y'_2 - y'_3$, both of length at most 2. Moreover, there are $k + 2$ rays in X_k . Two such rays start at y'_1 , one starts at y'_2 and the remaining $k - 1$ start at y'_3 .

Lemma 3.10 *Let T be the tree X_k , $k \geq 4$, in which all rays have lengths at most $l + 2$, $l \geq 1$. Suppose that the ray having length $l + 1$ and terminating at a' starts at y'_1 . Then $\Delta T \geq 0$.*

PROOF We use the notation of the proof of Lemma 3.9. Denote by $x_1, x_2, x_3, x_4, \dots, x_{k+2}$ the $k + 2$ vertices of S_{LT} corresponding to the first edges of rays starting at $y'_1, y'_1, y'_2, y'_3, \dots, y'_3$, respectively. The vertices x_1, x_2 and x_3 are in cliques of order 3, while x_4, \dots, x_{k+2} are in the clique of order k . Assume that $\bar{a} = x_1$. As shown in the proof of Lemma 3.9, we have $\sum_{u \in R_{LT}(x_1)} h_{LT}(u) = 1$. Further, $\sum_{u \in R_{LT}(x_2)} h_{LT}(u) \geq -2l - 3$ since $d(x_2, a) = l + 1$. The vertices corresponding to edges of $y'_1 - y'_2$ path contribute to $\sum_u h_{LT}(u)$ by at least $\min\{5d(e, a) + 1, 4d(e, a)\} = 4d(e, a) = 4l + 4$ as $d(e, a) = l + 1$. Finally, $\sum_{u \in R_{LT}(x_3)} h_{LT}(u) \geq \min\{-2, d(x_3, a) - 3l - 4\} \geq -2l - 2$ as $d(x_3, a) \geq l + 2$.

Since the vertices corresponding to edges of $y'_2 - y'_3$ path have degree $k + 1$ (in the case when the length of $y'_2 - y'_3$ is 1) or 3 and k (in the case when the length of $y'_2 - y'_3$ is 2), and since $h_{LT}(u) \geq 0$ if $d_u \geq 3$ by (1), the contribution of these vertices to $\sum_u h_{LT}(u)$ is nonnegative.

Finally, consider $\sum_{u \in R_{LT}(x_i)} h_{LT}(u)$ when $i \geq 4$. By Lemma 3.4 it follows that

$$\sum_{u \in R_{LT}(x_i)} h_{LT}(u) \geq \begin{cases} \left(\binom{k-1}{2} - 1 \right) d(x_i, a) + \binom{k-2}{2} - 2 & \text{if } |R_{LT}(x_i)| = 1, \\ \left(\binom{k}{2} - 2 \right) d(x_i, a) - 3l + \binom{k-1}{2} - 5 & \text{if } |R_{LT}(x_i)| \geq 2. \end{cases}$$

Since $d(x_i, a) \geq l + 3$, $k \geq 4$ and $l \geq 1$, we obtain $\sum_{u \in R_{LT}(x_i)} h_{LT}(u) \geq \min\{7, 11\} = 7$.

Summing up these inequalities we obtain

$$\sum_u h_{LT}(u) \geq 1 + (-2l - 3) + (4l + 4) + (-2l - 2) + 0 + (k - 1)7 = 7k - 7 \geq 3.$$

Since $d_a = 1$, by Proposition 2.2 we conclude that $\Delta T = \sum_u h_{LT}(u) - 3 \geq 0$. \square

Now we summarize the proof of Theorem 1.5

PROOF OF THEOREM 1.5 Let T be a tree, not homeomorphic to a path, claw $K_{1,3}$ and H . We prove that $D(T) = W(L^3(T)) - W(T) > 0$. Denote by $l+2$, $l \geq -1$, the length of a longest ray in T . If $l = -1$ then $D(T) = W(L^3(T)) - W(T) > 0$ by Theorem 1.4.

Suppose that $l \geq 0$ and suppose that the statement of the theorem is true for all trees (not homeomorphic to a path, claw $K_{1,3}$ and H) rays of which have lengths at most $l+1$. Let R'_1, R'_2, \dots, R'_t be rays of T having length $l+2$. Further, denote by c'_i the first vertex of R'_j , denote by b'_i its last vertex and denote by a'_i the neighbour of b'_i in T , $1 \leq i \leq t$. Finally, denote by T_i a tree obtained from T by removing the vertices $b'_{i+1}, b'_{i+2}, \dots, b'_t$ and edges $a'_{i+1}b'_{i+1}, a'_{i+2}b'_{i+2}, \dots, a'_t b'_t$, $0 \leq i \leq t$. Then $T_t = T$ and T_0 is a tree rays of which have length at most $l+1$. By induction we have $D(T_0) = W(L^3(T_0)) - W(T_0) > 0$. Let $\Delta T_i = D(T_{i+1}) - D(T_i)$, $0 \leq i \leq t-1$.

Suppose that $l = 0$. All rays of T_i have length at most $l+2$, and the ray R'_{i+1} terminating at a'_{i+1} has length $l+1$. Moreover, T_{i+1} is obtained from T_i by adding the vertex b'_{i+1} and the edge $a'_{i+1}b'_{i+1}$. Hence $\Delta T_i \geq 0$ by Lemma 3.2, $0 \leq i \leq t-1$, where the vertex a'_{i+1} and the tree T_i play the role of a and T , respectively. Consequently $\sum_{i=0}^{t-1} \Delta T_i \geq 0$. Since

$$0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we have $W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0$.

Now suppose that $l \geq 1$. In T_i shorten all interior paths of length at least 3 to paths of length 2, and denote the resulting graph by T_i^- . Analogously as T_{i+1} is obtained from T_i , the tree T_{i+1}^- is obtained from T_i^- by adding the vertex b'_{i+1} and the edge $a'_{i+1}b'_{i+1}$. We prove that $\Delta T_i^- = D(T_{i+1}^-) - D(T_i^-) \geq 0$ by induction on the number of vertices of degree at least 3. Observe that T_i^- , as well as T_i , is a tree, rays of which have length at most $l+2$ and the ray terminating at a'_{i+1} has length $l+1$, $0 \leq i \leq t-1$.

Denote by V_i^3 the set of vertices of degree at least 3 in T_i^- . We distinguish four cases.

- $|V_i^3| = 1$: Then T_i^- is homeomorphic to $K_{1,k}$. Since T is not homeomorphic to $K_{1,3}$, it follows that $k \geq 4$. By Lemma 3.7 we have $\Delta T_i^- \geq 0$.
- $|V_i^3| = 2$: If the degree of c'_{i+1} is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.8, since T is not homeomorphic to $H = H_{3,3}$. On the other hand if the degree of c'_{i+1} is $k \geq 4$,

then denote by c'' the other vertex of V_i^3 . Remove the rays starting at c'' from T_i^- , and denote the resulting graph by T'' . Then T'' is a tree, rays of which have length at most $l+2$, and T'' is homeomorphic to $K_{1,k}$. By Lemma 3.7 we have $\Delta T'' \geq 0$. If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5, while if the degree of c'' is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

- $|V_i^3| = 3$: Denote by T^* a graph obtained from T_i^- by removing the edges of all rays. Since T^* is a tree, it has at least two vertices of degree 1. (We remark that in this case T^* is a path.) Denote by c'' a pendant vertex in T^* , $c'' \neq c'_{i+1}$, the degree of which is the smallest possible in T_i^- . Finally, denote by T'' a tree obtained from T_i^- by removing all rays starting at c'' . We distinguish two subcases.
 - T'' is homeomorphic to H : If the degree of c'' is 3 in T then $\Delta T_i^- \geq 0$ by Lemma 3.9. Suppose that the degree of c'' is k , $k \geq 4$. By the choice of c'' , the vertex c'_{i+1} is a leaf of T^* . Hence T is X_k and c'_{i+1} is the vertex y'_1 in the notation of Lemma 3.10. Therefore $\Delta T_i^- \geq 0$ by Lemma 3.10.
 - T'' is homeomorphic to $H_{i,j}$, $i \leq j$ and $j \geq 4$: Since T'' is not homeomorphic to H , it follows that $\Delta T'' \geq 0$ by the previous case (the case $|V_i^3| = 2$). If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5; while if the degree of c'' is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

Thus we proved $\Delta T_i^- \geq 0$ for every tree T_i^- rays of which have length at most $l+2$ and $|V_i^3| = 3$.

- $|V_i^3| \geq 4$: Analogously as in the previous case, denote by T'' a tree obtained from T_i^- by removing all rays starting at a pendant vertex c'' of T^* , $c'' \neq c'_{i+1}$. By induction we assume that $\Delta T'' \geq 0$. If the degree of c'' is 3, then $\Delta T_i^- \geq 0$ by Lemma 3.5, while if the degree of c'' is at least 4, then $\Delta T_i^- \geq 0$ by Lemma 3.6.

Hence, in any case we have $\Delta T_i^- \geq 0$. If $T_i^- = T_i$, then it follows that also $\Delta T_i \geq 0$. Otherwise form a sequence $T_i^- = F_0, F_1, \dots, F_r = T_i$ such that F_{j+1} is obtained from F_j by subdividing one edge of one interior path, $0 \leq j \leq r-1$. By Lemma 3.3 we have $\Delta F_{j+1} - \Delta F_j \geq 0$. Hence $\sum_{j=0}^{r-1} (\Delta F_{j+1} - \Delta F_j) \geq 0$. Since

$$0 \leq \sum_{j=0}^{r-1} (\Delta F_{j+1} - \Delta F_j) = \Delta T_i - \Delta T_i^-,$$

we see that $\Delta T_i \geq \Delta T_i^- \geq 0$.

Thus we proved that $\Delta T_i \geq 0$ for every $i \in \{0, 1, \dots, t-1\}$. Hence $\sum_{i=0}^{t-1} \Delta T_i \geq 0$. Since

$$0 \leq \sum_{i=0}^{t-1} \Delta T_i = D(T_t) - D(T_0) = [W(L^3(T)) - W(T)] - [W(L^3(T_0)) - W(T_0)],$$

we conclude that $W(L^3(T)) - W(T) \geq W(L^3(T_0)) - W(T_0) > 0$. □

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References

- [1] F. Buckley, Mean distance in line graphs, *Congr. Numer.* **32** (1981), 153–162.
- [2] A. A. Dobrynin, Distance of iterated line graphs *Graph Theory Notes New York* **37** (1999), 50–54.
- [3] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: Theory and applications *Acta Appl. Math.* **66**(3) (2001), 211–249.
- [4] A. A. Dobrynin, I. Gutman, S. Klavžar, P. Žigert, Wiener index of hexagonal systems, *Acta Appl. Math.* **72** (2002), 247–294.
- [5] A. A. Dobrynin, L. S. Meĭnikov, Some results on the Wiener index of iterated line graphs, *Electronic notes in Discrete Mathematics* **22** (2005), 469–475.
- [6] R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, *Czechoslovak Math. J.* **26** (1976), 283–296.
- [7] I. Gutman, S. Klavžar, B. Mohar (eds), Fifty years of the Wiener index, *MATCH Commun Math. Comput. Chem.* **35** (1997), 1–259.
- [8] I. Gutman, S. Klavžar, B. Mohar (eds), Fiftieth Anniversary of the Wiener index, *Discrete Appl. Math.* **80**(1) (1997), 1–113.
- [9] I. Gutman, I. G. Zenkevich, Wiener index and vibrational energy, *Z. Naturforsch.* **57** A (2002), 824–828.
- [10] L. Niepel, M. Knor, L. Šoltés, Distances in iterated line graphs, *Ars Combinatoria* **43** (1996), 193–202.
- [11] M. Knor, P. Potočnik, R. Škrekovski, Wiener index in iterated line graphs, *submitted*, (see also IMFM preprint series **48**, article number 1128 (2010), <http://www.imfm.si/preprinti/index.php?langID=1>).
- [12] M. Knor, P. Potočnik, R. Škrekovski, Wiener index of iterated line graphs of trees homeomorphic to the claw $K_{1,3}$, *in preparation*.

- [13] M. Knor, P. Potočník, R. Škrekovski, Wiener index of iterated line graphs of trees homeomorphic to “ H ”, *in preparation*.
- [14] J. Plesník, On the sum of all distances in a graph or digraph, *J. Graph Theory* **8** (1984), 1–21.
- [15] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69**(1947), 17–20.