An enumeration of minimum genus orientable embeddings of some complete bipartite graphs.

M. J. Grannell
Department of Mathematics and Statistics
The Open University
Walton Hall, Milton Keynes MK7 6AA
UNITED KINGDOM
(m.j.grannell@open.ac.uk)

M. Knor
Department of Mathematics, Faculty of Civil Engineering
Slovak University of Technology
Radlinského 11, 813 68 Bratislava
SLOVAKIA
(knor@math.sk)

January 8, 2010

Abstract

We enumerate nonisomorphic minimum genus orientable embeddings of the complete bipartite graph $K_{m,n}$ for $2 \leq m, n \leq 7$ except for $(m, n) = (7, 7)$.

AMS classification: 05C10.
Keywords: Minimum genus embedding, complete bipartite graph.

1 Introduction

It was shown by Ringel [2] that a minimum genus orientable embedding of the complete bipartite graph $K_{m,n}$ $(m, n \geq 2)$ has genus g given by the formula

$$g = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

The number of faces f in such an embedding is given by Euler's formula as $f = mn + 2 - (m + n) - 2g$. If f_i denotes the number of faces of length i then

$f = \sum_{j\geq 2} f_{2j}$, and by counting their edges we also have $2mn = \sum_{j\geq 2} 2jf_{2j}$. From these equations we deduce that, in a minimum genus orientable embedding of $K_{m,n}$,

$$f_6 + 2f_8 + 3f_{10} + \ldots = 4g - (m-2)(n-2).$$

If we put $h = h(m,n) = 4g-(m-2)(n-2)$ then $h = 0, 1, 2$ or 3; we will refer to h as the *excess*. When $h = 0$, all faces are 4-gons, that is to say the embedding is quadrangular. When $h = 1$, there is one 6-gon and all remaining faces are 4-gons. When $h = 2$ there are two possibilities, namely one 8-gon or two 6-gons, with all remaining faces being 4-gons in both cases. When $h = 3$ there are three possibilities, namely one 10-gon, or one 8-gon and one 6-gon, or three 6-gons, with all remaining faces being 4-gons in each case. Table 1 shows the values of (g,h) for $2 \leq m \leq n \leq 7$.

$m \diagdown n$	2	3	4	5	6	7
2	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
3		$(1,3)$	$(1,2)$	$(1,1)$	$(1,0)$	$(2,3)$
4			$(1,0)$	$(2,2)$	$(2,0)$	$(3,2)$
5				$(3,3)$	$(3,0)$	$(4,1)$
6					$(4,0)$	$(5,0)$
7						$(7,3)$

Table 1. Genus and excess (g,h).

With the exception of the case $(m,n) = (7,7)$ we enumerate all nonisomorphic minimum genus orientable embeddings of $K_{m,n}$ for $2 \leq m, n \leq 7$. Somewhat to our surprise, no such enumeration seems to have been previously undertaken. Without loss of generality, it can be assumed that $m \leq n$. We will always take the bipartition $\{A, B\}$ where $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. We assume that the reader is familiar with the representation of embeddings by means of rotation schemes where the notation $a_i : b_1, b_2, \ldots, b_n$ means that at the point a_i the neighbouring points in cyclic order are $b_1, b_2, \ldots, b_n$. For background material on topological graph theory, we refer the reader to [1].

In determining possible isomorphisms between orientable embeddings when $m \neq n$, there are m possible images for a_1 and taking one of these, say a'_i, the rotations at a_1 and a'_i give n possible images for b_1 with either the orientation preserved or reversed (in the representation of embeddings by rotation schemes, reversal is only meaningful if $n > 2$). The rest of the potential isomorphism is then determined uniquely. Thus for $m \neq n$ (and $n > 2$), to determine if two rotation schemes represent isomorphic embeddings, there are just $2mn$ possible mappings to consider. When $m = n > 2$, there is the additional possibility of exchanging the vertex parts, and the number of mappings requiring examination rises to $4n^2$. A similar argument applies to the determination of automorphisms. An orientable embedding of $K_{n,n}$ is said to be *regular* if the order of the full automorphism group is as large as possible, namely $4n^2$.

2 Enumeration

We start by examining some particularly simple cases, namely when $m = 2$ and when $(m, n) = (3, 3)$ or $(3, 4)$. We then move on to the more general cases.

When $m = 2$, all faces are 4-gons and the embedding is planar. Figure 1 makes it clear that, up to isomorphism, there is just one minimum genus embedding of $K_{2,n}$ for each n.

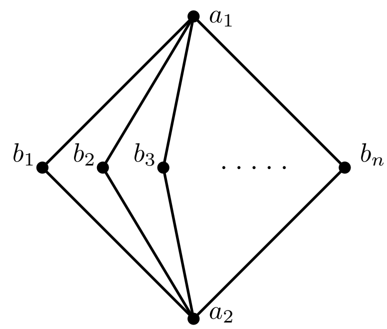

Figure 1: Quadrangular embedding of $K_{2,n}$.

In the case $(m, n) = (3, 3)$, we have $f = 3$ and we may assume that the rotation scheme has the outline form

$$\begin{array}{lllll} a_1 & : & b_1 & b_2 & b_3 \\ a_2 & : & b_1 & * & * \\ a_3 & : & b_1 & * & * \end{array} \qquad \begin{array}{lllll} b_1 & : & a_1 & a_2 & a_3 \\ b_2 & : & a_1 & * & * \\ b_3 & : & a_1 & * & * \end{array}$$

There are potentially 16 ways to complete this outline form, but this number is reduced to ten by taking account of the symmetry between A and B. It is easy to check by hand that exactly seven of these ten completions give a minimum genus embedding (the others give non-minimum genus embeddings). The seven solutions fall into two isomorphism classes, representatives of which are

$$\begin{array}{lllll} a_1 & : & b_1 & b_2 & b_3 \\ a_2 & : & b_1 & b_2 & b_3 \\ a_3 & : & b_1 & b_2 & b_3 \end{array} \qquad \begin{array}{lllll} b_1 & : & a_1 & a_2 & a_3 \\ b_2 & : & a_1 & a_2 & a_3 \\ b_3 & : & a_1 & a_2 & a_3 \end{array}$$

and

$$\begin{array}{lllll} a_1 & : & b_1 & b_2 & b_3 \\ a_2 & : & b_1 & b_2 & b_3 \\ a_3 & : & b_1 & b_3 & b_2 \end{array} \qquad \begin{array}{lllll} b_1 & : & a_1 & a_2 & a_3 \\ b_2 & : & a_1 & a_2 & a_3 \\ b_3 & : & a_1 & a_3 & a_2 \end{array}$$

The former corresponds to three 6-gon faces with the embedding having an automorphism group of order 36, where 9 automorphisms preserve the bipartition and orientation, 9 preserve the bipartition and reverse the orientation, 9 exchange the bipartition and preserve the orientation, and 9 exchange the bipartition and reverse the orientation. The latter corresponds to one 10-gon and two 4-gons with the embedding having an automorphism group of order 4, where there is one automorphism of each of the four types.

In the case $(m, n) = (3, 4)$, we have $f = 5$ and so at least one face is a 4-gon. Without loss of generality therefore, we may assume that the embedding contains the geometric configuration shown in Figure 2 where $a_1b_1a_2b_4$ is one of the faces.

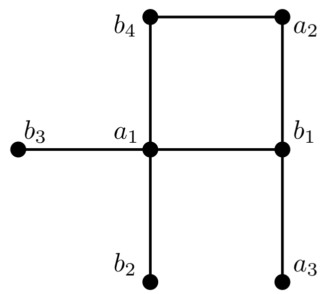

Figure 2: Configuration in an embedding of $K_{3,4}$.

Figure 2 gives the outline rotation scheme

$$\begin{array}{ccccccc} a_1 & : & b_1 & b_2 & b_3 & b_4 \\ a_2 & : & b_1 & b_4 & * & * \\ a_3 & : & b_1 & * & * & * \end{array} \qquad \begin{array}{ccccc} b_1 & : & a_1 & a_2 & a_3 \\ b_2 & : & a_1 & * & * \\ b_3 & : & a_1 & * & * \\ b_4 & : & a_1 & a_3 & a_2 \end{array}$$

This gives rise to eight minimum genus embeddings in just three isomorphism classes. A representative of each class is given below. The face type is described by $(f; f_4, f_6, f_8, f_{10})$ and $|Aut|$ gives the order of the automorphism group. In each case, half of the automorphisms preserve the orientation and the other half reverse it.

Class 1, face type $(5; 3, 2, 0, 0)$, $|Aut| = 6$.
A-rotations: $a_1 : b_1b_2b_3b_4$, $a_2 : b_1b_4b_2b_3$, $a_3 : b_1b_2b_4b_3$.
B-rotations: $b_1 : a_1a_2a_3$, $b_2 : a_1a_2a_3$, $b_3 : a_1a_2a_3$, $b_4 : a_1a_3a_2$.
Class 2, face type $(5; 4, 0, 1, 0)$, $|Aut| = 4$.
A-rotations: $a_1 : b_1b_2b_3b_4$, $a_2 : b_1b_4b_2b_3$, $a_3 : b_1b_3b_2b_4$.
B-rotations: $b_1 : a_1a_2a_3$, $b_2 : a_1a_2a_3$, $b_3 : a_1a_3a_2$, $b_4 : a_1a_3a_2$.
Class 3, face type $(5; 4, 0, 1, 0)$, $|Aut| = 8$.
A-rotations: $a_1 : b_1b_2b_3b_4$, $a_2 : b_1b_4b_3b_2$, $a_3 : b_1b_2b_3b_4$.
B-rotations: $b_1 : a_1a_2a_3$, $b_2 : a_1a_3a_2$, $b_3 : a_1a_2a_3$, $b_4 : a_1a_3a_2$.

In all other cases under consideration, $mn > 6f_6 + 8f_8 + 10f_{10}$ and consequently there is an edge which lies in two 4-gons. Then, without loss of generality, we may assume that the embedding contains the geometric configuration shown in Figure 3 where $a_1b_1a_2b_n$ and $a_mb_1a_1b_2$ are adjacent faces.

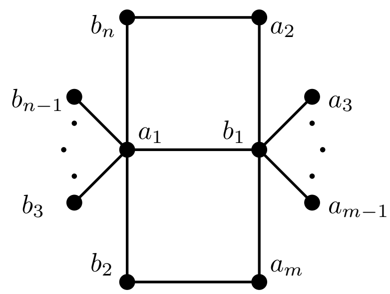

Figure 3: Configuration in an embedding of $K_{m,n}$
$(3 \le m \le n,\ (m,n) \ne (3,3),(3,4))$.

Figure 3 gives the outline rotation scheme

$$\begin{array}{ccccccccccccc}
a_1 & : & b_1 & b_2 & \dots & b_n & \qquad & b_1 & : & a_1 & a_2 & \dots & a_m \\
a_2 & : & b_1 & b_n & *** & * & & b_2 & : & a_1 & a_m & *** & * \\
a_3 & : & b_1 & * & *** & * & & b_3 & : & a_1 & * & *** & * \\
\vdots & & & & & & & \vdots & & & & & \\
a_m & : & b_1 & * & *** & b_2 & & b_n & : & a_1 & * & *** & a_2
\end{array}$$

The number of possible completions of the A-rotations is $((n-2)!)^2((n-1)!)^{m-3}$ and the number of possible completions of the B-rotations is $((m-2)!)^2((m-1)!)^{n-3}$. Potentially it seems necessary to examine each pair of these A- and B-rotations to see if a minimum genus embedding results. However, some simplifications are possible. Firstly, if $m = n$ we may exploit the symmetry between A and B.

Secondly, if there is an A-point and a B-point that only appear in 4-gons then we may take these points to be a_1 and b_1, and so if the rotation at a_i is $a_i : b_1 \dots b_j$, then the rotation at a_{i+1} is $a_{i+1} : b_1 b_j \dots$. A similar argument applies to the B-rotations. This considerably reduces the number of possible A- and B-rotations, and is particularly useful in the cases $(m,n) = (5,7)$ and $(6,7)$.

Thirdly, let $p(i,j)$ be the number of times that b_i immediately precedes b_j in the cyclic order of the A-rotations. If $h(m,n) = 0$ (that is, the embedding is quadrangular) then we must have $p(i,j) = p(j,i)$ because each 4-gon containing both b_i and b_j will have one vertex a_k with rotation $a_k : \dots b_i b_j \dots$ and another vertex a_l with rotation $a_l : \dots b_j b_i \dots$. Hence, if $h(m,n) = 0$ we require $S = \sum_{1 \le i < j \le n} |p(i,j) - p(j,i)| = 0$. When $h(m,n) > 0$ some degree of imbalance is possible. If $h(m,n) = 1$, at most three consecutive pairs $b_i b_j$ can be unbalanced and so, with S as defined previously, we require $S \le 3$. Similarly, if $h(m,n) = 2$, we require $S \le 6$ and if $h(m,n) = 3$, we require $S \le 9$. In all cases this reduces to $S \le 3h(m,n)$. The same procedure may be applied to the B-rotations.

For $m \le n \le 7$, apart from $(m,n) = (7,7)$, these simplifications enable us to compute all possible minimum genus orientable embeddings of $K_{m,n}$. As an example, consider the case $(m,n) = (6,7)$. The formulas above give the potential numbers of A- and B-rotations as 5 374 771 200 000 and 119 439 360 000 respectively. Applying the two simplifications reduces these numbers to 239 310 and 44 191 respectively. Of the $239\,310 \times 44\,191$ potential rotation schemes that result, only 35 882 actually generate minimum genus orientable embeddings. With this relatively modest number of candidates, it is then easy to determine that there are 584 isomorphism classes and to determine their automorphisms.

Table 2 summarizes the results. In each case we give the face types and number of isomorphism classes of each face type. All the results have been obtained by two independently written computer programs.

(m,n)	Face type	Number of isomorphism classes
$(3,3)$	$(3;2,0,0,1)$	1
	$(3;0,3,0,0)$	1
$(3,4)$	$(5;4,0,1,0)$	2
	$(5;3,2,0,0)$	1
$(3,5)$	$(7;6,1,0,0)$	1
$(3,6)$	$(9;9,0,0,0)$	1
$(3,7)$	$(9;8,0,0,1)$	5
	$(9;7,1,1,0)$	12
	$(9;6,3,0,0)$	8
$(4,4)$	$(8;8,0,0,0)$	2
$(4,5)$	$(9;8,0,1,0)$	9
	$(9;7,2,0,0)$	6
$(4,6)$	$(12;12,0,0,0)$	5
$(4,7)$	$(13;12,0,1,0)$	71
	$(13;11,2,0,0)$	102
$(5,5)$	$(11;10,0,0,1)$	15
	$(11;9,1,1,0)$	89
	$(11;8,3,0,0)$	63
$(5,6)$	$(15;15,0,0,0)$	6
$(5,7)$	$(17;16,1,0,0)$	204
$(6,6)$	$(18;18,0,0,0)$	53
$(6,7)$	$(21;21,0,0,0)$	584

Table 2. Summary of results.

It is not feasible to list further details of all the embeddings here but they are available from the authors with a representative rotation scheme for each isomorphism class and the automorphism group type. The overwhelming majority of the embeddings have small automorphism groups. Only eight of the embeddings in Table 2 have an automorphism group of order mn or greater. These are given in Table 3 with their automorphism group types $(t; t_1, t_2, t_3, t_4)$ where t is the total number of automorphisms, t_1 the number that preserve the bipartition and orientation, t_2 the number that preserve the bipartition and reverse the orientation, t_3 the number that exchange the bipartition and preserve the orientation, and t_4 the number that exchange the bipartition and reverse the orientation. The rotations in Table 3 are in a compact form where, for example, the entry for $(m, n) = (3, 6)$ that reads $A : \ldots, 163254, \ldots$ denotes the rotation $a_2 : b_1b_6b_3b_2b_5b_4$. Regular embeddings, those where $m = n$ and $|Aut| = 4n^2$, are noted.

(m, n)	Rotation scheme	Face type	Group type
$(3, 3)$	$A : 123, 123, 123.$ $B : 123, 123, 123.$	$(3; 0, 3, 0, 0)$	$(36; 9, 9, 9, 9)$ regular
$(3, 6)$	$A : 123456, 163254, 143652.$ $B : 123, 132, 123, 132, 123, 132.$	$(9; 9, 0, 0, 0)$	$(36; 18, 18, 0, 0)$
$(4, 4)$	$A : 1234, 1423, 1324, 1432.$ $B : 1234, 1423, 1324, 1432.$	$(8; 8, 0, 0, 0)$	$(32; 8, 8, 8, 8)$
$(4, 4)$	$A : 1234, 1432, 1234, 1432.$ $B : 1234, 1432, 1234, 1432.$	$(8; 8, 0, 0, 0)$	$(64; 16, 16, 16, 16)$ regular
$(4, 6)$	$A : 123456, 163254, 145236,$ $165432.$ $B : 1234, 1432, 1234, 1432,$ $1234, 1432.$	$(12; 12, 0, 0, 0)$	$(24; 12, 12, 0, 0)$
$(4, 6)$	$A : 123456, 165432, 123456,$ $165432.$ $B : 1234, 1432, 1234, 1432,$ $1234, 1432.$	$(12; 12, 0, 0, 0)$	$(48; 24, 24, 0, 0)$
$(6, 6)$	$A : 123456, 163254, 143652,$ $123456, 163254, 143652.$ $B : 123456, 165432, 123456,$ $165432, 123456, 165432.$	$(18; 18, 0, 0, 0)$	$(72; 36, 36, 0, 0)$
$(6, 6)$	$A : 123456, 165432, 123456,$ $165432, 123456, 165432.$ $B : 123456, 165432, 123456,$ $165432, 123456, 165432.$	$(18; 18, 0, 0, 0)$	$(144; 36, 36, 36, 36)$ regular

Table 3. Minimum genus orientable embeddings with $|Aut| \geq mn$.

Consideration was given to enumerating minimum genus orientable embeddings for $K_{7,7}$. There are three possible face types, namely $(23; 22, 0, 0, 1)$, $(23; 21, 1, 1, 0)$ and $(23; 20, 3, 0, 0)$. In the last two of these cases it is conceivable that no A- or B-points appear only in 4-gons and so one of the simplifying assumptions to which we referred above does not apply in these cases. To obtain all solutions for $(m, n) = (7, 7)$ by the current method would be likely to take several weeks of computer time with our current processing speed and require larger amounts of computer memory than we have available.

Acknowledgement This work commenced when the second author was visiting the first author and he thanks him for his hospitality. He also acknowledges partial support by Slovak research grants VEGA 1/0489/08, APVT-20-000704 and APVV-0040-06.

References

[1] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, John Wiley, New York (1987).

[2] G. Ringel, Das Geschlecht des vollständigen paaren Graphen, *Abh. Math. Sem. Univ. Hamburg* **28** (1965), 139–150.